

The vacuum impedance and unit systems

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In electromagnetism, the vacuum impedance Z_0 is a universal constant, which is as important as the velocity of light c in vacuum. Unfortunately, however, its significance does not seem to be appreciated so well and sometimes the presence itself is ignored. It is partly because in the Gaussian system of units, which has widely been used for long time, Z_0 is a dimensionless constant and of unit magnitude. In this paper, we clarify that Z_0 is a fundamental parameter in electromagnetism and plays major roles in the following scenes: reorganizing the structure of the electromagnetic formula in reference to the relativity; renormalizing the quantities toward natural unit systems starting from the SI unit system; and defining the magnitudes of electromagnetic units.

I. INTRODUCTION

The notion of vacuum impedance was introduced in late 1930's by Schelkunoff¹ in the study of wave propagation. It is defined as the amplitude ratio of the electric and magnetic fields of plane waves in vacuum, $Z_0 = E/H$, which has the dimension of electrical resistance. It is also called the characteristic impedance of vacuum or the wave resistance of vacuum. Due to the historical reasons, it has been recognized as a special parameter for engineers rather than a universal physical constant. Compared with the famous formula for the velocity of light in terms of the vacuum permittivity ϵ_0 and the vacuum permeability μ_0 ,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}, \quad (1)$$

the expression for the vacuum impedance

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (2)$$

is used far less often. It is obvious when you look up index pages of textbooks on electromagnetism. A possible reason is perhaps that the Gaussian system of units, in which Z_0 is a dimensionless constant and of unit magnitude, has been used for long time.

In this paper, we reexamine the structure of electromagnetism in view of the SI (The International System of Units) system and find that Z_0 plays very important roles as a universal constant. In this process we also find that a wide-spread belief that the Gaussian system of units is akin to natural unit systems and suitable for theoretical studies is not correct.

II. RELATIVISTIC PAIRS OF VARIABLES

In relativity, the space variable \mathbf{x} and the time variable t are combined to form a 4-vector (ct, \mathbf{x}) . The constant c , which has the dimension of velocity, serves as a factor matching the dimensions of time and space. When we

introduce a normalized variable $\tau \equiv ct$, the 4-vector is simplified as (τ, \mathbf{x}) .

In this form, space and time are represented by the same dimension. This can be done offhandedly by setting $c = 1$. It should be noted, however, this procedure is irreversible and the dimension for time is lost. There c becomes dimensionless. It is better to introduce normalized quantities such as τ when we compare the different systems of units.

When the ratio of two quantities X and Y is dimensionless (just a pure number), we write $X \stackrel{D}{\sim} Y$ and read X and Y are dimensionally equivalent. For example, we have $ct \stackrel{D}{\sim} \mathbf{x}$. If a quantity X can be measured in a unit u , we can write $X \stackrel{D}{\sim} u$.² For example, for $l = 2.5$ m we have $l \stackrel{D}{\sim} \text{m}$.

With this notation, we can repeat the above discussion. For $ct \stackrel{D}{\sim} \mathbf{x}$, instead of recasting $t \stackrel{D}{\sim} \mathbf{x}$ by forcibly setting $c = 1$, we introduce a new normalized quantity $\tau = ct$ and have $\tau \stackrel{D}{\sim} \mathbf{x}$. Then velocity v and c is normalized as $\tilde{v} = v/c$ and $\tilde{c} = c/c = 1$, respectively.

From the relativistic point of view, the scalar potential ϕ and the vector potential \mathbf{A} are respectively a part of a unified quantity. Considering $\phi \stackrel{D}{\sim} V$, $\mathbf{A} \stackrel{D}{\sim} \text{Vs/m}$, we can introduce a pair of quantities:

$$(\phi, c\mathbf{A}) \stackrel{D}{\sim} V. \quad (3)$$

Similarly, we introduce other pairs:

$$\begin{aligned} (\mathbf{E}, c\mathbf{B}) &\stackrel{D}{\sim} V/\text{m}, \\ (\mathbf{H}, c\mathbf{D}) &\stackrel{D}{\sim} A/\text{m}, \\ (\mathbf{J}, c\rho) &\stackrel{D}{\sim} A/\text{m}^2. \end{aligned} \quad (4)$$

where \mathbf{E} , \mathbf{B} , \mathbf{H} , \mathbf{D} , \mathbf{J} , and ρ represent electric field, magnetic flux density, magnetic field strength, electric flux density, and charge density, respectively. Mathematically those pairs are anti-symmetric tensors in the 4-dimensional space as defined in Appendix.

We have seen that the constant c appears when we form a relativistic tensor from a pair of non-relativistic electromagnetic quantities. In Table I, such relativistic pairs are listed according to their tensor orders. We will

<u>F series</u>		<u>S series</u>
$(\phi, c\mathbf{A})$		
$\downarrow d$	Y_0	
$(\mathbf{E}, c\mathbf{B}) \leftarrow * \rightarrow (\mathbf{H}, c\mathbf{D})$		
$\downarrow d$	Z_0	$\downarrow d$
0		$(\mathbf{J}, c\varrho)$
		$\downarrow d$
		0

TABLE I: Relativistic pairs of quantities are arranged as a diagram, the rows of which correspond to the orders of tensors ($n = 1, 2, 3, 4$). In the left column, the quantities related to the electromagnetic forces (the F series), and in the right column, the quantities related to the electromagnetic sources (the S series) are listed. The exterior derivative “d” connects two pairs of quantities. These differential relations correspond to the definition of (scalar and vector) potentials, the Maxwell’s four equations, and the charge conservation. Hodge’s star operator “*” connects $(\mathbf{E}, c\mathbf{B})$ and $(\mathbf{H}, c\mathbf{D})$ pairs. This corresponds to the constitutive relations for vacuum. Here appears the vacuum impedance $Z_0 = 1/Y_0$.

find that this table is very helpful to overview the structure of electromagnetism.^{3,4}

III. ROLES OF THE VACUUM IMPEDANCE

In this section we show some examples for which Z_0 plays important roles.

a. Source-field relation We know that the scalar potential $\Delta\phi$ induced by a charge $\Delta q = \varrho\Delta v$ is

$$\Delta\phi = \frac{1}{4\pi\epsilon_0} \frac{\varrho\Delta v}{r}, \quad (5)$$

where r is the distance between the source and the point of observation. The charge is presented as a product of charge density ϱ and a small volume Δv . Similarly a current moment (current by length) $\mathbf{J}\Delta v$ generates the vector potential

$$\Delta\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{J}\Delta v}{r}. \quad (6)$$

Considering the relations (3) and (4), these equations can be united as

$$\begin{bmatrix} \Delta\phi \\ \Delta(c\mathbf{A}) \end{bmatrix} = \frac{Z_0}{4\pi r} \begin{bmatrix} c\varrho \\ \mathbf{J} \end{bmatrix} \Delta v. \quad (7)$$

We see that the vacuum impedance Z_0 plays the role to relate the source $(\mathbf{J}, c\varrho)\Delta v$ and the resultant fields $\Delta(\phi, c\mathbf{A})$ in a unified manner.

b. Constitutive relation A similar role can be seen for the constitutive relations for vacuum; $\mathbf{D} = \epsilon_0\mathbf{E}$ and $\mathbf{H} = \mu_0^{-1}\mathbf{B}$ can be combined as

$$\begin{bmatrix} \mathbf{E} \\ c\mathbf{B} \end{bmatrix} = Z_0 \begin{bmatrix} c\mathbf{D} \\ \mathbf{H} \end{bmatrix}. \quad (8)$$

More precisely, with Hodge’s star operator “*” (see Appendix),^{5,6} it can be written as

$$(\mathbf{E}, c\mathbf{B}) = *Z_0(\mathbf{H}, c\mathbf{D}). \quad (9)$$

It should be noted that the electric relation and the magnetic relation are united under the sole parameter Z_0 .

c. Plane wave For linearly polarized plane waves in vacuum, a simple relation $E = cB$ holds. If we introduce $H (= \mu_0^{-1}B)$ instead of B , we have $E = Z_0H$. The latter relation was introduced by Schelkunoff¹ in 1938. The reason why H is used instead of B is as follows. The boundary conditions for magnetic fields at the interface of media 1 and 2 are $H_{1t} = H_{2t}$ (tangential) and $B_{1n} = B_{2n}$ (normal). For the case of normal incidence, which is most important practically, the latter condition becomes trivial and cannot be used. Therefore H is used more conveniently. The mixed use of the quantities (E and H) of the F and S series invite Z_0 unintentionally.

d. Magnetic monopole Let us compare the force between charges q and that between the magnetic monopoles g ($\overset{\text{D}}{\text{Vs}} = \text{Wb}$). If these forces are the same for equal distances r , i.e., $q^2/(4\pi\epsilon_0 r^2) = g^2/(4\pi\mu_0 r^2)$, we have the relation $g = Z_0q$. With this relation in mind, the Dirac monopole g_0 , whose quantization condition is $g_0e = h$, can be beautifully expressed in terms of the elementary charge e as

$$g_0 = \frac{h}{e} = \frac{h}{Z_0e^2}(Z_0e) = (2\alpha)^{-1}Z_0e \quad (10)$$

where $h = 2\pi\hbar$ is Planck’s constant. The dimensionless parameter $\alpha = Z_0e^2/2h = e^2/4\pi\epsilon_0\hbar c \sim 1/137$ is called the fine structure constant, whose value is independent of unit systems and characterize the strength of the electromagnetic interaction. The use of Z_0 helps to keep SI-formulae in simple forms.

e. The F series versus the S series Impedance (resistance) is a physical quantity by which voltage and current are related.⁷ In the SI system, the unit for voltage is V(= J/C) (volt) and the unit for current is A(= C/s) (ampere). We should note that the latter is proportional to and the former is inversely proportional to the unit of charge, C (coulomb). We also note in Table I that the units for quantities in the F series are proportional to the volt and those in the S series are proportional to the ampere. After all, the vacuum impedance Z_0 plays the role to connect the quantities in the F and S series. In the above cases we have found that the use of Z_0 (together with c) instead of ϵ_0 or μ_0 simplifies equations.

IV. THE MAGNITUDE OF THE UNIT OF RESISTANCE

Here we consider a hypothetical situation where we are allowed to redefine the magnitudes of units in electromagnetism.

The product of the unit of voltage and that of current should yield the unit of power, $1\text{ V} \times 1\text{ A} = 1\text{ W} = 1\text{ J/s}$,

(ϕ, \tilde{A}) V	(ϕ, \tilde{A}) V	$(\hat{\phi}, \hat{A})$ \sqrt{N}	$(\hat{\phi}, \hat{A})$ \sqrt{N}
(\mathbf{E}, \tilde{B}) V/m	(\mathbf{E}, \tilde{B}) V/m	$(\hat{\mathbf{E}}, \hat{B})$ \sqrt{N}/m	$(\hat{\mathbf{E}}, \hat{B})$ \sqrt{N}/m
(\mathbf{H}, \tilde{D}) A/m	$(\mathbf{H}^*, \tilde{D}^*)$ V/m	$(\hat{\mathbf{H}}, \hat{D})$ \sqrt{N}/m	$(\hat{\mathbf{H}}, \hat{D})$ \sqrt{N}/m
$(\mathbf{J}, \tilde{\varrho})$ A/m ²	$(\mathbf{J}^*, \tilde{\varrho}^*)$ V/m ²	$(c^{-1}\hat{\mathbf{J}}, \hat{\varrho})$ \sqrt{N}/m^2	$(\hat{\mathbf{J}}, \hat{\varrho})$ \sqrt{N}/m^2
(a) c -normalization	(b) (c, Z_0) -normalization	(c) Gaussian	(d) modified Gaussian

TABLE II: Pairs of quantities in electromagnetism and their units (dimension). Quantities X normalized with c and Z_0 are marked as $\tilde{X} = cX$ and $X^* = Z_0X$, respectively. As seen in (a) and (b), the variety of units is reduced owing to the normalization. Gaussian unit system (c) and the modified Gaussian unit system (d) are presented with normalized variables: $\hat{S} = S/\sqrt{\varepsilon_0}$, $\hat{F} = F\sqrt{\varepsilon_0}$, $\hat{S} = S/\sqrt{\mu_0}$, $\hat{F} = F\sqrt{\mu_0}$, where S (F) represents a quantity in the S (F) series. We notice an irregularity in the fourth row of the Gaussian system.

which is a fixed quantity determined mechanically, or outside of electromagnetism. Thus, a new volt V' and a new ampere A' must be defined so as to satisfy

$$A' = kA, \quad V' = k^{-1}V, \quad (11)$$

in terms of the currently used V and A, where k ($\neq 0$) is a scaling factor. Accordingly a new ohm Ω' must be redefined as

$$\Omega' = k^{-2}\Omega. \quad (12)$$

We denote the numerical value as $\{A\} \equiv A/u$, when we measure a physical quantity A with a unit u . For example, for $l = 1.3\text{ m}$ we write $\{l\} = l/m = 1.3$. We can have another numerical value $\{A\}' \equiv A/u'$, when we measure the same quantity A in a different unit u' . Now we have the relation

$$A = \{A\}u = \{A\}'u'. \quad (13)$$

It should be stressed that the physical quantity A itself is independent of the choice of units. What depends on the choice is the numerical value $\{A\}$.

In the SI system, from the definition of c and μ_0 , the vacuum impedance is represented as $Z_0 = \sqrt{\mu_0/\varepsilon_0} = c\mu_0 = (299\,792\,458\text{ m/s}) \times (4\pi \times 10^{-7}\text{ H/m}) = (119.916\,983\,2 \times \pi)\Omega \sim 377\Omega$. In our new system, with (12) we have

$$Z_0 = \{Z_0\}\Omega = k^2\{Z_0\}\Omega' = \{Z_0\}'\Omega'. \quad (14)$$

The numerical value must be changed from $\{Z_0\} = 377$ to $\{Z_0\}' = 377k^2$. For example, we could choose a new ohm Ω' so that $Z_0 = 1\Omega'$ is satisfied by setting $k \sim 1/\sqrt{377}$. Conversely, to fix $\{Z_0\}$ to a particular number implies the determination of the magnitude of units (Ω , V, A, and others) in electromagnetism.

Once k , or $\{Z_0\}$ is fixed, the numerical values for quantities in the F series are multiplied by k and those in the S series are divided by k . The sole parameter k or $\{Z_0\}$ determines the numerical relation between the F and S series.

Coulomb's law for charges q_1 and q_2 can be rewritten as

$$\begin{aligned} F &= \frac{1}{4\pi\varepsilon_0} \frac{q_1q_2}{r^2} = q_2E = \{q_2\}As \times \{E\}V/m \\ &= \{q_2\}'A's \times \{E\}'V'/m \end{aligned} \quad (15)$$

where $E = (4\pi\varepsilon_0)^{-1}(q_1/r^2)$ is the electric field induced by q_1 . We see

$$\{q_2\}' = k^{-1}\{q_2\}, \quad \{E\}' = k\{E\}, \quad (16)$$

and find that the numerical value for the charge q and that for the electric field E will be changed reciprocally. We also note $\{\varepsilon_0\}' = k^{-2}\{\varepsilon_0\}$ and $\{\mu_0\}' = k^2\{\mu_0\}$.

In the SI, the ampere is defined in terms of the force F between the parallel two wires carrying the same amplitude of current I . We have $F/l = \mu_0 I^2/(2\pi r)$, where r is the separation and l is the length of wires. Substituting $F = 2 \times 10^{-7}\text{ N}$, $r = l = 1\text{ m}$, $I = 1\text{ A}$, we get $\mu_0 = 4\pi \times 10^{-7}\text{ H/m}$. Thus the magnitude $\{\mu_0\}$ (or $\{Z_0\}$) are fixed.

We could determine $\{\varepsilon_0\}$ by the force between charges with the same magnitude. In Giorgi's unit system (1901), which is a predecessor of the MKSA unit system or the SI system, k was fixed by determining the magnitude of the ohm. The way of determination $\{Z_0\}$ has been and will be changed according to the development of high precision measurement technique.

V. TOWARD NATURAL UNIT SYSTEMS

As shown in Table II (a), by introducing a new set of normalized quantities, $\tilde{X} = cX$, derived from SI quantities X , we can reduce the number of fundamental dimensions. In this case, we only need three; the ampere, the volt, and the meter.

Further, as seen in Table II (b), when we introduce a set of normalized quantities, $X^* = Z_0X$, by multiplying Z_0 , only the volt and the meter are required. By normalizing the quantities with the fundamental constants, c and Z_0 , we have a simplified set of Maxwell's equations:

$$\begin{aligned} \nabla \cdot \tilde{D}^* &= \tilde{\varrho}^*, \quad \nabla \times \tilde{H}^* = \frac{\partial \tilde{D}^*}{\partial \tau} + \mathbf{J}^* \\ \nabla \cdot \tilde{B} &= 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \tilde{B}}{\partial \tau}. \end{aligned} \quad (17)$$

with $\tilde{D}^* = \mathbf{E}$ and $\mathbf{H}^* = \tilde{B}$. Considering $\tau = ct$, this set of equations resembles to the Maxwell's equations in the Gaussian system of units except for the rationalizing

factor 4π [See Eq. (29)]. However there is a significant difference; the factor $1/c$ is missing in the current density term. We will return to this point later. It should be stressed that a natural system of units can be reached from the SI system by normalizations without detouring via the Gaussian system.

The number of basic units has been reduced from four (m, kg, s, A) to two (m, V) by introducing the quantities normalized with c and Z_0 . For further reduction toward a natural unit system,^{8,9} \hbar and the gravitational constant G can be used for example.

VI. GAUSS AND HEAVISIDE-LORENTZ SYSTEMS OF UNITS

The SI and the cgs (esu, emu, Gaussian) systems differ in three respects. First, in the cgs unit systems, no fundamental dimensions are supplemented to the three fundamental dimensions for mechanics; length, mass, and time. On the other hand in the SI (MKSA) system, a new fundamental dimension that for electric current is introduced. The cgs systems contain three basic units, while the SI system contains four.

Secondly, the cgs systems are irrational systems; the factor $(1/4\pi)$ is erased from Coulomb's law but the factor 4π appears in the source terms of Maxwell's equations instead. The SI is a rational system, which has the opposite appearance.

Thirdly, the base mechanical system for the cgs systems is the cgs (centimeter, gram, and second) mechanical system. That for the SI system is the MKS (meter, kilogram, and second) system.

In order to focus all our attention on the first respect, i.e., the number of basic units, we will ignore the differences in the last two respects. *From now on, we pretend that all the cgs systems (esu, emu, and Gaussian) are constructed rationally on the MKS mechanical system.* (Actually the Heaviside-Lorentz system is an MKS version of the Gaussian system, namely, a three-unit, rational system based on the MKS system.)

To go from the SI system to the cgs systems, we have to reduce the number of basic units by normalization with a universal constant.

In the cgs electrostatic system of units (esu), Coulomb's law is expressed as

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} = \frac{1}{4\pi} \frac{\hat{q}_1 \hat{q}_2}{r^2}. \quad (18)$$

Thus, the normalized charge¹⁰

$$\hat{q} = \frac{q}{\sqrt{\epsilon_0}} \simeq \frac{C}{\sqrt{F/m}} = \sqrt{Jm} = \sqrt{Nm} \quad (19)$$

is a quantity expressed by mechanical dimensions only. The quantities in the S series, each of which is proportional to the coulomb, C, can be normalized by division with $\sqrt{\epsilon_0}$. On the other hand, the quantities in the F

series, each of which is inversely proportional to C, can be normalized by multiplication with $\sqrt{\epsilon_0}$. For example, \mathbf{E} , \mathbf{D} , \mathbf{B} , and \mathbf{H} , are normalized as

$$\begin{aligned} \hat{\mathbf{E}} &= \mathbf{E}\sqrt{\epsilon_0} \simeq \frac{\sqrt{N}}{m}, & \hat{\mathbf{D}} &= \frac{\mathbf{D}}{\sqrt{\epsilon_0}} \simeq \frac{\sqrt{N}}{m}, \\ \hat{\mathbf{B}} &= \mathbf{B}\sqrt{\epsilon_0} \simeq \frac{\sqrt{Ns}}{m^2}, & \hat{\mathbf{H}} &= \frac{\mathbf{H}}{\sqrt{\epsilon_0}} \simeq \frac{\sqrt{N}}{s}, \end{aligned} \quad (20)$$

respectively. We have the constitutive relation

$$\hat{\mathbf{D}} = \hat{\mathbf{E}}, \quad \hat{\mathbf{H}} = c^2 \hat{\mathbf{B}}, \quad (21)$$

and the normalized permittivity $\hat{\epsilon}_0 = 1$ and permeability $\hat{\mu}_0 = 1/c^2$. The normalized vacuum impedance is

$$\hat{Z}_0 = \sqrt{\frac{\hat{\mu}_0}{\hat{\epsilon}_0}} = \frac{1}{c} \simeq \frac{s}{m}. \quad (22)$$

For the cgs electromagnetic system of units (emu), S-series quantities are multiplied by $\sqrt{\mu_0}$ and F-series quantities are divided by $\sqrt{\mu_0}$. With this normalization, μ_0 is eliminated from the magnetic Coulomb law or the law of magnetic force between currents. The fields are normalized as

$$\begin{aligned} \hat{\mathbf{B}} &= \frac{\mathbf{B}}{\sqrt{\mu_0}} \simeq \frac{\sqrt{N}}{m}, & \hat{\mathbf{H}} &= \mathbf{H}\sqrt{\mu_0} \simeq \frac{\sqrt{N}}{m}, \\ \hat{\mathbf{E}} &= \frac{\mathbf{E}}{\sqrt{\mu_0}} \simeq \frac{\sqrt{N}}{s}, & \hat{\mathbf{D}} &= \mathbf{D}\sqrt{\mu_0} \simeq \frac{\sqrt{Ns}}{m^2}. \end{aligned} \quad (23)$$

The constitutive relations are

$$\hat{\mathbf{H}} = \hat{\mathbf{B}}, \quad \hat{\mathbf{D}} = c^{-2} \hat{\mathbf{E}}, \quad (24)$$

and we have the normalized permeability $\hat{\mu}_0 = 1$ and permittivity $\hat{\epsilon}_0 = 1/c^2$. The normalized vacuum impedance is

$$\hat{Z}_0 = \sqrt{\frac{\hat{\mu}_0}{\hat{\epsilon}_0}} = c \simeq \frac{m}{s}. \quad (25)$$

The Gaussian system of units is a combination of the esu and emu systems. For electrical quantities the esu normalization is used and for magnetic quantities the emu normalization is used. Namely we use $\hat{\mathbf{E}}$, $\hat{\mathbf{D}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{H}}$, all of which have the dimension \sqrt{N}/m . The constitutive relations are simplified as

$$\hat{\mathbf{D}} = \hat{\mathbf{E}}, \quad \hat{\mathbf{H}} = \hat{\mathbf{B}}, \quad (26)$$

and we have $\hat{\epsilon}_0 = 1$ and $\hat{\mu}_0 = 1$. So far it looks nice because electric and magnetic quantities are treated symmetrically. This appearance is the reason why the Gaussian system has been used so widely. However, there is an overlooked problem in the normalization of current density. It is normalized as $\hat{\mathbf{J}} = \mathbf{J}/\sqrt{\epsilon_0}$ in the Gaussian system. The current density is the quantity primarily connected to magnetic fields and therefore it should be

normalized as $\hat{\mathbf{J}} = \mathbf{J}\sqrt{\mu_0}$ as for the emu system. Because of this miscasting, we have an irregularity in the fourth row of the column (c) of Table II. The Gaussian normalization happens to make the pairs of quantities relativistic with exception of the (\mathbf{J}, ϱ) pair.

The relativistic expression for the conservation of charge should be

$$\frac{\partial \hat{\varrho}^*}{\partial(ct)} + \nabla \cdot \hat{\mathbf{J}}^* = 0, \quad (27)$$

as for the cases of c - or (c, Z_0) -normalization. In the Gaussian system, however, the non-relativistic expression

$$\frac{\partial \hat{\varrho}}{\partial t} + \nabla \cdot \hat{\mathbf{J}} = 0 \quad (28)$$

is adopted. As a practical system of units, it is a reasonable (and perhaps unique) choice.

This quirk can clearly be seen, when we compare the Maxwell's equations (17) in the natural system of units and that for the Gaussian system:

$$\begin{aligned} \nabla \cdot \hat{\mathbf{D}} &= \hat{\varrho}, & \nabla \times \hat{\mathbf{H}} &= \frac{1}{c} \frac{\partial \hat{\mathbf{D}}}{\partial t} + \frac{1}{c} \hat{\mathbf{J}}, \\ \nabla \cdot \hat{\mathbf{B}} &= 0, & \nabla \times \hat{\mathbf{E}} &= -\frac{1}{c} \frac{\partial \hat{\mathbf{B}}}{\partial t}. \end{aligned} \quad (29)$$

The factor $1/c$ in the current density term is a seam introduced when the esu and emu systems are joined into the Gaussian system.

The common belief that the Gaussian system is superior to the SI system because of the similarity to a natural unit system or because of the compatibility with relativity is almost pointless. We should remember that the Gaussian unit system was established in 1870s, when the relativity or the Lorentz transformation were not known yet.

The modified Gaussian system,¹¹ in which $\hat{\mathbf{J}}$ is adopted and the above seam is eliminated, has been proposed but is rarely used. Column (d) of Table II contains quantities in the modified Gaussian system. They differ uniformly by a factor $\sqrt{\varepsilon_0}$ from the (c, Z_0) -normalized quantities in Column (b).

VII. SUMMARY AND DISCUSSION

The important expression (1) for the velocity of light also holds for the esu and emu systems:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\hat{\mu}_0 \hat{\varepsilon}_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \quad (30)$$

but not for the Gaussian system; $1/\sqrt{\varepsilon_0 \mu_0} = 1 \neq c$.

The expression (2) for the vacuum impedance, is rewritten as $\hat{Z}_0 = 1/c$ and $\hat{Z}_0 = c$ for the esu and emu systems, respectively. Maxwell himself worked with

the emu system when he found that light is an electromagnetic disturbance propagated according to electromagnetic laws.¹² For the emu system the dimensions of resistance and velocity are degenerate. For the Gaussian system, the vacuum impedance reduces to unity, $\sqrt{\hat{\mu}_0/\hat{\varepsilon}_0} = 1$.

Thus there is no room for the vacuum impedance in the cgs systems, which contains only three basic units. However when we move to a unit system with four basic units, the vacuum impedance Z_0 should be recognized as a fundamental constant as important as the velocity of light c . It has been underestimated or ignored for long time. It is due to the fact that the Gaussian system, for which Z_0 is dimensionless and of unit magnitude, has been used for long time even after the introduction of the MKSA and the SI systems.

As has been pointed out by Sommerfeld,³ the Gaussian system tends to veil the significance of D and H in vacuum. Sometimes it is told that in vacuum only E and B have their significance and D and H lose their meaning. This argument is strongly misled by the use of Gaussian system of units. Considering the tensorial nature of quantities as in (9), the constitutive relations for the Gaussian system are expressed as

$$(\hat{\mathbf{E}}, \hat{\mathbf{B}}) = *(\hat{\mathbf{H}}, \hat{\mathbf{D}}). \quad (31)$$

with Hodge's star operator. This relation represents important geometrical relations of electromagnetic fields, which can hardly be suggested by the simple vector relations (26).

Now we have understood that without the help of Gaussian system, we can reach natural systems of units directly from the SI system. We believe it's time to say goodbye to the Gaussian system of units. You won't miss its simpleness if you have the vacuum impedance Z_0 as a key parameter.

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APPENDIX: TENSOR NOTATIONS

In this appendix, we will explain mathematically the entities of Table I. With basis vectors $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, a space-time vector can be represented as

$$(ct, \mathbf{x}) = (ct)\mathbf{e}_0 + \mathbf{x}, \quad \mathbf{x} = \sum_{i=1}^3 x^i \mathbf{e}_i. \quad (\text{A.1})$$

The basis vectors satisfy $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$). The nonzero elements of g are $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$. We also introduce the dual basis vectors: $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$, with $\mathbf{e}^0 = -\mathbf{e}_0$, $\mathbf{e}^i = \mathbf{e}_i$ ($i = 1, 2, 3$).

The quantities in electromagnetism are expressed by antisymmetric tensors of rank n (n -forms) in the four-dimensional space.^{4,5,6} For scalar fields α , β and 3-dimensional vector fields \mathbf{X} , \mathbf{Y} , n -forms ($n = 1, 2, 3$) are defined as

$$\begin{aligned}(\alpha; \mathbf{X})_1 &= \alpha e^0 + \mathbf{X}, \\ (\mathbf{X}; \mathbf{Y})_2 &= e^0 \wedge \mathbf{X} + \mathbf{Y} \\ (\mathbf{Y}; \beta)_3 &= e^0 \wedge \mathbf{Y} + \beta \sigma,\end{aligned}\quad (\text{A.2})$$

where $\sigma = e^1 \wedge e^2 \wedge e^3$ is the 3-form representing the volume element and “ \wedge ” represents the antisymmetric tensor product. $\mathbf{Y} = \sum_{j,k=0}^3 \epsilon_{ijk} Y_i e_j \wedge e_k$ is a 2-form (in three dimensional space) derived from \mathbf{Y} . ϵ_{ijk} is the Levi-Civita symbol. With these, the pair quantities in Table I are defined as

$$\begin{aligned}(\phi, c\mathbf{A}) &= (-\phi; c\mathbf{A})_1, & (\mathbf{E}, c\mathbf{B}) &= (-\mathbf{E}; c\mathbf{B})_2, \\ (\mathbf{H}, c\mathbf{D}) &= (\mathbf{H}; c\mathbf{D})_2, & (\mathbf{J}, c\varrho) &= (-\mathbf{J}; c\varrho)_3.\end{aligned}\quad (\text{A.3})$$

The differential operator d is defined as

$$d = e^0 \wedge \frac{\partial}{\partial x^0} + \sum_{i=1}^3 e^i \wedge \frac{\partial}{\partial x^i}.\quad (\text{A.4})$$

and the application to an n -form results in an $(n+1)$ -form. For example, we have $d(\phi, c\mathbf{A}) = (\mathbf{E}, c\mathbf{B})$, which is equivalent to $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The successive applications of d always yield zero ($dd = 0$), therefore we have $d(\mathbf{E}, c\mathbf{B}) = 0$, which corresponds to $\partial\mathbf{B}/\partial t + \nabla \times \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$. Furthermore, $d(\mathbf{H}, c\mathbf{D}) = (\mathbf{J}, c\varrho)$ yields $-\partial\mathbf{D}/\partial t + \nabla \times \mathbf{H} = \mathbf{J}$, $\nabla \cdot \mathbf{D} = \varrho$, and $d(\mathbf{J}, c\varrho) = 0$ yields the conservation of charge: $\partial\varrho/\partial t + \nabla \cdot \mathbf{J} = 0$.

Another important notation is Hodge’s star operator “ $*$ ”, which converts an n -form into a $(4-n)$ -form;

$$\begin{aligned}* (\alpha; \mathbf{X})_1 &= (\mathbf{X}; \alpha)_3, \\ * (\mathbf{X}; \mathbf{Y})_2 &= (-\mathbf{Y}; \mathbf{X})_2, \\ * (\mathbf{Y}; \beta)_3 &= (\beta; \mathbf{Y})_1.\end{aligned}\quad (\text{A.5})$$

From the second relation and Eq. (A.3), the constitutive relations are represented as $(\mathbf{E}, c\mathbf{B}) = *Z_0(\mathbf{H}, c\mathbf{D})$.

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¹ S. A. Schelkunoff, “The Impedance Concept and Its Application to Problems of Reflection, Refraction, Shielding, and Power Absorption,” Bell System Tech. J. **17**, 17 (1938).

² Usually, in order to avoid the use of a specific unit system, a dimension is represented universally in terms of L (length), M (mass), T (time), and I (current), such as $[f] = LMT^{-2}$. Here, for simplicity, we use SI units representing dimensions as $f \stackrel{D}{\sim} N = \text{kg m/s}^2$.

³ A. Sommerfeld, *Electrodynamics — Lectures on Theoretical Physics* (Academic Press, New York, 1952), pp. 45–54, 212–222.

⁴ G. A. Deschamps, “Electromagnetics and Differential Forms,” Proc. IEEE **69**, 676 (1981).

⁵ T. Frankel, *The Geometry of Physics: An Introduction* (Cambridge, 2004), 2nd Ed., pp. 118–123.

⁶ F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics, Charge, Flux, and Metric* (Birkhäuser, Boston, 2003), pp. 143–162.

⁷ Consider a rectangular, uniform sheet with size $w \times l$. Two electrodes are attached along the edges of length w . The resistance R between the electrodes is proportional to l and inversely proportional to w : $R = \sigma l/w$. The proportionality constant σ is called the surface resistivity of the sheet and $\sigma = Rw/l \stackrel{D}{\sim} \Omega\text{m/m} = \Omega$. The surface resistivity and

the resistance happen to share the same dimension $\stackrel{D}{\sim} \Omega$ but their physical meaning is rather different. Dimensionally the vacuum impedance resembles to the former because it is introduced in terms of the fields of plane wave as $Z_0 = E/H \stackrel{D}{\sim} (\text{V/m})/(\text{A/m}) = \Omega\text{m/m}$. See Reference 1.

⁸ M. J. Duff, L. B. Okun, and G. Veneziano, “Dialogue on the number of fundamental constants,” J. High Energy Phys. PHEP03, 023 (2002); arXiv:physics/0110060 (2002).

⁹ F. Wilczek, “On Absolute Units, I: Choices,” Physics Today, October, 12 (2005).

¹⁰ To be precise the real esu charge \hat{q}_{esu} is

$$\hat{q}_{\text{esu}} = \frac{\hat{q}}{\sqrt{4\pi}} \frac{\sqrt{\text{dyn cm}}}{\sqrt{N} \text{ m}} = \frac{\hat{q}}{\sqrt{4\pi \times 10^5}},$$

where $\text{dyn} = \text{g} \cdot \text{cm/s}^2$ is the cgs unit of force. We will hereafter ignore factors such as $(4\pi \times 10^{-5})^{-1/2}$.

¹¹ J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, New York, 1998), 3rd Ed., pp. 782–784. See footnotes for Tables 3 and 4 in Appendix.

¹² J. C. Maxwell, “A Dynamical Theory of Electromagnetic Field,” Royal Society Transactions **155**, 526 (1864); T. K. Simpson, *Maxwell on the Electromagnetic Field* (Rutgers University Press, New Brunswick, 1997), p. 284.