PAPER

A New Construction of Asymmetric ZCZ Sequence Sets*

Li CUI[†,††a)], **Xiaoyu CHEN**[†], *Nonmembers*, *and* **Yubo LI**[†], *Member*

SUMMARY An asymmetric zero correlation zone (A-ZCZ) sequence set can be regarded as a special type of ZCZ sequence set, which consists of multiple sequence subsets. Each subset is a ZCZ sequence set, and have a common zero cross-correlation zone (ZCCZ) between sequences from different subsets. This paper supplements an existing construction of A-ZCZ sequence sets and further improves the research results. Besides, a new construction of A-ZCZ sequence sets is proposed by matrices transformation. The obtained sequence sets are optimal with respect to theoretical bound, and the parameters can be chosen more flexibly, such as the number of subsets and the lengths of ZCCZ between sequences from different subsets. Moreover, as the diversity of the orthogonal matrices and the flexibility of initial matrix, more A-ZCZ sequence sets can be obtained. The resultant sequence sets presented in this paper can be applied to multi-cell quasi-synchronous code-division multiple-access (QS-CDMA) systems, to eliminate the interference not only from the same cell but also from adjacent cells.

key words: *quasi-synchronous CDMA, zero correlation zone (ZCZ), sequence set, asymmetric zero correlation zone (A-ZCZ)*

1. Introduction

1.1 Background

In quasi-synchronous code-division multiple-access (QS-CDMA) systems, the requirement for synchronization is not as strict as synchronous CDMA systems, and the relative time delay between the signals of different users is allowed to vary in a domain around the origin. In order to utilize this advantage, zero correlation zone (ZCZ) sequence sets are used as spreading sequences and applied to QS-CDMA systems [1]. More precisely, if their relative time delay does not exceed a certain region known as ZCZ length, then ZCZ sequence sets with good correlation property can be used to eliminate co-channel multi-path interference (MPI) and multi-access interference (MAI) in same cell (intra-cell). To date, binary, ternary, and polyphase ZCZ sequences have been widely investigated [3]–[6].

As a matter of fact, the interference between signals of different users not only comes from intra-cell, but also from different cells (inter-cell). In [7], Gong introduced that the latter interference was referred to as correlation of multiple sequence sets, and proposed two constructions of multiple sequence sets with low correlation, which have been well developed by many researchers in [8]–[10]. Tang et al. constructed a type of binary ZCZ sequence sets that consist of multiple sequence subsets in [8]. The most special property was that it had a common ZCZ between sequences from different subsets, which is referred to as inter-set zero cross-correlation zone (ZCCZ). This statement is to distinguish ZCZ between sequences from the same subset, which is referred to as intra-set ZCZ. This type of sequence sets can be used to eliminate both the inter-cell interference and intra-cell interference in multi-cell QS-CDMA systems.

The multiple ZCZ sequence sets with inter-set ZCCZ are also named asymmetric ZCZ (A-ZCZ) sequence sets, which was proposed by Torii et al. in [11]. Up to now, certain constructions of A-ZCZ sequence sets have been reported [11]–[21]. Torii et al. proposed several types of polyphase A-ZCZ sequence sets based on perfect sequences in [11]–[16], and based on DFT matrices in [17]. These resultant sequence sets have larger ZCCZ lengths than the mathematical upper bound of conventional ZCZ sequence sets. However, the lengths of perfect sequences are required certain constraints. For example, in [15], the period of initial perfect sequences L were required to be $L = Nq(2k+1)$, which means that perfect sequences with prime lengths cannot satisfy the condition. Moreover, total number of the A-ZCZ sequence sets obtained by [11], [12], [14], [15] cannot exceed the period of the perfect sequences, which have been resolved in [16]. Based on interleaved technique and uncorrelated ZCZ sequence sets, uncorrelated A-ZCZ sequence sets can be generated by [13]. The parameters of the A-ZCZ sequence sets can approach or almost approach the theoretical bound only if the uncorrelated ZCZ sequence set was optimal or quasi-optimal. In [17], polyphase A-ZCZ sequence sets were constructed from the P-dimension DFT matrices and the orthogonal matrices of order M. The number of subsets is determined by $\lfloor P/M \rfloor$, which represents that the closer of P and M means the less of number of subsets. In addition, the obtained A-ZCZ sequence sets were optimal when $M = \lfloor P/N \rfloor$. In [8], [19]–[21], several types of binary and ternary A-ZCZ sequence sets were proposed by Tang and Hayashi et al. However, most resultant sequence sets are quasi-optimal [8], [12]–[16], [19]–[21]. A new construction of optimal polyphase A-ZCZ sequence sets were proposed

Manuscript received November 27, 2021.
Manuscript revised February 10, 2022.
Manuscript publicized March 29, 2022.

[†]The authors are with School of Information Science and Engineering, Yanshan University, Qinhuangdao 066004, China.
[††]The author is School of Mathematics and Information Science and Technology, Hebei Normal University of Science and Technology, Qinhuangdao 066004, China.

*This paper was supported by China National Natural Science Foundation (No.61671402); Natural Science Foundation of Hebei Province, China (No.F2020203043, No.F2021203078).

a) E-mail: cuili3456@163.com

DOI: 10.1587/transfun.2021EAP1159

based on DFT matrices and non-repeated frequency-hopping sequence (FHS) sets with no-hit zone (NHZ) in [22]. For NHZ FHS set, its Hamming cross-correlations and out-of-phase Hamming auto-correlations are equal to 0 as long as the time delay does not exceed a certain region know as NHZ. If all of the frequency slots in each sequence are different from each other, it is called non-repeated FHS set.

1.2 Contributions

The aim presented in this paper is to design optimal A-ZCZ sequence sets. It requires that the performance parameters of both intra-set and inter-set of the obtained sequence sets can achieve the theoretical bound [8]. The research motivation of this paper is from literature [22]. By investigating [22], on one hand, the obtained FHS sets cannot satisfy the non-repeatability, which is the basis for constructing A-ZCZ sequence sets. On the other hand, the number of A-ZCZ sequence sets were not rich enough. The contributions of this paper are twofold: the first is to supplement the conditions for constructing optimal A-ZCZ sequence set in [22]. The other contribution is to propose a new construction of A-ZCZ sequence sets which is inspired by [22]. It can be simply described as follows. The rows of a DFT matrix are reordered in turn, according to the different rows of a constructed matrix obtained from matrices transformation. The constructed matrix has the similar characteristic as non-repeated NHZ-FHS set. On this basis, combined with an orthogonal matrix, an A-ZCZ sequence set can be generated, by using the good uncorrelation between different rows of DFT matrix. In this paper, we propose that it is not necessary to use two DFT matrices, a DFT matrix and an orthogonal matrix are sufficient. Due to the diversity of orthogonal matrices, more sequence sets can be constructed.

1.3 Organization and Notations

The rest of the paper is organized as follows. After summarizing the notations in this paper, some useful preliminaries are given in Sect. 2. In Sect. 3, supplementary properties are given, so as to obtain FHS sets suitable for constructing A-ZCZ sequence sets. Then a new construction of A-ZCZ sequence sets with optimal parameters is proposed, and an example of the construction is given. Finally, Sect. 5 concludes this paper.

The following notations will be used throughout this paper:

- $\omega_N = \exp(2\pi\sqrt{-1}/N), N \geq 2$.
- $(\cdot)^*$ denotes complex conjugate of $(\cdot)$.
- $\lfloor\cdot\rfloor$ denotes the floor function.
- $\mathbb{Z}_N$ denotes the set $\{0, 1, \cdots, N-1\}$.
- $\gcd(a, b)$ denotes the greatest common divisor of a and b.
- $\mathbf{A} = [a_j^i]_{M\times N}$ denotes an $M \times N$ matrix, where $\mathbf{a}^i$ denotes the i-th row of $\mathbf{A}$, and a_j^i denotes the j-th column of the i-th row, $0 \leq i \leq M-1$, $0 \leq j \leq N-1$. Note that both superscript i and j start from 0.

2. Preliminaries

Definition 1: Let $\mathbf{a} = (a(0), a(1), \cdots, a(L-1))$ and $\mathbf{b} = (b(0), b(1), \cdots, b(L-1))$ be two complex sequences of period L. The periodic cross-correlation function (CCF) $R_{\mathbf{a},\mathbf{b}}(\tau)$ between $\mathbf{a}$ and $\mathbf{b}$ at shift τ is defined as:

$$R_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{l=0}^{L-1} a(l) \cdot b^*(l+\tau), 0 \leq \tau \leq L-1 \tag{1}$$

where $l+\tau$ is performed modulo L. If $\mathbf{a} = \mathbf{b}$, then $R_{\mathbf{a},\mathbf{b}}(\tau)$ is called the periodic auto-correlation function (ACF) of $\mathbf{a}$, denoted by $R_{\mathbf{a}}(\tau)$.

2.1 Matrix

Lemma 1: Let $\mathbf{F} = [f_j^i]_{N\times N}$ be a DFT matrix, and $f_j^i = (\omega_N)^{i\cdot j}$, then all of the rows in $\mathbf{F}$ satisfy

$$R_{\mathbf{f}^{i_0},\mathbf{f}^{i_1}}(\tau) = 0, 0 \leq \tau \leq N-1, i_0 \neq i_1. \tag{2}$$

Lemma 2: Let $\mathbf{Q} = [q_j^i]_{N\times N}$ be an orthogonal matrix. Then all of the rows in $\mathbf{Q}$ satisfy

$$R_{\mathbf{q}^{i_0},\mathbf{q}^{i_1}}(0) = 0, i_0 \neq i_1. \tag{3}$$

Definition 2: Let $\mathbf{a} = (a_0, a_1, \cdots, a_{N-1})$ and $\mathbf{b} = (b_0, b_1, \cdots, b_{N-1})$ be two vectors with same length N, then $\mathbf{a} \odot \mathbf{b}$ is defined as:

$$\mathbf{a} \odot \mathbf{b} = (a_0b_0, a_1b_1, \cdots, a_{N-1}b_{N-1}). \tag{4}$$

Definition 3: Let $\mathbf{a} = (a_0, a_1, \cdots, a_{M-1})$ be a complex-valued sequence and $\mathbf{e} = (e_0, e_1, \cdots, e_{N-1})$ be a sequence over $\mathbb{Z}_N$, an $M \times N$ matrix $\mathbf{U}$ can be obtained as follows:

$$\mathbf{U} = \begin{bmatrix} a_{0+e_0} & a_{0+e_1} & \cdots & a_{0+e_{N-1}} \\ a_{1+e_0} & a_{1+e_1} & \cdots & a_{1+e_{N-1}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M-1+e_0} & a_{M-1+e_1} & \cdots & a_{M-1+e_{N-1}} \end{bmatrix} \tag{5}$$

where the additions in subscripts are performed modulo M. For convenience, the matrix $\mathbf{U}$ in (5) is rewritten as:

$$\mathbf{U} = [L^{e_0}(\mathbf{a}), L^{e_1}(\mathbf{a}), \cdots, L^{e_{N-1}}(\mathbf{a})] \tag{6}$$

where L^i denotes left cyclical shift operator, i.e., $L^i(\mathbf{a}) = (a_i, a_{i+1}, \cdots, a_{M-1}, a_0, \cdots, a_{i-1})$, $0 \leq i \leq N-1$, and the sequences $\mathbf{a}$ and $\mathbf{e}$ are called component sequence and shift sequence of $\mathbf{U}$ respectively.

2.2 Zero Correlation Zone Sequence Set

Definition 4: Let $\mathbb{S} = \{\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{N-1}\}$ be a set of N sequences of length L, each sequence can be represented as

$\mathbf{s}_n = (s_n(0), s_n(1), \cdots, s_n(L-1))$, $0 \le n \le N-1$. For $\mathbf{s}_{n_1}, \mathbf{s}_{n_2} \in \mathbb{S}$, where $0 \le n_1, n_2 \le N-1$, the sequence set $\mathbb{S}$ is called a (conventional) ZCZ sequence set, denoted by $(L, N, Z) - \mathrm{ZCZ}$ if

$$R_{\mathbf{s}_{n_1}, \mathbf{s}_{n_2}}(\tau) = \begin{cases} E, & n_1 = n_2, \tau = 0 \\ 0, & n_1 = n_2, 0 < |\tau| \le Z-1 \\ 0, & n_1 \ne n_2, 0 \le |\tau| \le Z-1 \end{cases} \tag{7}$$

where $E = \sum_{l=0}^{L-1} |s_{n_1}(l)|^2$, Z is the ZCZ length of $\mathbb{S}$.

Lemma 3 ([23]): For a sequence set $(L, N, Z) - \mathrm{ZCZ}$, the performance parameter η satisfies the following relationship:

$$\eta = \frac{NZ}{L} \le 1. \tag{8}$$

If $\eta = 1$, then $(L, N, Z) - \mathrm{ZCZ}$ is called an optimal ZCZ sequence set.

2.3 Asymmetric ZCZ Sequence Set

Definition 5: Supposed that $\mathcal{S} = \{\mathbb{S}^0, \mathbb{S}^1, \cdots, \mathbb{S}^{M-1}\}$ is a set with M sequence subsets, the subset $\mathbb{S}^m$ with N sequences of length L can be represented as $\mathbb{S}^m = \{\mathbf{s}_0^m, \mathbf{s}_1^m, \cdots, \mathbf{s}_{N-1}^m\}$, and $\mathbf{s}_n^m = (s_n^m(0), s_n^m(1), \cdots, s_n^m(L-1))$, for $0 \le m \le M-1$, $0 \le n \le N-1$. Let $\mathbf{s}_{n_1}^{m_1} \in \mathbb{S}^{m_1}$ and $\mathbf{s}_{n_2}^{m_2} \in \mathbb{S}^{m_2}$ be two arbitrary sequences, the sequence set $\mathcal{S}$ is called an A-ZCZ sequence set, denoted by $Z_A(L, [N, M], [Z, Z_{\mathrm{CCZ}}])$ if

$$R_{\mathbf{s}_{n_1}^{m_1}, \mathbf{s}_{n_2}^{m_2}}(\tau) = \begin{cases} E, & m_1 = m_2, n_1 = n_2, \tau = 0 \\ 0, & m_1 = m_2, n_1 = n_2, 0 < |\tau| \le Z-1 \\ 0, & m_1 = m_2, n_1 \ne n_2, 0 \le |\tau| \le Z-1 \\ 0, & m_1 \ne m_2, 0 \le |\tau| \le Z_{\mathrm{CCZ}}-1 \end{cases} \tag{9}$$

where $E = \sum_{l=0}^{L-1} |s_{n_1}^{m_1}(l)|^2$, Z is the ZCZ length of each subset, and Z_{CCZ} is the ZCCZ length between sequences from different subsets.

Lemma 4 ([8]): Let $\mathcal{S}$ be $Z_A(L, [N, M], [Z, Z_{\mathrm{CCZ}}])$. $\mathcal{S}$ is called an optimal A-ZCZ sequence set, if the parameters satisfy the following two conditions:

1. Each subet $\mathbb{S}^m$ is an optimal ZCZ sequence set $(L, N, Z) - \mathrm{ZCZ}$. The performance parameters of intra-set η_1 satisfies $\eta_1 = \frac{NZ}{L} = 1$.
2. The union of M sequence subsets is an optimal ZCZ sequence set $(L, MN, min\{Z, Z_{\mathrm{CCZ}}\}) - \mathrm{ZCZ}$. The inter-set performance parameter η_2 satisfies $\eta_2 = \frac{MN\min\{Z, Z_{\mathrm{CCZ}}\}}{L} = 1$.

2.4 Frequency-Hopping Sequence Set

Definition 6: Let $\mathbb{F} = \{f_0, f_1, \cdots, f_{q-1}\}$ be a set of available frequency slots with size q. A sequence $\mathbf{x} = (x(0), x(1), \cdots, x(N-1))$ with length N is called a frequency-hopping sequence over $\mathbb{F}$ if $x(n) \in \mathbb{F}$ for all $0 \le n \le N-1$. For two FHSs $\mathbf{x}$ and $\mathbf{y}$ of length N over $\mathbb{F}$, the Hamming correlation between $\mathbf{x}$ and $\mathbf{y}$ is defined as:

$$H_{\mathbf{x},\mathbf{y}}(\tau) = \sum_{n=0}^{N-1} h[(x(n), y(n+\tau)], 0 \le \tau \le N-1 \tag{10}$$

where $n + \tau$ is calculated modulo N, and $h[f_i, f_j] = \begin{cases} 1, & f_i = f_j \\ 0, & f_i \ne f_j \end{cases}$. If $\mathbf{x} = \mathbf{y}$, $H_{x,y}(\tau)$ is called Hamming auto-correlation of $\mathbf{x}$, denoted by $H_x(\tau)$.

Let $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^{M-1}\}$ be a set of M FHSs with length N over a frequency set with size q. Assume that $H_{\mathbf{x}^{m_1}, \mathbf{x}^{m_2}}(\tau) = 0$, for $0 < \tau \le L-1$ when $m_1 = m_2$ and $0 \le \tau \le L-1$ when $m_1 \ne m_2$. Then $\mathbb{X}$ is called a NHZ-FHS set, and the length of NHZ is L.

Definition 7: Given an FHS $\mathbf{x} = (x(0), x(1), \cdots, x(N-1))$ with length N. For arbitrary integer s and t with $0 \le s \ne t \le N-1$, if $x(s) \ne x(t)$, then $\mathbf{x}$ is called non-repeated FHS.

3. Construction of A-ZCZ Sequence Sets

3.1 The Existing Construction of A-ZCZ Sequence Sets

In [22], the A-ZCZ sequence sets are constructed based on non-repeated NHZ FHS sets. The proof of *Lemma 5* is provided in [22].

Lemma 5 ([22]): Let $\mathbb{F} = \{f_0, f_1, \cdots, f_{q-1}\}$ be a set of available frequency slots with size q. M and Z are positive integer, which satisfy $q = M(Z+1)$. For $f_h \in \mathbb{F}$, it can be represented as $f_h = f_{a,b}$, where $a = \lfloor h/(Z+1) \rfloor$, $b = h \bmod (Z+1)$. Construct a sequence set $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^{M-1}\}$ with M sequences of length $M(Z+1)$, and each sequence can be represented as $\mathbf{x}^m = (x^m(0), x^m(1), \cdots, x^m(M(Z+1)-1))$, $0 \le m \le M-1$. The element of each sequence is calculated as:

$$x^m(n) = f_{\delta, j} \tag{11}$$

where $\delta = (m+i) \oplus i(j+1)$, $i = \lfloor n/(Z+1) \rfloor$, $j = n \bmod (Z+1)$, $0 \le n \le M(Z+1)-1$, $\oplus$ represents modulo M addition. The sequence set $\mathbb{X}$ resolved by (11) is an NHZ FHS set with NHZ of length Z.

Remark 1: As the basis of A-ZCZ seqeuence sets, the non-repeated property of each FHS is very important. However, it was not be verified in [22]. Unfortunately, the parameters under the known conditions cannot assure each FHS is non-repeated, which can be known as the following example.

Example 1: Set $M = 6, Z = 2$ and $q = 18$, then $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^5\}$ is an FHS set constructed from *Lemma 5*. The FHSs in $\mathbb{X}$ are listed as follows:

$$\mathbf{x}^0 = (0, 1, 2, 6, 10, 14, 12, 1, 8, 0, 10, 2, 6, 1, 14, 12, 10, 8)$$

$$\mathbf{x}^1 = (3, 4, 5, 9, 13, 17, 15, 4, 11, 3, 13, 5, 9, 4, 17, 15, 13, 11)$$

$$\mathbf{x}^2 = (6,7,8,12,16,2,0,7,14,6,16,8,12,7,2,0,16,14)$$
$$\mathbf{x}^3 = (9,10,11,15,1,5,3,10,17,9,1,11,15,10,5,3,1,17)$$
$$\mathbf{x}^4 = (12,13,14,0,4,8,6,13,2,12,4,14,0,13,8,6,4,2)$$
$$\mathbf{x}^5 = (15,16,17,3,7,11,9,16,5,15,7,17,3,16,11,9,7,5)$$

Obviously, there are repeated frequency slots in each FHS.

Example 2: Set $M = 5, Z = 3$ and $q = 20$, then $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4\}$ is a FHS set constructed from *Lemma 5*. The FHSs in $\mathbb{X}$ are listed as follows:

$$\mathbf{x}^0 = (0,1,2,3,8,13,18,3,16,5,14,3,4,17,10,3,12,9,6,3)$$
$$\mathbf{x}^1 = (4,5,6,7,12,17,2,7,0,9,18,7,8,1,14,7,16,13,10,7)$$
$$\mathbf{x}^2 = (8,9,10,11,16,1,6,11,4,13,2,11,12,5,18,11,0,17,14,11)$$
$$\mathbf{x}^3 = (12,13,14,15,0,5,10,15,8,17,6,15,16,9,2,15,4,1,18,15)$$
$$\mathbf{x}^4 = (16,17,18,19,4,9,14,19,12,1,10,19,0,13,6,19,8,5,2,19)$$

There are repeated frequency slots still in each FHS.

Based on *Lemma 5*, the following conditions are complemented so as to obtain the non-repeated FHS sets.

Lemma 6: Each FHS in the set of $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^{M-1}\}$ obtained by *Lemma 5* is non-repeated sequence, if the parameters of $\mathbb{X}$ satisfy one of the following conditions:

1. M satisfies $\gcd(M, c) = 1$, where c is an arbitrary integer in $\{1, 2, \cdots, 2 + Z\}$.
2. $M > Z + 2$ and M is a prime number.

Proof: Let $\mathbf{x}^m$ be an arbitrary sequence in the set $\mathbb{X}$. Assume that $x^m(n_1) = x^m(n_2)$, where $0 \le m \le M-1$, $n_1 \ne n_2$, then according to (11), we have

$$f_{(m+i_1)\oplus i_1(j_1+1), j_1} = f_{(m+i_2)\oplus i_2(j_2+1), j_2} \tag{12}$$

where $i_1 = \lfloor n_1/(Z+1) \rfloor$, $j_1 = n_1 \bmod (Z+1)$, $i_2 = \lfloor n_2/(Z+1) \rfloor$, $j_2 = n_2 \bmod (Z+1)$. It can be derived that

$$\begin{cases} j_1 = j_2 \\ m \oplus i_1 (j_1 + 2) = m \oplus i_2 (j_2 + 2). \end{cases} \tag{13}$$

Since $0 \le j_1 = j_2 \le Z$, then $2 \le j_1 + 2 \le Z + 2$, it holds that $\gcd(M, j_1 + 2) = 1$, according to the first condition. Then we have, $i_1 \bmod M = i_2 \bmod M$. In addition, $0 \le i_1, i_2 \le M - 1$, thus $i_1 = i_2$. Note that $j_1 = j_2$, hence $n_1 = n_2$, which violates the assumption above. Therefore, the FHS $\mathbf{x}^m$ is non-repeated.

The second condition is a special case of the first one, so the proof process is omitted. □

Next, we give an example of *Lemma* 6.

Example 3: Set $M = 7, Z = 2$ and $q = 21$, then the FHS set $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^6\}$ obtained by *Lemma 6* is listed as follows:

$$\mathbf{x}^0 = (0,1,2,6,10,14,12,19,5,18,7,17,3,16,8,9,4,20,15,13,11)$$
$$\mathbf{x}^1 = (3,4,5,9,13,17,15,1,8,0,10,20,6,19,11,12,7,2,18,16,14)$$
$$\mathbf{x}^2 = (6,7,8,12,16,20,18,4,11,3,13,2,9,1,14,15,10,5,0,19,17)$$
$$\mathbf{x}^3 = (9,10,11,15,19,2,0,7,14,6,16,5,12,4,17,18,13,8,3,1,20)$$
$$\mathbf{x}^4 = (12,13,14,18,1,5,3,10,17,9,19,8,15,7,20,0,16,11,6,4,2)$$
$$\mathbf{x}^5 = (15,16,17,0,4,8,6,13,20,12,1,11,18,10,2,3,19,14,9,7,5)$$
$$\mathbf{x}^6 = (18,19,20,3,7,11,9,16,2,15,4,14,0,13,5,6,1,17,12,10,8)$$

It can be verified that all the above FHSs in $\mathbb{X}$ are non-repeated.

Finally, the construction of A-ZCZ sequence sets based on non-repeated NHZ FHS set in [22] is introduced briefly for readers convenience.

Lemma 7 ([22]): Let $\mathbb{F}$ be a frequency set with q available frequency slot, and $\mathbb{X} = \{\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^{M-1}\}$ be a set of non-repeated NHZ FHS set with M sequences of length $M(Z+1)$ over $\mathbb{F}$. Given two DFT matrices $\mathbf{G} = [g_j^i]_{N\times N}$ and $V = [v_j^i]_{N\times N}$, where $N = M(Z+1)$, $0 \le i, j \le N-1$. Construct $\mathcal{S} = \{\mathbb{S}^0, \mathbb{S}^1, \cdots, \mathbb{S}^{M-1}\}$ with M sequence subsets, each subset $\mathbb{S}^m = \{\mathbf{s}_0^m, \mathbf{s}_1^m, \cdots, \mathbf{s}_{N-1}^m\}$ consist of N sequences with period N^2, i.e., $\mathbf{s}_n^m = (s_n^m(0), s_n^m(1), \cdots, s_n^m(N^2-1))$, where $0 \le m \le M-1$ and $0 \le n \le N-1$. Each sequence element is constructed as follows:

$$s_n^m(l) = g_{l_1}^{x^m(l_0)} \cdot v_{l_0}^n \tag{14}$$

where $0 \le l \le N^2 - 1$, $l = Nl_1 + l_0$, $0 \le l_0 \le N-1$. The sequence set $\mathcal{S}$ is an optimal A-ZCZ sequence set that is represented as $Z_A(N^2, [N, M], [N, Z+1])$.

Lemma 8: When $q = M(Z+1)$ in *Lemma 7*, the obtained ZCZ sequence sets are optimal.

Under the condition $q = M(Z+1)$ and $N = M(Z+1)$, it is clear that every frequency slot in $\mathbb{F}$ will appear once and only once in each FHS. We can assume that $q > N$, then the order of DFT matrix $\mathbf{G}$ need to be $q \times q$, which can know from (14). Then the obtained A-ZCZ sequence set is represented as $Z_A(Nq, [N, M], [N, Z+1])$, according to (14). Then $\eta_1 = \frac{N\cdot N}{Nq} < 1$, and $\eta_2 = \frac{MN\cdot Z}{Nq} < 1$. So the A-ZCZ sequence set is not optimal.

Remark 2: From *Lemma 6*, it is clear that the number of subsets M need to be odd. In addition, the resultant sequence sets only depend on M and Z. This paper proposes a new construction of A-ZCZ sequence sets, which can provide more flexible parameters and richer results.

3.2 A New Construction of A-ZCZ Sequence Set

In this section, a new construction of A-ZCZ sequence sets is proposed by matrices transformation.

Step 1: Given a DFT matrix $\mathbf{F} = [f_j^i]_{N\times N}$ and an orthogonal matrix $\mathbf{Q} = [q_j^i]_{N\times N}$. Set two positive integers Z and M, which satisfy $M = N/Z$.

Step 2: Arbitrarily permute all the integers over $\mathbb{Z}_N$ to form a matrix $\mathbf{A} = [a_j^i]_{M\times Z}$.

Step 3: Choose two sequences with length M, $\mathbf{d} = (d_0, d_1, \cdots, d_{M-1})$ and $\mathbf{e} = (e_0, e_1, \cdots, e_{M-1})$. Both of the two sequences are arbitrary permutations over $\mathbb{Z}_M$. Construct a matrix $\mathbf{B} = [b_j^i]_{M\times M}$ from $\mathbf{d}$ and $\mathbf{e}$ as follows:

$$\mathbf{B} = \left[L^{e_0}(\mathbf{d}), L^{e_1}(\mathbf{d}), \cdots, L^{e_{M-1}}(\mathbf{d})\right]. \tag{15}$$

Step 4: Construct a matrix $\mathbf{C} = \left[c_n^m\right]_{M\times N}$, the element c_n^m is calculated by

$$c_n^m = a_{n \bmod Z}^{b^m_{\lfloor n/Z\rfloor}\oplus n \bmod Z} \tag{16}$$

where $\oplus$ represents modulo M addition.

Step 5: Construct a matrix set $\mathbb{H}$ with M matrices of order $N\times N$, $\mathbb{H} = \{\mathbf{H}(0), \mathbf{H}(1), \cdots, \mathbf{H}(M-1)\}$, and the matrix $\mathbf{H}(m)$ is generated by transforming the matrix $\mathbf{F}$ based on the m-th row of matrix $\mathbf{C}$, where $0 \le m \le M-1$.

$$\mathbf{H}(m) = \begin{bmatrix} f_0^{c_0^m} & f_0^{c_1^m} & \cdots & f_0^{c_{N-1}^m} \\ f_1^{c_0^m} & f_1^{c_1^m} & \cdots & f_1^{c_{N-1}^m} \\ \vdots & \vdots & \vdots & \vdots \\ f_{N-1}^{c_0^m} & f_{N-1}^{c_1^m} & \cdots & f_{N-1}^{c_{N-1}^m} \end{bmatrix} \tag{17}$$

The i-th row of $\mathbf{H}(m)$ is represented as follows:

$$\mathbf{h}^i(m) = (f_i^{c_0^m}, f_i^{c_1^m}, \cdots, f_i^{c_{N-1}^m}). \tag{18}$$

Step 6: Construct $\mathcal{S} = \left\{\mathbb{S}^0, \mathbb{S}^1, \cdots, \mathbb{S}^{M-1}\right\}$ with M sequence subsets, and $\mathbb{S}^m = \{\mathbf{s}_0^m, \mathbf{s}_1^m, \cdots, \mathbf{s}_{N-1}^m\}$ consists of N sequences. Each sequence is calculated as follows:

$$\mathbf{s}_n^m = (\mathbf{h}^0(m)\odot q^n, \mathbf{h}^1(m)\odot q^n, \cdots, \mathbf{h}^{N-1}(m)\odot q^n) \tag{19}$$

where $0 \le m \le M-1$ and $0 \le n \le N-1$. The length of $\mathbf{s}_n^m$ is N^2, $\mathbf{s}_n^m = \left(s_n^m(0), s_n^m(1), \cdots, s_n^m\left(N^2-1\right)\right)$, the element of sequence $s_n^m(l)$ is calculated as follows:

$$s_n^m(l) = f_{l_1}^{c_{l_0}^m} \cdot q_{l_0}^n \tag{20}$$

where $0 \le l \le N^2-1$, $l_1 = \lfloor l/N \rfloor$, $l_0 = l \bmod N$.

Theorem 1: $\mathcal{S}$ is an A-ZCZ sequence set with parameters $Z_A\left(N^2, [N, M], [N, Z]\right)$, which means that $\mathcal{S}$ has the following properties:

1. Each subset $\mathbb{S}^m$ is a conventional ZCZ sequence set $\left(N^2, N, N\right)$ − ZCZ.
2. The different subsets have a common ZCCZ length Z.

Proof: According to *Step 6*, the set of $\mathcal{S}$ contains M subset, and each subset contains N sequences of length N^2.

Next, two more points need to be proved. One is the ZCZ length of each subset is N, and the other is the length of inter-set ZCCZ is Z.

Consider arbitrary two sequences $\mathbf{s}_{n_1}^{m_1} \in \mathbb{S}^{m_1}$, $\mathbf{s}_{n_2}^{m_2} \in \mathbb{S}^{m_2}$, where $0 \le m_1, m_2 \le M-1$ and $0 \le n_1, n_2 \le N-1$. Set $\tau = N\tau_1 + \tau_0$, $0 \le \tau_0, \tau_1 \le N-1$, the periodic CCF between $\mathbf{s}_{n_1}^{m_1}$ and $\mathbf{s}_{n_2}^{m_2}$ can be calculated as follows:

$$\begin{aligned} &R_{\mathbf{s}_{n_1}^{m_1}, \mathbf{s}_{n_2}^{m_2}}(\tau) \\ &= \sum_{l=0}^{N^2-1} f_{\lfloor l/N\rfloor}^{c^{m_1}_{l \bmod N}} \cdot q^{n_1}_{l \bmod N} \cdot \left(f_{\lfloor (l+\tau)/N\rfloor}^{c^{m_2}_{(l+\tau) \bmod N}} \cdot q^{n_2}_{(l+\tau) \bmod N}\right)^* \\ &= \sum_{l_0=0}^{N-1} q_{l_0}^{n_1}\cdot \left(q_{l_0'}^{n_2}\right)^* \sum_{l_1=0}^{N-1} f_{l_1}^{c_{l_0}^{m_1}} \cdot \left(f_{l_1'}^{c_{l_0'}^{m_2}}\right)^* \end{aligned} \tag{21}$$

where $l_1' = (l_1 + \tau_1 + \lfloor (l_0+\tau_0)/N \rfloor) \bmod N$, $l_0' = (l_0 + \tau_0) \bmod N$. Consider the following cases:

Case 1: : $m_1 = m_2, n_1 = n_2, \tau \in \left(0, N^2-1\right]$.

Note that $\mathbf{s}_{n_1}^{m_1}$ and $\mathbf{s}_{n_2}^{m_2}$ are the same, then $R_{\mathbf{s}_{n_1}^{m_1}, \mathbf{s}_{n_2}^{m_2}}(\tau) = R_{\mathbf{s}_{n_1}^{m_1}}(\tau)$. Firstly, we only consider $\tau \in (0, N-1]$ and $\tau = N$.

When $\tau \in (0, N-1]$, we have $\tau_1 = 0$, $0 < \tau_0 \le N-1$. Since $0 \le l_0 \le N-1$, then $0 < l_0+\tau_0 \le 2N-2$. To calculate $R_{\mathbf{s}_{n_1}^{m_1}}(\tau)$ for $0 < l_0+\tau_0 \le N-1$ and $N \le l_0+\tau_0 \le 2N-2$ respectively, consider the following situations:
(1). If $0 < l_0 + \tau_0 \le N-1$, then $l_1' = l_1$, $l_0' = l_0 + \tau_0$. Assume that $c_{l_0}^{m_1} = c_{l_0'}^{m_2}$. According to (16), we have $a_{l_0 \bmod Z}^{b^{m_1}_{\lfloor l_0/Z\rfloor}\oplus l_0 \bmod Z} = a_{l_0' \bmod Z}^{b^{m_2}_{\lfloor l_0'/Z\rfloor}\oplus l_0' \bmod Z}$. Since the elements value of matrix $\mathbf{A}$ are different from each other, we can conclude that

$$\begin{cases} l_0 \bmod Z = l_0' \bmod Z \\ b^{m_1}_{\lfloor l_0/Z\rfloor} \oplus l_0 \bmod Z = b^{m_2}_{\lfloor l_0'/Z\rfloor} \oplus l_0' \bmod Z. \end{cases} \tag{22}$$

It can be derived that $b^{m_1}_{\lfloor l_0/Z\rfloor} = b^{m_2}_{\lfloor l_0'/Z\rfloor}$. According to (15), when $m_1 = m_2$, it holds that $\lfloor l_0/Z\rfloor = \lfloor l_0'/Z\rfloor$, then $l_0 = l_0'$, equivalent to $\tau_0 = 0$, which is contradicted against $0 < \tau_0 \le N-1$. Therefore $c_{l_0}^{m_1} \ne c_{l_0'}^{m_2}$. According to (2), it follows that $\sum_{l_1=0}^{N-1} f_{l_1}^{c_{l_0}^{m_1}} \cdot (f_{l_1'}^{c_{l_0'}^{m_2}})^* = R_{\mathbf{f}^{c_{l_0}^{m_1}}, \mathbf{f}^{c_{l_0'}^{m_2}}}(0) = 0$. As a result, we have $R_{\mathbf{s}_{n_1}^{m_1}}(\tau) = 0$ for $\tau \in (0, N-1]$ and $0 < l_0 + \tau_0 \le N-1$.
(2). If $N \le l_0 + \tau_0 \le 2N-2$, then $l_1' = (l_1+1) \bmod N$, $l_0' = l_0 + \tau_0 - N$. Assume that $c_{l_0}^{m_1} = c_{l_0'}^{m_2}$. Similar to above, we can get $l_0 = l_0'$, then $\tau_0 = N$, which is contradicted against $0 < \tau_0 \le N-1$. Therefore $c_{l_0}^{m_1} \ne c_{l_0'}^{m_2}$. According to (2), it follows that $\sum_{l_1=0}^{N-1} f_{l_1}^{c_{l_0}^{m_1}} \cdot (f_{l_1'}^{c_{l_0'}^{m_2}})^* = R_{\mathbf{f}^{c_{l_0}^{m_1}}, \mathbf{f}^{c_{l_0'}^{m_2}}}(1) = 0$

As a result, we have $R_{\mathbf{s}_{n_1}^{m_1}}(\tau) = 0$ for $\tau \in (0, N-1]$ and $N \le l_0 + \tau_0 \le 2N-2$.

Combining above two situations, we can conclude that $R_{\mathbf{s}_{n_1}^{m_1}}(\tau) = 0$ for $\tau \in (0, N-1]$.

When $\tau = N$, we have $\tau_1 = 1$, $\tau_0 = 0$, $l_1{}' = (l_1+1) \bmod N$, $l_0{}' = l_0$. Note that all the elements in $c_{l_0}^{m_1}$ are different from each other over $\mathbb{Z}_N$, according to (15). Therefore, it can be derived that $\sum_{l_0=0}^{N-1} f_{l_1}^{c_{l_0}^{m_1}} \cdot \left(f_{(l_1+1) \bmod N}^{c_{l_0'}^{m_2}}\right)^* = \sum_{l_0=0}^{N-1} f_{c_{l_0}^{m_1}}^{l_1} \cdot \left(f_{c_{l_0}^{m_1}}^{(l_1+1) \bmod N}\right)^* = 0$. Thus, $R_{\mathbf{s}_{n_1}^{m_1}}(N) = 0$.

According to the above proof process, we can deduce $R_{\mathbf{s}_{n_1}^{m_1}}(\tau) = 0$, when $\tau \in [N+1, N^2-1]$. The process is omitted here.

So far, we can conclude that $R_{\mathbf{s}_{n_1}^{m_1}}(N) = 0$ for $\tau \in (0, N^2-1]$.

Case 2: $m_1 = m_2, n_1 \ne n_2, \tau \in [0, N-1]$.

Note that $\mathbf{s}_{n_1}^{m_1}$ and $\mathbf{s}_{n_2}^{m_2}$ are two different sequences from the same subset. According to $\tau \in [0, N-1]$, it holds that $\tau_1 = 0, 0 \le \tau_0 \le N-1$. To calculate $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau)$ for $\tau_0 = 0$ and $0 < \tau_0 \le N-1$ respectively, consider the following situations.

(1). If $\tau_0 = 0$, then $l_1{}' = l_1$, $l_0{}' = l_0$.

According to (3), it follows that $\sum_{l_0=0}^{N-1} q_{l_0}^{n_1} \cdot \left(q_{l_0'}^{n_2}\right)^* = R_{\mathbf{q}^{n_1},\mathbf{q}^{n_2}}(0) = 0$. As a result, we have $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(0) = 0$.

(2). If $0 < \tau_0 \le N-1$.

Similar to *Case 1*, it can be derived that $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau) = 0$ for $\tau \in (0, N-1]$.

Combining above situations of *Case 2*, we can conclude that $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau) = 0$ for $\tau \in [0, N-1]$.

From *Case 1* and *Case 2*, it is clear that each subset $\mathbb{S}^m$ of $\mathcal{S}$ has a ZCZ of length N. Therefore $\mathbb{S}^m$ can be represented as $(N^2, N, N) - \text{ZCZ}$.

Case 3: : $m_1 \ne m_2, \tau \in [0, Z-1]$.

Note that $\mathbf{s}_{n_1}^{m_1}$ and $\mathbf{s}_{n_2}^{m_2}$ are sequences from different subsets. According to $\tau \in [0, Z-1]$, we have $\tau_1 = 0$, $0 \le \tau_0 \le Z-1$. To calculate $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau)$ for $\tau_0 = 0$ and $0 < \tau_0 \le Z-1$ respectively, consider the following situations:

(1). If $\tau_0 = 0$, then $l_1{}' = l_1$ and $l_0{}' = l_0$.

Assume that $c_{l_0}^{m_1} = c_{l_0'}^{m_2}$. Similar to the first situation of *Case 1*, it can be derived that $m_1 = m_2$, which violates $m_1 \ne m_2$. Hence $c_{l_0}^{m_1} \ne c_{l_0'}^{m_2}$. Similarly, we have $R_{s_{n_1}^{m_1},s_{n_2}^{m_2}}(0) = 0$.

(2). If $0 < \tau_0 \le Z-1$, then $0 < l_0 + \tau_0 \le N + Z - 2$.

To calculate $R_{s_{n_1}^{m_1},s_{n_2}^{m_2}}(\tau)$ for $0 < l_0 + \tau_0 \le N-1$ and $N \le l_0 + \tau_0 \le N + Z - 2$ respectively, consider the following states:

a) If $0 < l_0 + \tau_0 \le N-1$, then $l_1{}' = l_1$, $l_0{}' = l_0 + \tau_0$. When $0 < \tau_0 \le Z-1$, we have $l_0 \bmod Z \ne (l_0 + \tau_0) \bmod Z$, then $l_0 \bmod Z \ne l_0{}' \bmod Z$. Since the elements value of matrix $\mathbf{A}$ are different from each other, then $a_{l_0 \bmod Z}^{b_{\lfloor l_0/Z \rfloor}^{m_1} \oplus l_0 \bmod Z} \ne a_{l_0' \bmod Z}^{b_{\lfloor l_0'/Z \rfloor}^{m_2} \oplus l_0' \bmod Z}$, thus $c_{l_0}^{m_1} \ne c_{l_0'}^{m_2}$. Similarly, $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau) = 0$ for $0 < l_0 + \tau_0 \le N-1$.

b) If $N \le l_0 + \tau_0 \le N + Z - 2$, then $l_1{}' = (l_1 + 1) \bmod N$, $l_0{}' = l_0 + \tau_0 - N$. It can be derived that $l_0{}' \bmod Z = (l_0 + \tau_0 + N) \bmod Z = (l_0 + \tau_0) \bmod Z$. When $0 < \tau_0 \le Z-1$, we have $l_0{}' \bmod Z \ne l_0 \bmod Z$. Similar to above $c_{l_0}^{m_1} \ne c_{l_0'}^{m_2}$, we have $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau) = 0$ for $N \le l_0 + \tau_0 \le N + Z - 2$.

From the two situations of *Case 3*, it can be seen that $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau) = 0$ for $\tau \in [0, Z-1]$.

In order to ensure that the length of inter-set ZCCZ is Z strictly, that is, never be larger than Z, we can consider $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(\tau)$ for $\tau = Z$. Thus, we have $\tau_1 = 0$, $\tau_0 = Z$, $l_1{}' = (l_1 + \lfloor (l_0 + Z)/N \rfloor) \bmod N$ and $l_0{}' = (l_0 + Z) \bmod N$. Since $l_0 \bmod Z = l_0{}' \bmod Z$, then from (16), it holds that $c_{l_0}^{m_1} = c_{l_0'}^{m_2}$ if $b_{\lfloor l_0/Z \rfloor}^{m_1} = b_{\lfloor l_0'/Z \rfloor}^{m_2}$. According to (15), $b_{\lfloor l_0/Z \rfloor}^{m_1} = b_{\lfloor l_0'/Z \rfloor}^{m_2}$ will occur inevitably. Hence, among all the cross-correlation functions of inter-set, there will always be some ones whose values are not equal to 0 when $\tau = Z$. That is to say, the situation of $R_{\mathbf{s}_{n_1}^{m_1},\mathbf{s}_{n_2}^{m_2}}(Z) \ne 0$ will definitely happen.

From *Case 3*, it is clear that the length of inter-set ZCCZ is Z.

Combining three cases above, $\mathcal{S}$ is a A-ZCZ sequence set $Z_A\left(N^2, [N, M], [N, Z]\right)$. □

Theorem 2: The A-ZCZ sequence set obtained by *Theorem 1* is an optimal A-ZCZ sequence set.

Proof: Note that, each subset $\mathbb{S}^m$ of $\mathcal{S}$ is an A-ZCZ sequence set with parameters $\left(N^2, N, N\right) - \text{ZCZ}$. From *Lemma 4*, we have $\eta_1 = \frac{N \cdot N}{N^2} = 1$, then each subset $\mathbb{S}^m$ is an optimal ZCZ sequence set. Moreover, there is an inter-set ZCCZ of length Z, and $\min\{N, Z\} = Z$, hence the union of MN sequences is a large ZCZ sequence set with parameters $\left(N^2, MN, Z\right) - \text{ZCZ}$, where $N = MZ$. It holds that $\eta_2 = \frac{NMZ}{N^2} = \frac{N \cdot N}{N^2} = 1$. As a result, $\mathcal{S}$ is an optimal A-ZCZ sequence set. □

Example 4: Suppose that $N = 6$, for writing convenience, the 6×6 orthogonal matrix $\mathbf{Q}$ is also denoted by 6×6 DFT matrix $\mathbf{F}$, which can be represented as follows:

$$\mathbf{F} = \mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 0 & 4 & 2 & 0 & 4 & 2 \\ 0 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \tag{23}$$

where each element represents a power of $\exp(2\pi\sqrt{-1}/6)$.

Set $Z = 3$ and $M = 2$. then 2×3 matrix $\mathbf{A}$ is given as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix}. \tag{24}$$

Choose a component sequence $\mathbf{d} = (0, 1)$ and a shift sequence $\mathbf{e} = (0, 1)$. Then the 2×2 matrix $\mathbf{B}$ is generated as follows:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

Based on matrices $\mathbf{A}$ and $\mathbf{B}$, the 2×6 matrix $\mathbf{C}$ is constructed as follows:

$$\mathbf{C} = \begin{bmatrix} 1 & 5 & 2 & 0 & 4 & 3 \\ 0 & 4 & 3 & 1 & 5 & 2 \end{bmatrix}. \tag{26}$$

According to $\mathbf{C}$ and $\mathbf{F}$, matrices set $\mathbb{H}$ with two matrices $\mathbf{H}(0)$ and $\mathbf{H}(1)$ are constructed as follows:

$$\mathbf{H}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 2 & 0 & 4 & 3 \\ 2 & 4 & 4 & 0 & 2 & 0 \\ 3 & 3 & 0 & 0 & 0 & 3 \\ 4 & 2 & 2 & 0 & 4 & 0 \\ 5 & 1 & 4 & 0 & 2 & 3 \end{bmatrix}, \tag{27}$$

$$\mathbf{H}(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 1 & 5 & 2 \\ 0 & 2 & 0 & 2 & 4 & 4 \\ 0 & 0 & 3 & 3 & 3 & 0 \\ 0 & 4 & 0 & 4 & 2 & 2 \\ 0 & 2 & 3 & 5 & 1 & 4 \end{bmatrix} \tag{28}$$

where each element represents a power of $\exp(2\pi\sqrt{-1}/6)$.

An A-ZCZ sequence set $\mathcal{S} = \{\mathbb{S}^0, \mathbb{S}^1\}$ including two subsets is constructed from the new construction. Each subset consists of 6 sequences with length 36, i.e., $\mathbb{S}^k = \{\mathbf{s}_0^k, \mathbf{s}_1^k, \mathbf{s}_2^k, \mathbf{s}_3^k, \mathbf{s}_4^k, \mathbf{s}_5^k\}$, $k \in \{0, 1\}$. Each element of sequences represents a power of $\exp(2\pi\sqrt{-1}/6)$. For sake of page limitations, just a part of sequences are listed as follows:

$$\begin{aligned}
\mathbf{s}_0^0 = (&0,0,0,0,0,0,1,5,2,0,4,3,2,4,4,0,2,0,\\ &3,3,0,0,0,3,4,2,2,0,4,0,5,1,4,0,2,3)\\
\mathbf{s}_1^0 = (&0,1,2,3,4,5,1,0,4,3,2,2,2,5,0,3,0,5,\\ &3,4,2,3,4,2,4,3,4,3,2,5,5,2,0,3,0,2)\\
\mathbf{s}_2^0 = (&0,2,4,0,2,4,1,1,0,0,0,1,2,0,2,0,4,4,\\ &3,5,4,0,2,1,4,4,0,0,0,4,5,3,2,0,4,1)\\
\mathbf{s}_3^0 = (&0,3,0,3,0,3,1,2,2,3,4,0,2,1,4,3,2,3,\\ &3,0,0,3,0,0,4,5,2,3,4,3,5,4,4,3,2,0)\\
\mathbf{s}_4^0 = (&0,4,2,0,4,2,1,3,4,0,2,5,2,2,0,0,0,2,\\ &3,1,2,0,4,5,4,0,4,0,2,2,5,5,0,0,0,5)\\
\mathbf{s}_5^0 = (&0,5,4,3,2,1,1,4,0,3,0,4,2,3,2,3,4,1,\\ &3,2,4,3,2,4,4,1,0,3,0,1,5,0,2,3,4,4)\\
\mathbf{s}_0^1 = (&0,0,0,0,0,0,4,3,1,5,2,0,2,0,2,4,4,\\ &0,0,3,3,3,0,0,4,0,4,2,2,0,2,3,5,1,4)\\
\mathbf{s}_1^1 = (&0,1,2,3,4,5,0,5,5,4,3,1,0,3,2,5,2,3,\\ &0,1,5,0,1,5,0,5,2,1,0,1,0,3,5,2,5,3)\\
\mathbf{s}_2^1 = (&0,2,4,0,2,4,0,0,1,1,1,0,0,4,4,2,0,2,\\ &0,2,1,3,5,4,0,0,4,4,4,0,0,4,1,5,3,2)
\end{aligned}$$

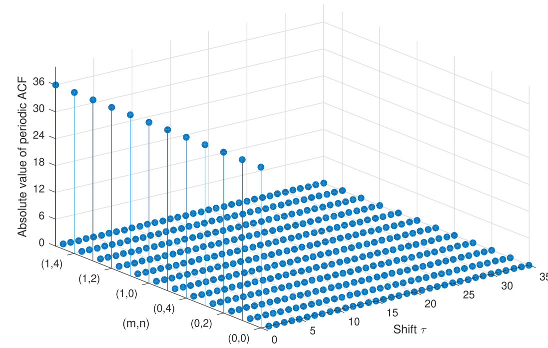


Fig. 1 Period auto-correlation properties of $\mathbf{s}_n^m$, $0 \le m \le 1$, $0 \le n \le 5$.

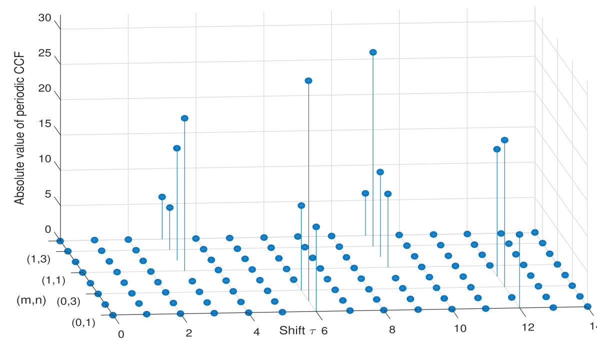


Fig. 2 Period cross-correlation properties of $\mathbf{s}_0^0$ and $\mathbf{s}_n^m$, $0 \le m \le 1$, $1 \le n \le 4$, $0 \le \tau \le 14$.

$$\begin{aligned}
\mathbf{s}_3^1 = (&0,3,0,3,0,3,0,1,3,4,5,5,0,5,0,5,4,1,\\ &0,3,3,0,3,3,0,1,0,1,2,5,0,5,3,2,1,1)\\
\mathbf{s}_4^1 = (&0,4,2,0,4,2,0,2,5,1,3,4,0,0,2,2,2,0,\\ &0,4,5,3,1,2,0,2,2,4,0,4,0,0,5,5,5,0)\\
\mathbf{s}_5^1 = (&0,5,4,3,2,1,0,3,1,4,1,3,0,1,4,5,0,5,\\ &0,5,1,0,5,1,0,3,4,1,4,3,0,1,1,2,3,5)
\end{aligned}$$

We provided Fig. 1 to show the ACF properties of this example. From Fig. 1, it can be verified that the absolute value of the ACF of each sequence in $\mathcal{S}$ is equal to 0 at shift $\tau \in (0, 35]$. Therefore, the resultant sequence set has good auto-correlation property, which can be used to completely eliminate MPI.

We provided Fig. 2 to show the CCF properties of $\mathbf{s}_0^0$ and some other sequences respectively, which are from the same subset $\mathbb{S}^0$ and different subset $\mathbb{S}^1$. In order to show the ZCZ clearly, the shift τ is given between 0 to 14. On one hand, it can be verified that the absolute value of CCF of $\mathbf{s}_0^0$ and each sequence in $\{\mathbf{s}_1^0, \mathbf{s}_2^0, \mathbf{s}_3^0, \mathbf{s}_4^0\}$ is equal to 0 at shift $\tau \in [0, 5]$. It can be known that each subset is a conventional ZCZ sequence set with parameters$(36, 6, 6) - \mathrm{ZCZ}$. Furthermore, the parameters of $\mathbb{S}^0$ is optimal, i.e., $\eta_1 = \frac{6 \times 6}{36} = 1$. On the other hand, it can be verified that the absolute value of CCF of $\mathbf{s}_0^0$ and each sequence in $\{\mathbf{s}_1^1, \mathbf{s}_2^1, \mathbf{s}_3^1, \mathbf{s}_4^1\}$ is equal to 0 at shift $\tau \in [0, 2]$. Then we can get that the ZCCZ length is 3. Let $\mathbb{T} = \mathbb{S}^0 \cup \mathbb{S}^1$, then $\mathbb{T}$ is a $(36, 12, 3) - \mathrm{ZCZ}$, it holds that $\eta_2 = \frac{12 \times 3}{36} = 1$. Therefore, the parameters of sequence

Table 1 Comparison of several constructions of A-ZCZ sequence sets.

Constructions	The parameters of intra-set-ZCZ	Number of subsets	Z_{CCZ}	Intra-set optimal or not	Inter-set optimal or not	Constraints
in [14]	$(LN_1, N_1, N_1L_1 - 1)$	N	$N_1L_1 - 1 + N_1$	Not	Quasi-optimal when $\lfloor L/N \rfloor = N_1L_1$	$\left\lfloor \frac{L}{N_1L_1} \right\rfloor = N$ $L_1 \geq 1$
in [15]	$(TL, T, 2k + 1)$	N	$q(2k + 1) + q - 1$	Optimal when $N = 1, q = 1$	Optimal when $q = 1$	$L = Nq(2k + 1)$ $N, q, k \geq 1$
in [16]	$(kLM, kM, M - 1)$	N	$2M - 1$	Not	Quasi-optimal when $\lfloor L/N \rfloor = M$	$\lfloor L/M \rfloor = N$ $K \geq 2, M \geq 2$
in [17]	(PM, M, M)	N	PM	Not	Optimal when $M = \lfloor P/N \rfloor$	$N = \lfloor P/M \rfloor$
in [18]	$(2NZ, 2M, LZ)$	P	Z	Optimal when $M = \frac{1}{2}\lfloor 2N/L \rfloor$	Not	$2 < L < N$
in [22] with *Lemma* 6	$\left(N^2, N, N\right)$	M	Z	Optimal	Optimal	*Lemma*6
Theorem 1	$\left(N^2, N, N\right)$	M	Z	Optimal	Optimal	$M = N/Z$

set $\mathbb{T}$ are optimal. We can conclude that $\mathcal{S}$ is an A-ZCZ $Z_A(36, [6, 2], [6, 3])$ whose parameters are optimal for both intra-set and inter-set.

4. Comparison with Existing Constructions

The objective of this paper is to construct A-ZCZ sequence sets with optimal parameters. As a comparison, we list some parameters, constraints of parameters and parameters performance from the known constructions and the ones we proposed in Table 1. Constructions in [14]–[16] were based on perfect sequence sets, whose lengths L were constrained differently. When $\lfloor L/N \rfloor = N_1L_1$ in [14] and $\lfloor L/N \rfloor = M$ in [16], the inter-set parameter performance is quasi-optimal, where N denotes the number of subsets. In [15], the length of initial perfect sequences L are required to be $L = Nq(2k + 1)$, which means that perfect sequences with prime length cannot satisfy the condition. Moreover, the obtained A-ZCZ sequence sets were optimal when $N = 1$ and $q = 1$. A construction of Gaussian A-ZCZ sequence sets were proposed in [18], the parameters of each subset are optimal if and only if $M = \frac{1}{2}\lfloor 2N/L \rfloor$. In [17], polyphase A-ZCZ sequence sets were generated from the P-dimension DFT matrices and the orthogonal matrices of order M, and parameters must satisfy $N = \lfloor P/M \rfloor$, where N is the number of subsets. It means when the value of M and P are closer, the number of subsets will be less. On the contrary, the dimensions of DFT matrices are equal to the orders of orthogonal matrices in this paper, and the number of subsets is determined by the ratio of the dimension of DFT matrix to the length of ZCZ. According to above references, it is clear that only one of the intra-set and inter-set parameters performance can achieve optimal or quasi-optimal under certain conditions in [14], [16]–[18].

In [22], the optimal A-ZCZ sequence sets are presented with *Lemma* 6. However, the number of subsets M need to be odd according to *Lemma 6*. Furthermore, if the values of M and Z is determined, the sequence set obtained is unique. In order to enlarge the number of A-ZCZ sequence sets, this paper presents a new construction of A-ZCZ sequence sets. As the diversity of the orthogonal matrices and the flexibility of the initial matrix, component and shift sequence, this construction can generate great deal of A-ZCZ sequence sets. In addition, the number of subsets M and the length Z of inter-set ZCCZ can be chosen flexibly. Besides, the resultant sequence sets obtained in this paper, not only the parameters of intra-set are optimal, but also the inter-set are optimal. As a result, this paper can provide more available sequences for communication systems and enrich the research results of A-ZCZ sequence sets.

5. Conclusion

In this paper, we proposed that the NHZ FHS sets obtained by [22] should satisfy one of the two conditions in *Lemma 6*, so as to construct A-ZCZ sequence set. A condition for constructing the optimal A-ZCZ sequence set based on non-repeated NHZ FHS set is supplemented. Moreover, a new construction of optimal A-ZCZ sequence sets is proposed by matrices transformation. The length of inter-set ZCCZ and the number of subsets can be chosen flexibly. As the diversity of the orthogonal matrices and the flexibility of initial matrix, the new constructions can generate more sequence sets than [22]. As a result, this paper can provide more available sequences for communication systems and enriched the research results of A-ZCZ sequence sets.

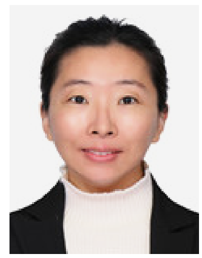
Acknowledgments

This paper was supported by Nation Natural Science Foundation of China (61671402); Natural Science Foundation of Hebei Province (F2020203043, F2021203078).

References

[1] X.M. Deng and P.Z. Fan, "Spreading sequence sets with zero correlation zone," Electron. Lett., vol.36, no.11, pp.993–994, May 2000.
[2] T. Liu, C.Q. Xu, and Y.B. Li, "Construction of optimal mutually orthogonal sets of binary zero correlation zone sequences," Journal of Electronics and Information Technology, vol.39, no.10, pp.2442–2448, Oct. 2017.
[3] M. Addad and A. Djebbari, "Ternary ZCZ codes with inter-subgroup

zero-zones for multiuser commun.," Proc. 6th Int. Conf. Image Signal Process. Appl., Mostaganem, Algeria, pp.1–4, 2019.
[4] S. Das, U. Parampalli, S. Majhi, and Z. Liu, "Near-optimal zero correlation zone sequence sets from paraunitary matrices," Proc. IEEE International Symposium on Information Theory, Paris, France, pp.2284–2288, 2019.
[5] X.H. Tang and W.H. Mow, "A new systematic construction of zero correlation zone sequences based on interleaved perfect sequences," IEEE Trans. Inf. Theory, vol.54, no.12, pp.5729–5734, 2008.
[6] X.Y. Chen, Q.C. Gao, and Y.J. Li, "Construction of optimal zero correlation zone sequence set," Journal on Commun., vol.41, no.8, pp.215–222, Aug. 2020.
[7] G. Gong, "Constructions of multiple shift-distinct signal sets with low correlation," Proc. Int. Symp. Information Theory, pp.2306–2310, Nice, France, June 2007.
[8] X.H. Tang, P.Z. Fan, and J. Lindner, "Multiple binary ZCZ sequence sets with good cross-correlation property based on complementary sequence sets," IEEE Trans. Inf. Theory, vol.56, no.8, pp.4038–4045, Aug. 2010.
[9] Y.B. Li, L.Y. Tian, and T. Liu, "Constructions of polyphase ZCZ sequence sets with low cross-correlation property," IET Communication, vol.13, no.6, pp.733–740, April 2019.
[10] D. Zhang, M.G. Parker, and T. Helleseth, "Polyphase zero correlation zone sequences from generalised bent functions," Cryptogr. Commun., vol.12, no.3, pp.325–335, May 2020.
[11] H. Torii and M. Nakamura, "A study of asymmetric ZCZ sequence sets," Proc. Recent Researches in Multimedia Systems, Signal Processing, Robotics, Control and Manufacturing Technology, Venice, pp.79–86, March 2011.
[12] T. Hayashi, T. Maeda, S. Kanemoto, and S. Matsufuji, "A novel construction of zero-correlation zone sequence set with wide intersubset zero-correlation zone," Proc. Int. Workshop Signal Des. Its Appl. Commun., IWSDA, Guilin, China, pp.25–28, Oct. 2011.
[13] L.Y. Wang, X.L. Zeng, and H. Wen, "Asymmetric ZCZ sequence sets with inter-subset uncorrelated sequences via interleaved technique," IEICE Trans. Fundamentals, vol.E100-A, no.2, pp.751–756, Feb. 2017.
[14] H. Torii, T. Matsumoto, and M. Nakamura, "A new method for constructing asymmetric ZCZ sequence sets," IEICE Trans. Fundamentals, vol.E95-A, no.9, pp.1577–1586, Sept. 2012.
[15] L.Y. Wang, X.L. Zeng, and H. Wen, "A novel construction of asymmetric ZCZ sequence sets from interleaving perfect sequence," IEICE Trans. Fundamentals, vol.E97-A, no.12, pp.2556–2561, Dec. 2014.
[16] H. Torii, T. Matsumoto, and M. Nakamura, "Extension of methods for constructing polyphase asymmetric ZCZ sequence sets," IEICE Trans. Fundamentals, vol.E96-A, no.11, pp.2244–2252, Nov. 2013.
[17] H. Torii, M. Nakamurai, and Makoto, "Optimal polyphase asymmetric ZCZ sets including uncorrelated sequences," Journal Signal Process, vol.16, no.6, pp.487–494, Nov. 2012.
[18] X.Y. Chen, H.R. Su, Y.B. Li, and X.P. Peng, "Construction of asymmetric gaussian integer ZCZ sequence sets," IEICE Trans. Fundamentals, vol.E102-A, no.2, pp.471–475, Feb. 2019.
[19] T. Liu, C.Q. Xu, Y.B. Li, and X.Y. Chen, "New constructions of multiple binary ZCZ sequence sets with inter-set zero cross-correlation zone," IEICE Trans. Fundamentals, vol.E100-A, no.12, pp.3007–3015, Dec. 2017.
[20] T. Hayashi, T. Maeda, and S. Okawa, "A generalized construction of zero correlation zone sequence set with sequence subsets," IEICE Trans. Fundamentals, vol.E94-A, no.7, pp.1957–1602, July 2011.
[21] T. Hayashi, T. Maeda, S. Matsufuji, and S. Okawa, "A ternary zero correlation zone sequence set having wide inter-subsetzero correlation zone," IEICE Trans. Fundamentals, vol.E94-A, no.11, pp.2230–2235, Nov. 2011.
[22] X.Y. Chen, X.C. Gao, and X.Y. Peng, "Construction of multiple optimal polyphase zero correlation zone sequence sets with inter-set zero cross-correlation zone," IEEE Commun. Lett., vol.25, no.9, pp.2795–2799, Sept. 2021.
[23] X.H. Tang and P.Z. Fan, "Lower bounds on correlation of spreading sequence set with low or zero correlation zone," Electron. Lett., vol.36, no.6, pp.551–552, March 2000.



Li Cui was born in Hebei, China, in 1980. She received the B.S. degree in computer science and technology from Hebei Normal University of Science and Technology, China, in 2004, and the M.S. degree in computer application technology from Yanshan University, China in 2007. She is currently pursued the Ph.D. degree with the School of Information Science and Engineering, Yanshan University. Since 2007, she has been in Hebei Normal University of Science and Technology as a Lecturer. Her research interests include sequence design and coding theory.

Xiaoyu Chen was born in 1983. She received the B.S. and Ph.D. degrees in circuit and system in communication engineering from Yanshan University in 2006 and 2013, respectively. Currently, she is an associate professor at the School of Information Science and Engineering, Yanshan University, Hebei, China. Her research interests include sequence design and network coding theory.

Yubo Li was born in Hebei, China, in 1985. He received the B.S. and Ph.D. degrees in circuit and system in communication engineering from Yanshan University in 2007 and 2012, respectively. Currently, he is an associate professor at the School of Information Science and Engineering, Yanshan University, Hebei, China. His research interests include sequence design for wireless communication and spread spectrum systems.