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LETTER

Quantum Frequency Arrangements, Quantum Mixed Orthogonal Arrays and Entangled States*

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SUMMARY In this work, we introduce notions of quantum frequency arrangements consisting of quantum frequency squares, cubes, hypercubes and a notion of orthogonality between them. We also propose a notion of quantum mixed orthogonal array (QMOA). By using irredundant mixed orthogonal array proposed by Goyeneche et al. we can obtain k-uniform states of heterogeneous systems from quantum frequency arrangements and QMOAs. Furthermore, some examples are presented to illustrate our method.

key words: *quantum frequency arrangements, quantum mixed orthogonal array, irredundant orthogonal array, k-uniform states*

1. Introduction

The phenomenon of entanglement is a remarkable feature of quantum physics that has been identified as a key ingredient in many areas of quantum information theory including quantum key distribution [4], superdense coding [1], and teleportation [2]. However, the general problem of how to construct genuinely multipartite entangled states remains unresolved. There has been some progress towards a solution [5]–[7], [10], [20], but the task at hand is generally considered a difficult one.

As is often the case [15], [17], combinatorics can be useful to quantum information theory, and orthogonal arrays (OAs) are fundamental ingredients in the construction of other useful combinatorial objects [9]. Recently, many new methods of constructing OAs of strength k, especially mixed orthogonal arrays (MOAs), have been presented, and many new classes of OAs have been obtained [3], [16], [18], [19]. It is these new developments in OAs that suggest the possibility of constructing infinitely many new genuinely multipartite entangled states. A highly entangled quantum state of heterogeneous multipartite systems consisting of $N > 2$ parties is said to be k-uniform if every reduction to k parties is maximally mixed [6]. These states are closely related to quantum error correction codes over mixed alphabets. Recently, quantum Latin squares, cubes, hypercubes, and quantum orthogonal arrays have been introduced by the authors in [8], [11], [12], [22]. They also demonstrated that k-uniform states constructed from quantum Latin arrangements have high persistency of entanglement,which makes them ideal candidates for quantum information protocols. However, these combinatorial designs were only used to construct k-uniform states of homogeneous systems [8]. Therefore, more mathematical tools need to be discovered to construct k-uniform states of heterogeneous systems in [6].

In this work, we introduce notions of quantum frequency arrangements consisting of quantum frequency squares, cubes, hypercubes and a notion of orthogonality between them. We also propose a notion of quantum mixed orthogonal array (QMOA). By using irredundant mixed orthogonal array which is proposed in [6] we can obtain k-uniform states of heterogeneous systems from quantum frequency arrangements and QMOAs. Furthermore, some examples are presented to illustrate our method.

2. Preliminaries

Let A^T be the transposition of matrix A and $(d) = (0, 1, \ldots, d-1)^T$. Let 0_r and 1_r denote the $r \times 1$ vectors of $0s$ and $1s$, respectively. If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{u\times v}$ with elements from a Galois field with binary operations (+ and ·), the Kronecker product $A \otimes B$ is defined as $A \otimes B = (a_{ij} \cdot B)_{mu\times nv}$, where $a_{ij} \cdot B$ represents the $u \times v$ matrix with entries $a_{ij} \cdot b_{rs}$ $(1 \le r \le u, 1 \le s \le v)$. A matrix A can often be identified with a set of its row vectors if necessary. Let $\mathcal{H}_d^{\otimes m}$ be $\underbrace{\mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d}_{m}$.

Definition 1: ([9]) An orthogonal array OA$(r, N, d_1 d_2 \cdots d_N, k)$ is an $r \times N$ matrix, with the property that, in any $r \times k$ submatrix, all possible combinations of k symbols appear equally often as a row. The orthogonal array is called symmetrical if $d_1 = d_2 = \cdots = d_N$. Otherwise, the array is called a MOA.

Definition 2: ([6]) A MOA$(r, N, d_1 d_2 \ldots d_N, k)$ is called irredundant, written IrMOA, if every subset of $N-k$ columns contains a different sequence of $N-k$ symbols in every row.

Definition 3: ([20]) Let $S^l = \{(v_1, \ldots, v_l) | v_i \in S,\ i = 1, 2, \ldots, l\}$. The Hamming distance HD(u, v) between two vectors $u = (u_1, \ldots, u_l)$, $v = (v_1, \ldots, v_l) \in S^l$ is defined as the number of positions in which they differ. The minimal distance of a matrix A, written MD(A), is defined to be the minimal Hamming distance between its distinct rows.

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HD(A) is used to represent all the values of the Hamming distances between two distinct rows of A.

Definition 4: ([13]) Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\rho_A \equiv \mathrm{Tr}_B(\rho^{AB}),$$

where Tr_B is a map of operators known as the partial trace over system B. The partial trace is defined by

$$\mathrm{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|\mathrm{Tr}(|b_1\rangle\langle b_2|),$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the state space of A, and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the state space of B. The trace operation appearing on the right hand side is the usual trace operation for system B, so $\mathrm{Tr}(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$.

Definition 5: ([8]) A quantum orthogonal array QOA(r, N, d, k) is an arrangement consisting of r rows composed by N-partite normalized pure quantum states $|\varphi_j\rangle \in \mathcal{H}_d^{\otimes N}$, having d internal levels each, such that

$$d^k \sum_{i,j=0}^{r-1} \mathrm{Tr}_{i_1,\ldots,i_{N-k}}(|\varphi_i\rangle\langle\varphi_j|) = r\mathbb{I}_{d^k}$$

for every subset of $N-k$ parties $\{i_1, \ldots, i_{N-k}\}$.

Definition 6: The $3n$ slices in three directions of a cubic matrix $A = \{a_{\alpha\beta\gamma} \in F_d^w | \alpha, \beta, \gamma \in F_n\}$ can be expressed as
frontal slice: $A_{\alpha,:,:} = \{a_{\alpha\beta\gamma} \in F_d^w | \beta, \gamma \in F_n\},\ \alpha \in F_n$,
lateral slice: $A_{:,\beta,:} = \{a_{\alpha\beta\gamma} \in F_d^w | \alpha, \gamma \in F_n\},\ \beta \in F_n$, and
horizontal slice: $A_{:,:,\gamma} = \{a_{\alpha\beta\gamma} \in F_d^w | \alpha, \beta \in F_n\},\ \gamma \in F_n$.

Lemma 1: A MOA($r, N, d_1 \cdots d_N, k$) is irredundant if and only if its minimal distance is greater than k.

Proof. It follows from the definition of IrMOA.

Lemma 2: ([9]) Taking the runs in an $A = \mathrm{OA}(r, N, d_1 d_2 \cdots d_N, k)$ that begin with 0 (or any other particular symbol) and omitting the first column yields an OA($r/d_1, N-1, d_2 \cdots d_N, k-1$), denoted by A_0. Similarly, we can obtain $A_1, A_2, \ldots, A_{d_1-1}$ are all OA($r/d_1, N-1, d_2 \cdots d_N, k-1$). Then A can be written as follows.

$$A = \begin{pmatrix} 0_{r/d_1} & A_0 \\ 1_{r/d_1} & A_1 \\ \vdots & \vdots \\ (d_1-1)_{r/d_1} & A_{d_1-1} \end{pmatrix}.$$

Lemma 3: Let A as in Lemma 2 be an IrMOA($r, N, d_1 d_2 \cdots d_N, k$). Then A_i is an IrMOA($r/d_1, N-1, d_2 \cdots d_N, k-1$) and MD($A_i$) $\geq k+1$ for $i = 0, 1, \ldots, d_1-1$.

Proof. It follows from Lemma 2 that A_i is an OA($r/d_1, N-1, d_2 \cdots d_N, k-1$). By Lemma 1, we have MD(A) $\geq k+1$. Then MD(A_i) $\geq k+1 > k$. Therefore, A_i is an IrMOA($r/d_1, N-1, d_2 \cdots d_N, k-1$) by Lemma 1.

Lemma 4: Suppose A is an IrMOA($n^k, m+k, n^k d^m, k$). Then,
(1) If $d \leq n$, then $m \geq k$.
(2) If $d < n$, then $m > k$.
(3) If $d = n$, then $m \geq k$.

Proof. We only prove the second case. Assume $m \leq k$, then after removing the first k columns of A, the $n^k \times m$ subarray contains two same rows, which is a contradiction.

Unless stated otherwise, we only consider the case of $n > d$.

3. Main Results

Definition 7: A quantum frequency square of size n is an arrangement

$$\mathrm{QFS}(n, d) = \begin{pmatrix} |\psi_{0,0}\rangle & \ldots & |\psi_{0,n-1}\rangle \\ \vdots & & \vdots \\ |\psi_{n-1,0}\rangle & \ldots & |\psi_{n-1,n-1}\rangle \end{pmatrix}$$

composed of n^2 single-particle quantum states $|\psi_{i,j}\rangle \in \mathcal{H}_d$, $i, j \in \{0, \ldots, n-1\}$, such that each row and each column determine n/d orthonormal bases for a qudit system.

Definition 8: A set of n^2 pure quantum states $|\psi_{i,j}\rangle \in \mathcal{H}_d^{\otimes m}$, $m > 2$ arranged as

$$\begin{pmatrix} |\psi_{0,0}\rangle & \ldots & |\psi_{0,n-1}\rangle \\ \vdots & & \vdots \\ |\psi_{n-1,0}\rangle & \ldots & |\psi_{n-1,n-1}\rangle \end{pmatrix},$$

forms a set of m mutually orthogonal quantum frequency squares (m MOQFS(n, d)) if the following properties hold:

(1) The set of n^2 states $\{|\psi_{i,j}\rangle | i, j = 0, \ldots, n-1\}$ are orthogonal.

(2) The sum of every row in the array, i.e., $\sum_{j=0}^{n-1} |\psi_{i,j}\rangle$, is a 1-uniform state.

(3) The sum of every column in the array, i.e., $\sum_{i=0}^{n-1} |\psi_{i,j}\rangle$, is a 1-uniform state.

Definition 9: A quantum mixed orthogonal array QMOA($r, N, d_1 d_2 \ldots d_N, k$) is an arrangement consisting of r rows composed by N-partite normalized pure quantum states $|\varphi_j\rangle \in \mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_N}$, such that

$$d_{l_{N-k+1}} \cdots d_{l_N} \sum_{i,j=0}^{r-1} \mathrm{Tr}_{l_1,\ldots,l_{N-k}}(|\varphi_i\rangle\langle\varphi_j|) = r\mathbb{I}_{d_{l_{N-k+1}} \cdots d_{l_N}},$$

for every subset of $N-k$ parties $\{l_1, \ldots, l_{N-k}\}$.

Theorem 1: (1) From an IrMOA($n^2, m+2, n^2 d^m, 2$), we can construct a QMOA($n^2, m+2, n^2 d^m, 2$) and a set of m MOQFS(n, d).

(2) From the set of MOQFS in (1), we can define a QMOA($n^2, m+2, n^2 d^m, 2$).

(3) From the QMOA in (1), we can generate the set of m MOQFS(n, d).

Proof. From IrMOA($n^2, m+2, n^2 d^m, 2$) by a sequence of permutations of the columns, the rows, and the levels of each factor, we can obtain $A = (a, b, C)$, where $a = (n) \otimes 1_n$, $b = 1_n \otimes (n)$, and $C = (c_{ij})_{n^2 \times m}$ for $i = 0, 1, \cdots, n^2 - 1$, $j = 3, 4, \cdots, m+2$.

Construction. From A, we can obtain B = QMOA$(n^2, m+2, n^2 d^m, 2)$ and a set of m MOQFS(n, d) M as follows.

$$B = \begin{pmatrix} |0\rangle & |0\rangle & |c_{0,3}c_{0,4}\dots c_{0,m+2}\rangle \\ \vdots & \vdots & \vdots \\ |0\rangle & |n-1\rangle & |c_{n-1,3}c_{n-1,4}\dots c_{n-1,m+2}\rangle \\ \vdots & \vdots & \vdots \\ |n-1\rangle & |0\rangle & |c_{(n-1)n,3}c_{(n-1)n,4}\dots c_{(n-1)n,m+2}\rangle \\ \vdots & \vdots & \vdots \\ |n-1\rangle & |n-1\rangle & |c_{n^2-1,3}c_{n^2-1,4}\dots c_{n^2-1,m+2}\rangle \end{pmatrix},$$

$$M = \begin{pmatrix} |c_{0,3}\dots c_{0,m+2}\rangle & \cdots & |c_{n-1,3}\dots c_{n-1,m+2}\rangle \\ \vdots & & \vdots \\ |c_{(n-1)n,3}\dots c_{(n-1)n,m+2}\rangle & \cdots & |c_{n^2-1,3}\dots c_{n^2-1,m+2}\rangle \end{pmatrix}.$$

Verification. (1) Firstly, we prove B is a QMOA$(n^2, m+2, n^2 d^m, 2)$.

Let $|\varphi_i\rangle = |a_i b_i c_{i,3} \dots c_{i,m+2}\rangle$ and $T_{i,j} = \mathrm{Tr}_{l_1,\dots,l_m}(|\varphi_i\rangle\langle\varphi_j|)$, where $i, j = 0, 1, \dots, n^2-1$ and $\{l_1, l_2, \dots, l_m, l_{m+1}, l_{m+2}\} = \{1, 2, \dots, m+2\}$. Since A is an IrMOA$(n^2, m+2, n^2 d^m, 2)$, $T_{i,j} = 0$ for $i \neq j$. Therefore, we will consider the following three cases.

Case 1. If $1, 2 \notin \{l_1, l_2, \cdots, l_m\}$, then

$$\begin{aligned} T_{i,i} &= \mathrm{Tr}_{3,4,\dots,m+2}(|\varphi_i\rangle\langle\varphi_i|) \\ &= \langle c_{i,3}c_{i,4}\dots c_{i,m+2}|c_{i,3}c_{i,4}\dots c_{i,m+2}\rangle |a_i b_i\rangle\langle a_i b_i| \\ &= |a_i b_i\rangle\langle a_i b_i|. \end{aligned}$$

In the columns (a, b), each of all possible pairs as a row occurs with frequency $n^2/n^2 = 1$, then $\sum_{i=0}^{n^2-1} T_{i,i} = I_{n^2}$. We have

$$d_{l_{m+1}} d_{l_{m+2}} \sum_{i,j=0}^{r-1} T_{i,j} = d_a d_b \sum_{i=0}^{n^2-1} T_{i,i} = n^2 I_{n^2} = n^2 I_{d_{l_{m+1}} d_{l_{m+2}}}.$$

Case 2. If $\{1, 2\} \subseteq \{l_1, l_2, \cdots, l_m\}$, we can without loss of generality assume that $\{l_1, l_2\} = \{1, 2\}$. Then

$$\begin{aligned} T_{i,i} &= \mathrm{Tr}_{l_1,l_2,\dots,l_m}(|\varphi_i\rangle\langle\varphi_i|) \\ &= \langle a_i b_i c_{i,l_3} \cdots c_{i,l_m}|a_i b_i c_{i,l_3} \cdots c_{i,l_m}\rangle |c_{i,l_{m+1}} c_{i,l_{m+2}}\rangle\langle c_{i,l_{m+1}} c_{i,l_{m+2}}| \\ &= |c_{i,l_{m+1}} c_{i,l_{m+2}}\rangle\langle c_{i,l_{m+1}} c_{i,l_{m+2}}|. \end{aligned}$$

Since each of all possible pairs as a row occurs n^2/d^2 times in $(c_{l_{m+1}}, c_{l_{m+2}})$, $\sum_{i=0}^{n^2-1} T_{i,i} = n^2/d^2\, I_{d^2}$. We have

$$\begin{aligned} d_{l_{m+1}} d_{l_{m+2}} \sum_{i,j=0}^{n^2-1} T_{i,j} &= d^2 \sum_{i=0}^{n^2-1} T_{i,i} = d^2 (n^2/d^2) I_{d^2} = n^2 I_{d^2} \\ &= n^2 I_{d_{l_{m+1}} d_{l_{m+2}}}. \end{aligned}$$

Case 3. If $1 \in \{l_1, l_2, \cdots, l_m\}$ and $2 \notin \{l_1, l_2, \cdots, l_m\}$, or $1 \notin \{l_1, l_2, \cdots, l_m\}$ and $2 \in \{l_1, l_2, \cdots, l_m\}$, here we only prove the first case. Without loss of generality we may assume that $l_1 = 1$ and $l_{m+1} = 2$. Then

$$\begin{aligned} T_{i,i} &= \mathrm{Tr}_{l_1,\dots,l_m}(|\varphi_i\rangle\langle\varphi_i|) \\ &= \langle a_i c_{i,l_2} c_{i,l_3} \dots c_{i,l_m}|a_i c_{i,l_2} c_{i,l_3} \dots c_{i,l_m}\rangle |b_i c_{i,l_{m+2}}\rangle\langle b_i c_{i,l_{m+2}}| \\ &= |b_i c_{i,l_{m+2}}\rangle\langle b_i c_{i,l_{m+2}}|. \end{aligned}$$

Each of all possible pairs as a row in $(b, c_{l_{m+2}})$ occurs $n^2/(nd) = n/d$ times, then $\sum_{i=0}^{n^2-1} T_{i,i} = (n/d) I_{nd}$. Thus

$$\begin{aligned} d_{l_{m+1}} d_{l_{m+2}} \sum_{i,j=0}^{n^2-1} T_{i,j} &= nd \sum_{i=0}^{n^2-1} T_{i,i} = nd\, \frac{n}{d} I_{nd} = n^2 I_{nd} \\ &= n^2 I_{d_{l_{m+1}} d_{l_{m+2}}}. \end{aligned}$$

It follows from Definition 9 that B is a QMOA$(n^2, m+2, n^2 d^m, 2)$.

Secondly, we prove M is a set of m MOQFS(n, d). Here we need to consider the three properties as follows.

A is an IrMOA$(n^2, m+2, n^2 d^m, 2)$, then any two rows of the submatrix $(c_3, c_4, \dots, c_{m+2})$ are different. Therefore, the n^2 quantum states of M correspond to the n^2 rows of A, that is, the n^2 quantum states of M are pairwise orthogonal.

Taking the runs in A that begin with 0 yields a submatrix denoted by A_1.

$$A_1 = \begin{pmatrix} 0 & 0 & c_{0,3} & c_{0,4} & \dots & c_{0,m+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & n-1 & c_{n-1,3} & c_{n-1,4} & \dots & c_{n-1,m+2} \end{pmatrix} = (a_1, b_1, C_1),$$

where $a_1 = 0_n$ and $b_1 = (n)$. By Lemma 3, (b_1, C_1) is an IrMOA$(n, m+1, nd^m, 1)$ with MD ≥ 3 and then C_1 is an IrOA$(n, m, d, 1)$. The sum of the first row in M, namely $|c_{0,3}c_{0,4}\dots c_{0,m+2}\rangle + \cdots + |c_{n-1,3}c_{n-1,4}\dots c_{n-1,m+2}\rangle$, is a 1-uniform state. Similarly, the sum of every row in M is a 1-uniform state.

By the same argument, the sum of every column in M is a 1-uniform state.

Therefore, M is a set of m MOQFS(n, d).

(2) Obviously, as the M in (1) is a set of m MOQFS(n, d) constructed from A, we can obtain the B in (1).

(3) We can easily use the B in (1) to generate the M since B is a QMOA$(n^2, m+2, n^2 d^m, 2)$ constructed from A.

Example 1: We can construct a QMOA$(16, m+2, 4^2 2^m, 2)$ and a set of m MOQFS$(4, 2)$ for $4 \leq m \leq 9$.

Proof. From the OA$(16, 5, 4, 2)$ in [14] and OA$(4, 3, 2, 2)$ = $\begin{pmatrix} 000 \\ 011 \\ 101 \\ 110 \end{pmatrix}$, then a saturated MOA$(16, 11, 4^2 2^9, 2)$ can be obtained by using the expansive replacement method as follows. A MOA$(r, N, d_1 d_2 \cdots d_N, 2)$ is called saturated if $\sum_{i=1}^{N}(d_i - 1) = r - 1$ [14]. L = MOA$(16, 11, 4^2 2^9, 2)$ =

$$(a, b, c_1, c_2, \dots, c_9) = \begin{pmatrix} 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3 \\ 0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3 \\ 0&0&1&1&0&0&1&1&1&1&0&0&1&1&0&0 \\ 0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0 \\ 0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0 \\ 0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1 \\ 0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0 \\ 0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1 \\ 0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0 \\ 0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1 \\ 0&0&1&1&1&1&0&0&1&1&0&0&0&0&1&1 \end{pmatrix}^T$$

(1) We have HD(L) = 6, 7 by Ref. [21]. Then we can obtain an IrMOA$(16, m+2, 4^2 2^m, 2)$ for $6 \leq m \leq 9$ by Lemma 1. Thus, by Theorem 1 we can construct a QMOA$(16, m+2, 4^2 2^m, 2)$ and a set of m MOQFS$(4, 2)$ for $6 \leq m \leq 9$.

(2) An IrMOA$(16, 6, 4^2 2^4, 2)$ can be obtained from a MOA$(16, 4, 4, 2)$ and an OA$(4, 2, 2, 2)$ by using the expansive replacement method. Then we can generate a QMOA$(16, 6, 4^2 2^4, 2)$ and a set of 4 MOQFS$(4, 2)$ by Theorem 1. Additionally, we can add a 2 level column to the IrMOA$(16, 6, 4^2 2^4, 2)$ to obtain a MOA$(16, 7, 4^2 2^5, 2)$. It is evident that the MOA$(16, 7, 4^2 2^5, 2)$ is irredundant. Therefore, by Theorem 1, we can construct a QMOA$(16, 7, 4^2 2^5, 2)$ and a set of 5 MOQFS$(4, 2)$.

Definition 10: A quantum frequency cube QFC(n, d) of size n is a cubic arrangement composed of n^3 single-particle quantum pure states $|\psi_{i,j,k}\rangle \in \mathcal{H}_d$, $i, j, k \in \{0, \cdots, n-1\}$, such that every edge (row, column, file) determines n/d orthogonal bases.

Definition 11: A set of n^3 m-qudit pure states $|\psi_{i,j,k}\rangle$, $i, j, k \in \{0, 1, \ldots, n-1\}$, belonging to a composed Hilbert space $\mathcal{H}_d^{\otimes m}$, $m > 3$ forms m triplewise orthogonal quantum frequency cubes (m MOQFC$(n, d, 3)$) if the following properties hold:

(1) The set of n^3 states are orthogonal.

(2) The sum of every edge in this array, i.e., $\sum_{k=0}^{n-1} |\psi_{i,j,k}\rangle$, $\sum_{j=0}^{n-1} |\psi_{i,j,k}\rangle$, $\sum_{i=0}^{n-1} |\psi_{i,j,k}\rangle$, is a 1-uniform state, respectively.

(3) The $(m+2)$-qudit quantum states $\sum_{j,k=0}^{n-1} |j\rangle|k\rangle|\psi_{i,j,k}\rangle$ for $i = 0, 1, \ldots, n-1$, $\sum_{i,k=0}^{n-1} |i\rangle|k\rangle|\psi_{i,j,k}\rangle$ for $j = 0, 1, \ldots, n-1$, and $\sum_{i,j=0}^{n-1} |i\rangle|j\rangle|\psi_{i,j,k}\rangle$ for $k = 0, 1, \ldots, n-1$ are 2-uniform states, respectively.

Theorem 2: (1) From an IrMOA$(n^3, m+3, n^3 d^m, 3)$, we can construct a QMOA$(n^3, m+3, n^3 d^m, 3)$ and a set of m MOQFC$(n, d, 3)$.

(2) We can use the QMOA$(n^3, m+3, n^3 d^m, 3)$ in (1) to generate the set of m MOQFC$(n, d, 3)$ in (1).

(3) We can define the QMOA$(n^3, m+3, n^3 d^m, 3)$ in (1) from the set of m MOQFC$(n, d, 3)$ in (1).

Proof. From an IrMOA$(n^3, m+3, n^3 d^m, 3)$ by a sequence of permutations of the columns, the rows, and the levels of each factor, we can obtain a matrix $A = (a, b, c, C)$, where $a = (n) \otimes 1_{n^2}$, $b = 1_n \otimes (n) \otimes 1_n$, $c = 1_{n^2} \otimes (n)$ and $C = (c_{ij})_{n^3 \times m}$ for $i = 0, 1, \cdots, n^3 - 1$, $j = 4, \cdots, m+3$.

Construction. From A, we can obtain an $n^3 \times (m+3)$ matrix B and a cubic matrix M of size m. B and the u^{th} frontal slice of the M are as follows:

$$B = \begin{pmatrix} |0\rangle & |0\rangle & |0\rangle & |c_{0,4} \ldots c_{0,m+3}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |0\rangle & |0\rangle & |n-1\rangle & |c_{n-1,4} \ldots c_{n-1,m+3}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |n-1\rangle & |n-1\rangle & |0\rangle & |c_{(n-1)n,4} \ldots c_{(n-1)n,m+3}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |n-1\rangle & |n-1\rangle & |n-1\rangle & |c_{n^3-1,4} \ldots c_{n^3-1,m+3}\rangle \end{pmatrix},$$

$$M_{u,:,:} = \begin{pmatrix} |c_{un^2,4} \ldots c_{un^2,m+3}\rangle & \cdots & |c_{un^2+n-1,4} \ldots c_{un^2+n-1,m+3}\rangle \\ \vdots & & \vdots \\ |c_{un^2+(n-1)n,4} \ldots c_{un^2+(n-1)n,m+3}\rangle & \cdots & |c_{(u+1)n^2-1,4} \ldots c_{(u+1)n^2-1,m+3}\rangle \end{pmatrix}.$$

Then B and M are the QMOA$(n^3, m+3, n^3 d^m, 3)$ and the set of m MOQFC$(n, d, 3)$ needed, respectively.

Verification. (1) The proof that B is a QMOA$(n^3, m+3, n^3 d^m, 3)$ follows from the first part of the proof of the Theorem 1.

Then we prove M is a set of m MOQFC$(n, d, 3)$. From Theorem 1 and Definition 11, the first two properties hold. Here we consider the third property as follows.

Let A^0 be the submatrix consisting of the rows in A which begin with 0. Then A^0 can be written as

$$A^0 = (a^0, b^0, c^0, C^0) = \begin{pmatrix} 0 & 0 & 0 & c_{0,4} & c_{0,5} & \cdots & c_{0,m+3} \\ 0 & 0 & 1 & c_{1,4} & c_{1,5} & \cdots & c_{1,m+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & n-1 & n-1 & c_{n^2-1,4} & c_{n^2-1,5} & \cdots & c_{n^2-1,m+3} \end{pmatrix},$$

where $a^0 = 0_{n^2}$, $b^0 = (n) \otimes 1_n$, $c^0 = 1_n \otimes (n)$, and $C^0 = (c_{ij})$ corresponds to the first frontal slice in M. By Lemma 3, (b^0, c^0, C^0) is an IrMOA$(n^2, m+2, n^2 d^m, 2)$ with $MD \geq 4$, then $\sum_{j,k=0}^{n-1} |j\rangle|k\rangle|\psi_{i,j,k}\rangle$ is a 2-uniform state for $i = 0$, so is the sum for each $i = 1, \ldots, n-1$. Similarly, taking the submatrices consisting of the rows in A with the elements j in the second column or with the elements k in the third column, respectively, we can prove that $\sum_{i,k=0}^{n-1} |i\rangle|k\rangle|\psi_{i,j,k}\rangle$ and $\sum_{i,j=0}^{n-1} |i\rangle|j\rangle|\psi_{i,j,k}\rangle$ are 2-uniform states for every $0 \leq j, k \leq n-1$.

Therefore, M is the set desired.

The proofs of (2) and (3) are similar to the proofs of (2) and (3) in Theorem 1, respectively.

Definition 12: A quantum frequency hypercube with d levels and size n in dimension k, denoted QFH(n, d, k), is an arrangement composed of n^k single-particle quantum states $|\psi_{i_1,\ldots,i_k}\rangle \in \mathcal{H}_d$, $i_1, \ldots, i_k \in \{0, \ldots, n-1\}$, such that all states belonging to an edge of hypercube determine n/d orthogonal bases.

Definition 13: Let $m > k$. A set of m mutually orthogonal quantum frequency hypercubes with d levels and size n in dimension k, namely m MOQFH(n, d, k), is a k-dimensional arrangement composed of n^k m-qudit states $|\psi_{i_1,\ldots,i_k}\rangle \in \mathcal{H}_d^{\otimes m}$, $i_1, \ldots, i_k \in \{0, \ldots, n-1\}$, such that the following properties hold:

(1) The set of n^k states $\{|\psi_{i_1,\ldots,i_k}\rangle \in \mathcal{H}_d\}$ are orthogonal.

(2) The sum of n states belonging to the same edge of the hypercube, i.e. $\sum_{i_s=0}^{n-1} |\psi_{i_1,\ldots,i_s,\ldots,i_k}\rangle$, for every $1 \leq s \leq k$, forms a 1-uniform state.

(3) For any $2 \leq v \leq k-1$ and every subset $\{i_{s_1}, i_{s_2}, \ldots, i_{s_v}\} \subseteq \{i_1, \ldots, i_k\}$, the sum of the n^v quantum states, denoted by $\sum_{i_{s_1}, i_{s_2}, \ldots, i_{s_v}=0}^{n-1} |i_{s_1} i_{s_2} \cdots i_{s_v} \psi_{i_1,\ldots,i_{s_1},\ldots,i_{s_v},\ldots,i_k}\rangle$, is a v-uniform state.

Theorem 3: From an IrMOA$(n^k, m+k, n^k d^m, k)$, we can construct a set of m MOQFH(n, d, k) and a QMOA$(n^k, m+k, n^k d^m, k)$, one of which can be obtained from the other.

Proof: This follows from the arguments analogous to the proof of Theorem 2.

Theorem 4: The sum of rows of a QMOA($r, N, d_1 d_2 \cdots d_N, k$) produces a k-uniform state of a quantum system composed of N parties with $d_1, d_2, \ldots, d_N$ levels respectively.

Proof: A positive operator valued measure (POVM) is a set of positive semidefinite operators such that they sum up to identity, determining a generalized quantum measurement [13]. Every reduction to k columns of a QMOA($r, N, d_1 d_2 \cdots d_N, k$) defines a POVM, and thus the sum of its elements produces the identity operator.

Example 2: We can construct a set of 6 MOQFC(4, 2, 3) and a 3-uniform state of a system $4^3 \times 2^6$.

Proof. The frontal slices of the set of 6 MOQFC(4, 2, 3) are as follows.

$$K_{0,:,:} = \begin{pmatrix} |000000\rangle & |111101\rangle & |010110\rangle & |101011\rangle \\ |110111\rangle & |001010\rangle & |100001\rangle & |011100\rangle \\ |011001\rangle & |100100\rangle & |001111\rangle & |110010\rangle \\ |101110\rangle & |010011\rangle & |111000\rangle & |000101\rangle \end{pmatrix},$$

$$K_{1,:,:} = \begin{pmatrix} |011111\rangle & |100010\rangle & |001001\rangle & |110100\rangle \\ |101000\rangle & |010101\rangle & |111110\rangle & |000011\rangle \\ |000110\rangle & |111011\rangle & |010000\rangle & |101101\rangle \\ |110001\rangle & |001100\rangle & |100111\rangle & |011010\rangle \end{pmatrix},$$

$$K_{2,:,:} = \begin{pmatrix} |100101\rangle & |011000\rangle & |110011\rangle & |001110\rangle \\ |010010\rangle & |101111\rangle & |000100\rangle & |111001\rangle \\ |111100\rangle & |000001\rangle & |101010\rangle & |010111\rangle \\ |001011\rangle & |110110\rangle & |011101\rangle & |100000\rangle \end{pmatrix},$$

$$K_{3,:,:} = \begin{pmatrix} |111010\rangle & |000111\rangle & |101100\rangle & |010001\rangle \\ |001101\rangle & |110000\rangle & |011011\rangle & |100110\rangle \\ |100011\rangle & |011110\rangle & |110101\rangle & |001000\rangle \\ |010100\rangle & |101001\rangle & |000010\rangle & |111111\rangle \end{pmatrix}.$$

Then by Theorems 2 and 4, we can construct a 3-uniform state of a system $4^3 \times 2^6$ as follows.

$$\begin{aligned} |\phi_{4^3 \times 2^6}\rangle = {} & |000000000\rangle + |001111101\rangle + |002010110\rangle + |003101011\rangle \\ & + |010110111\rangle + |011001010\rangle + |012100001\rangle + |013011100\rangle \\ & + |020011001\rangle + |021100100\rangle + |022001111\rangle + |023110010\rangle \\ & + |030101110\rangle + |031010011\rangle + |032111000\rangle + |033000101\rangle \\ & + |100011111\rangle + |101100010\rangle + |102001001\rangle + |103110100\rangle \\ & + |110101000\rangle + |111010101\rangle + |112111110\rangle + |113000011\rangle \\ & + |120000110\rangle + |121111011\rangle + |122010000\rangle + |123101101\rangle \\ & + |130110001\rangle + |131001100\rangle + |132100111\rangle + |133011010\rangle \\ & + |200100101\rangle + |201011000\rangle + |202110011\rangle + |203001110\rangle \\ & + |210010010\rangle + |211101111\rangle + |212000100\rangle + |213111001\rangle \\ & + |220111100\rangle + |221000001\rangle + |222101010\rangle + |223010111\rangle \\ & + |230001011\rangle + |231110110\rangle + |232011101\rangle + |233100000\rangle \\ & + |300111010\rangle + |301000111\rangle + |302101100\rangle + |303010001\rangle \\ & + |310001101\rangle + |311110000\rangle + |312011011\rangle + |313100110\rangle \\ & + |320100011\rangle + |321011110\rangle + |322110101\rangle + |323001000\rangle \\ & + |330010100\rangle + |331101001\rangle + |332000010\rangle + |333111111\rangle. \end{aligned}$$

4. Conclusion

In this letter, we define QMOAs, MOQFS, MOQFC and MOQFH. After setting up the quantum combinatorial tools we present our method for constructing k-uniform states. The further work is to find more QMOAs, MOQFS, MOQFC and MOQFH to construct k-uniform states of heterogeneous systems.

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