LETTER

Constructions of Optimal Single-Parity Locally Repairable Codes with Multiple Repair Sets*

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SUMMARY Locally repairable codes have attracted lots of interest in Distributed Storage Systems. If a symbol of a code can be repaired respectively by t disjoint groups of other symbols, each groups has size at most r, we say that the code symbol has (r, t)-locality. In this paper, we employ parity-check matrix to construct information single-parity (r, t)-locality LRCs. All our codes attain the Singleton-like bound of LRCs where each repair group contains a single parity symbol and thus are optimal.

key words: *locally repairable codes, availability, distributed storage systems, linear codes*

1. Introduction

Distributed storage systems are developing quickly in recent years. The traditional way of introducing redundancy in distributed storage systems is replication, which caused large storage overhead. An alternative solution is using erasure codes to ensure high data reliability. However, the repair cost of erasure codes is much higher than replication. Recently, Locally repairable codes (LRCs for short) have been proposed to reduce the repair cost [2]. When we using LRCs in distributed storage system, it can apparently reduce the cost and make the repair process more efficiently [7].

For a code C of length n, dimension k and minimum distance d is called an $[n, k, d]$ code [13]. A symbol with locality r means it can be repaired by at most r other symbols. When the k information symbols have locality r, Gopalan et al. [2] proved that the minimum distance d satisfies

$$d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2. \tag{1}$$

Some LRCs attaining the bound (1) are proposed in [1], [4], [10], [11], [14], [17], [21]. However, when a node fails and some nodes in repair group also fails, the failed node could not be repaired locally. Therefore, many researchers have studied the code with more than one repair groups [3], [8], [12], [15], [16], [19]. A symbol of a code is said to have (r, t)-locality if there exist t disjoint groups of other symbols to repair this symbol and each group has size at most r. For a systematic $[n, k]$-linear code, namely the first k symbols are its information symbols. When the k information symbols have (r, t)-locality, the code is called (n, k, r, t)-LRC. Tamo et al. [17] and Wang et al. [19] have gave two bounds for the minimum distance of such codes respectively. If each repair group contains only one parity symbol, the codes are called single-parity (n, k, r, t)-LRCs. Rawat et al. [16] proved that the minimum distances of such codes satisfy

$$d \leq n - k - \left\lceil \frac{kt}{r} \right\rceil + t + 1, \tag{2}$$

A single-parity (n, k, r, t)-LRC achieving this bound is called optimal. Rawat et al. [16] have presented the generator matrix of a kind of optimal single-parity LRCs through MDS codes. Zhang et al. [20] proposed several types of single-parity LRCs from combinatorial designs. Other constructions of single-parity (n, k, r, t)-LRCs were proposed in [5], [6]. Inspired by the works of [5], [6], [20], we construct two classes of optimal single-parity (n, k, r, t)-LRCs through parity-check matrix with large minimum distance. Table 1 shows the parameters of our codes and their's.

Table 1 Parameters of optimal single-parity LRCs.

$(n, k, r, t)_q$	Constraints	Ref.
$(2t^2 - 2, t^2 - 1, t, t)_2$	$t = 2^s, s > 0$	[5]
$(2t^2 - 1, t^2 - 1, t, t)_2$	$t = 2^s, s > 0$	[6]
$(r^2 + rt + 1, r^2, r, t)_2$	$2 \mid t, 2 \nmid r$	[6]
$(N + \frac{kt}{r}, k, r, t)_q$	$r \mid k, k \leq N + t \leq q$	[16]
$(N + tr, r^2, r, t)_q$	$r^2 < N + t \leq q$	[20]
$(k + \frac{kt}{r} + s, k, r, t)_q$	$q = l^e, e \geq k \geq s \geq 0, r \leq k$	Thm 3.1
$(\frac{t(t+1)}{2} + t + 3, t + 1, 2, t)_q$	$q \geq t + 1 > 4, 2 \nmid q$	Thm 3.5

The rest of the paper is organized as follows. In Sect. 2, we introduce some definitions and results on LRCs. We present two classes optimal single-parity (r, t)-locality LRCs through parity-check matrix in Sect. 3.

2. Preliminary

Let C be an $[n, k, d]$ systematic code, the generator matrix of C be $G = [I_k, P]$, then the parity-check matrix is $H = [-P^T, -I_{n-k}]$, where P is an $k \times (n - k)$ matrix. Let $[n]$ denote the set of $\{1, 2, \cdots, n\}$ and $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ denote a codeword. Now we give the formal definition of the (r, t)-locality of C ([16]).

Definition 2.1. *Let C be an $[n, k, d]$ linear code, a symbol of a code c_i is said to have (r, t)-locality if it satisfies the*

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following three properties:

1) There exist t subsets $\Phi_1(i), \Phi_2(i), ..., \Phi_t(i) \subseteq [n] \backslash \{i\}$ such that $\Phi_j(i) \cap \Phi_l(i) = \emptyset$ and c_i can be recovered from the code symbols indexed by $\Phi_j(i), j \neq l \in [t]$.

2) $|\Phi_j(i)| \leq r$, for all $j \in [t]$.

When the k information symbols of a code have (r,t)-locality, then the code is called an (n,k,r,t)-LRC.

If all repair groups for information symbol only have one parity symbol, the resulting code is called a single-parity (n,k,r,t)-LRC. In this paper, we are all discussing about information symbol single-parity (n,k,r,t)-LRCs.

Next, we interpret the above definition by a group of desirable codewords in $C^{\perp}$.

Lemma 2.2. *[6] A code symbol c_i is said to have (r,t)-locality if there exist at least t codewords $\mathbf{c}^{\perp}(j) \in C^{\perp}, 1 \leq j \leq t$, such that:*

1) $i \in Supp(\mathbf{c}^{\perp}(j))$, for all $j \in [t]$

2) $wt(\mathbf{c}^{\perp}(j)) \leq r+1$, for all $j \in [t]$,

3) $Supp(\mathbf{c}^{\perp}(j)) \cap Supp(\mathbf{c}^{\perp}(l)) = \{i\}$, for all $j \neq l \in [t]$

In [5], the authors give an explicit construction of optimal single-parity (n,k,r,t)-LRC with minimum distance $d = t+1$ by using (r,t)-regular matrix, namely each row has uniform weight r and each column has uniform weight t.

Lemma 2.3. *[5] Let $H = [P_{(n-k)\times k}, I_{n-k}]$, where P is an $(r,t)-$regular matrix and the supports of any two row vectors of P share at most one common coordinate, I_{n-k} is an identity matrix. Then the linear code C with parity-check matrix H is a single-parity $(n,k,r,t)-$LRC with minimum distance $d = t+1$.*

From Lemma 2.3, P should be a regular matrix satisfying the intersection set of support set of any two row vectors has at most one element. In the following, we introduce a way to construct such matrix P.

Definition 2.4. *[9] A pair $(\mathcal{X}, \mathcal{B})$, where $\mathcal{X} = (x_1, \cdots, x_m)$ is a $m-$element set and $\mathcal{B} = (B_1, \cdots, B_k)$ is a family of subsets(blocks) of $\mathcal{X}$, is called a $2-(m,k,r,t,\lambda)$-resolvable design if it satisfies the following three properties:*

i) $|B_i| = t$, for all $i \in [k]$.

2) Every pair $(x,y) \subset \mathcal{X}$ is present in exactly λ subset(blocks) in $\mathcal{B}$ e 3) $\mathcal{B}$ comprises t parallel class partitions $\mathcal{E}_1, \cdots, \mathcal{E}_r \in \mathcal{B}$ such that

$$|\{(i,j,i_a)|x_j \in B_{i_a}, B_{i_a} \in \mathcal{E}_i\}| = 1.$$

We can define the incidence matrix $J_{(\mathcal{X},\mathcal{B})}$ of a 2-resolvable design $(\mathcal{X}, \mathcal{B})$ as

$$J_{(\mathcal{X},\mathcal{B})}(i,j) = \begin{cases} 1, & if\ x_i \in B_j \\ 0, & otherwise. \end{cases}$$

It is clearly that the incident matrix of a 2-resolvable design for $\lambda = 1$ is the matrix that we are looking for. We now present an example of 2-$(m,k,r,t,1)$-resolvable design. Some explicit 2-resolvable design were presented in [9].

Table 2 2-(4,6,3,2,1)-resolvable design.

$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$
{1,2}	{1,3}	{1,4}
{3,4}	{2,4}	{2,3}

Example 2.5. *Let $\mathcal{X} = \{1,2,3,4\}$, $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\{2,3\}\}$. The following Table 2 represents a 2-$(4,6,3,2,1)$- resolvable design with 3 parallel classes partitions $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ the set $\mathcal{X}$.*

then the incidence matrix of 2-(4,6,3,2,1)-resolvable design is

$$J_{(\mathcal{X},\mathcal{B})} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

At the end of this section, we presented some known results of linear codes and the Moore matrix.

Lemma 2.6. *[18] Let $G \in F_q^{k\times n}$ be a generator matrix of an $[n,k]_q$-linear code C. Then*

i) $d(C) \leq w$ iff there exists a set of $n-w$ columns of G that forms a submatrix of rank less than k.

ii) $d(C) \geq w$ iff any $n-w+1$ columns of G forms a submatrix of rank k.

Definition 2.7. *Let l be a power of q. For elements $\alpha_1, \ldots, \alpha_h \in F_\ell$, the Moore matrix is defined by*

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_h \\ \alpha_1^q & \alpha_2^q & \cdots & \alpha_h^q \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{q^{h-1}} & \alpha_2^{q^{h-1}} & \cdots & \alpha_h^{q^{h-1}} \end{pmatrix} \in F_l^{h\times h}.$$

It is well-known that $\det(M) \neq 0$ if and only if $\alpha_1, \ldots, \alpha_h$ are F_q-linearly independent.

3. Our Constructions

In this section, we focus on the constructions of single-parity (n,k,r,t)-LRCs. We present two classes of optimal single-parity (n,k,r,t)-LRCs based on the work in [5], [6], [20].

As we know, when LRCs attains the bound (2), then

$$n - k - \left\lceil \frac{kt}{r} \right\rceil = d - t - 1 \tag{3}$$

From Eq. (3), if $n = k + \lceil \frac{kt}{r} \rceil$, then $d = t+1$. Hao et al. constructed optimal single-parity $(2t^2-2, t^2-1, t, t)$-LRCs through the parity-check matrix $H = [P\ I]$, where P is an (t,t)-regular. If $n = k + \lceil \frac{kt}{r} \rceil + 1$, then $d = t+2$. Hao et al. constructed two classes optimal single-parity LRCs through parity check matrix [6]. Therefore, it is interesting to construct optimal LRCs with minimal distance $d \geq t+3$.

Now, we give our first construction.

Theorem 3.1. *Let $Q = q^e$, q be a prime power, and s, k, r, t*

be positive integers with $e \geq k \geq s$. Let $m = \lceil \frac{kt}{r} \rceil$. Suppose that

$$H = \begin{bmatrix} P & I_m & 0 \\ V_{s\times k} & 0 & I_s \end{bmatrix}_{(m+s)\times(k+s+m)} \tag{4}$$

where P is an incidence matrix of a $2-(m,k,r,t,1)$-resolvable design, I_m is an $m\times m$ identity matrix, I_s is an $s\times s$ identity matrix, and $V_{s\times k} \in F_Q^{s\times k}$ is a Moore matrix with the elements in the first row are linearly independent over F_q. Then C with parity-check matrix H is an optimal single-parity $(k+m+s,k,r,t)$-LRC with minimum distance $d = t+s+1$.

Proof. The length and the dimension of C are clearly. Now we want to show that C has information (r,t)-locality. Recall that the first k columns correspond to the information symbols, and the reminder correspond to the parity check symbols. Since P is an incidence matrix of a 2-$(m,k,r,t,1)$-resolvable design, which implies every information symbol has t repair groups and the supports of any two row vectors of P share at most one common coordinate, namely, the repair groups for a fixed information symbols are disjoint. Since $[P\ I_m]$ has uniform row weight $r+1$, the code C has locality r and each repair group contains exactly one parity-check symbol. Therefore, C has information (r,t)-locality.

Next we want to prove that the minimum distance d of C satisfies $d \geq t+s+1$. It is easy to see that the generator matrix of C is $G = [I_k, -P^\top_{k\times m}, -V^\top_{k\times s}]$, we just need to prove the rank of any $k+m-t$ columns in G is equal to k by Lemma 2.6. Let G' be a submatrix of G with $k+m-t$ columns, we divide it into two cases:

i) The columns of G' are all from $[I_k, -P^\top_{k\times m}]$. Note that $[I_k, -P^\top_{k\times m}]$ is same as the generator matrix in Lemma 2.3. Since the minimum distance d of code in Lemma 2.3 is equal to $t+1$, so any $k+m-t$ columns from $[I_k, -P^\top_{k\times m}]$ consists a matrix with rank k.

ii) $k+m-t-l$ columns of G' are from $[I_k, -P^\top_{k\times m}]$, where $1 \leq l \leq s$. From case (i), the rank of these $k+m-t-l$ columns $\geq k-l$. Assume that the rank of these $k+m-t-l$ columns is equal to $k-a$, WLOG, by elementary column transformation, we can get

$$G' \sim \begin{bmatrix} I_{k-a} & \underline{0} & -V^\top_{k\times l} \\ S_a & & \end{bmatrix}, \tag{5}$$

where I_{k-a} is an $(k-a)\times(k-a)$ identity matrix, and $\underline{0}$ is a null matrix, $S_a = (s_{i,j})_{1\leq i\leq a, 1\leq j\leq k-a}$ For every $v_i^{q^j}$ in $-V^\top_{k\times l}$, add $v_i^{q^j}$ times of the ith column in I_{k-l} to $V_i^{q^j}$, $1 \leq i \leq k-a$, $1 \leq j \leq l$. The elements of the first $k-a$ row of $-V^\top_{k\times s}$ become 0 and the elements of the last a row of $-V^\top_{k\times l}$ become

$$\begin{bmatrix} v_{k-a+1} + \sum_{i=1}^{k-a} s_{1,i}v_i, & \cdots, & v_{k-a+1}^{q^{l-1}} + \sum_{i=1}^{k-a} s_{1,i}v_i^{q^{l-1}} \\ \vdots & & \vdots \\ v_k + \sum_{i=1}^{k-a} s_{l,i}v_i, & \cdots, & v_k^{q^{l-1}} + \sum_{i=1}^{k-a} s_{l,i}v_i^{q^{l-1}} \end{bmatrix} \tag{6}$$

The matrix in above is also a Moore matrix. Since $v_1, v_2, \cdots, v_k$ are F_q-linearly independent, $v_{k-a+1} + \sum_{i=1}^{k-a} s_{1,i}v_i, \cdots, v_k + \sum_{i=1}^{k-a} s_{l,i}v_i$ are also F_q-linearly independent, the determinant of the any a columns of (6) is nonzero. Thus the rank of the G' is k.

Therefore, we have prove $d \geq t+s+1$. On the other hand, from (2), we have

$$\begin{aligned} d &\leq n-k-\left\lceil \frac{kt}{r} \right\rceil + t + 1 \\ &= t+s+1 \end{aligned}$$

then C is an optimal single-parity $(k+m+s,k,r,t)$-LRC with minimum distance $d = t+s+1$. □

Remark 3.2. *Note that the coordinates of submatix P in parity-check matrix is 0 or 1, which menns that the recovery function in our construction is more simple.*

Example 3.3. *Let $q = 2^4$, α is a primitive element of F_q satisfying $\alpha^4+\alpha^3+\alpha^2+\alpha+1 = 0$. Take $r = 2, t = 3, s = 4, k = 4$, let P be the transpose matrix of the matrix in Example 2.5, let*

$$V = \begin{bmatrix} \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ \alpha^2 & \alpha^4 & \alpha^8 & \alpha^{12} \\ \alpha^4 & \alpha^8 & \alpha^{12} & \alpha^1 \\ \alpha^8 & \alpha^1 & \alpha^9 & \alpha^2 \end{bmatrix}.$$

Then the code C with parity-check matrix H in Thm 3.1 is an optimal single-parity $(14,4,2,3)$-LRC with minimum distance $d = 8$.

In Distributed Storage Systems, peoples are interesting in the case for $r = 2$. Therefore, in the rest of the paper, we give an explicit construction of optimal single-parity LRCs with $(2,t)$-locality, and minimum distance $d = t+3$. For an positive integer k, let T_k be the incidence matrix of a 2-$(\frac{k(k-1)}{2}, k, 2, k-1, 1)$-resolvable design. Since each row of T_k correspond of 2-subsets of $[k]$, we can arrange these 2-subsets in a partial order, i.e. $(i_1, i_j) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ or $i_1 = i_2, j_1 \leq j_2$, then

$$T_k = \begin{bmatrix} 1 & 1 & & & & & \\ \vdots & & \ddots & & & & \\ 1 & & & & & 1 & \\ & & \vdots & & & & \\ & & & & 1 & 1 & \\ & & & & 1 & & 1 \\ & & & & & 1 & 1 \end{bmatrix}_{\frac{k(k-1)}{2}\times k}.$$

Lemma 3.4. *Let k be a positive integer, for any integer j with $2 \leq j \leq k$, let $wt(j)$ denote the minimum weight of the vector which is the linear combination of any j columns in T_k. Then we have*

$$wt(j) \geq j\cdot(k-1) - \sum_{x=1}^{j-1} 2x.$$

Proof. It is easy to see that each column of T_k with weight $k-1$ and each row of T_k with weight 2. Note that every rows of T_k correspond of 2-subsets of $[k]$ and every numbers have been used $k-1$ times, and any two row vectors of T_k share at most one common coordinate. We can get the weight of the combination of any j columns in P should minus at most $2+4+\cdots+2(j-1)=\sum_{x=1}^{j-1}2x$ by induction. Therefore, we have

$$wt(j)\geq j(k-1)-\sum_{x=1}^{j-1}2x.$$

□

Now we give the construction of optimal single-parity $(n,k,2,t)$-LRCs.

Theorem 3.5. *Let $q\geq k>3$, $2\nmid q$, let $\alpha_1,\cdots,\alpha_k$ be different elements in F_q, $\boldsymbol{\alpha}=[\alpha_1,\cdots,\alpha_k]$. Let*

$$H=\left[\begin{array}{c|c} T_k & \\ \boldsymbol{1} & I \\ \boldsymbol{\alpha} & \end{array}\right],$$

where $\boldsymbol{1}$ be all 1 vector with length k, I be an $(\frac{k(k-1)}{2}+2)\times(\frac{k(k-1)}{2}+2)$ identity matrix. Then C with parity-check matrix H is an q-ary optimal single-parity $(\frac{k(k-1)}{2}+k+2,k,2,k-1)$-LRC with minimum distance $d=k+2$.

Proof. The length and the dimension of C are clearly. From the expression of T_k, it is easy to see that C has information symbol $(2,k-1)$-locality. Next we want to prove that the minimum distance d of C satisfies $d\geq k+2$. Namely, we need to show that any $k+1=t+2$ columns of H are linearly independent. Let $H_1=(\mathbf{h}_1,\cdots,\mathbf{h}_{t+2})$ be any submatrix of H with $t+2$ columns, suppose that there exist $\lambda_j\in F_q$ such that $\sum_{j=1}^{t+2}\lambda_j\mathbf{h}_j=0$, we want to show that $\lambda_1=\cdots=\lambda_{t+2}=0$. We divide it into three cases:

i) If $t+2$ columns are all from I, clearly.

ii) If only $t+1$ columns are from I. WLOG, we may assume that $\mathbf{h}_2,\cdots,\mathbf{h}_{(t+2)}$ are from I. If $\lambda_1=0$, we have $\lambda_2=\cdots=\lambda_{t+2}=0$. Now assume that $\lambda_1\neq 0$, then $\lambda_1\mathbf{h}_{i_1}=-\sum_{j=2}^{t+2}\lambda_j\mathbf{h}_{i_j}$. Note that $wt(\lambda_1\mathbf{h}_1)=k+1=t+2$ and $wt(\sum_{j=2}^{t+2}\lambda_j\mathbf{h}_{i_j})\leq(t+1)$, we get contradiction.

iii) If $t+2-l$ columns are from I, where $2\leq l\leq t$. WLOG, we may assume $\mathbf{h}_{l+1},\cdots,\mathbf{h}_{t+2}\in I$. We have $\sum_{j=1}^{l}\lambda_j\mathbf{h}_j=\sum_{j=l+1}^{t+2}\lambda_j\mathbf{h}_j$. If $\lambda_1=\cdots=\lambda_l=0$, we have $\lambda_{l+1}=\cdots=\lambda_{t+2}=0$. This is because $\mathbf{h}_{(l+1)},\cdots,\mathbf{h}_{(t+2)}$ are linearly independent. Now suppose that $\lambda_1,\cdots,\lambda_l$ are not all zero. Then from Lemma 3.3, we have $wt(\sum_{j=1}^{l}\lambda_j\mathbf{h}_j)\geq l(k-1)-\sum_{j=1}^{l-1}2j\geq l(t+1-l)>t+2-l$, where the last inequality is because $t>2$. On the other hand, we have $wt(\sum_{j=l+1}^{t+2}\lambda_j\mathbf{h}_j)\leq t+2-l$. This is contradiction.

iv) If only one column comes from I, which means that we choose all columns of T_k. WLOG, we may assume that $\mathbf{h}_{t+2}\in I$. Then $H_1=\left[\begin{array}{c|c} A & \\ B & \mathbf{h}_{t+2} \\ C & \end{array}\right]$ where

$$A=\begin{bmatrix} 1 & 1 & & & \\ \vdots & & & \ddots & \\ 1 & & \cdots & & 1 \\ & 1 & 1 & & \end{bmatrix}, B=\begin{bmatrix} 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix}.$$

1) If the support of $\mathbf{h}_{t+2}$ belongs to the rows of A. We can see $\lambda_{t-1}=\lambda_t=\lambda_{t+1}=0$, cause

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}\neq 0,$$

If there exist $\lambda_j\neq 0, 1\leq j\leq t-2$, then the weight of the remaining $t-2$ columns satisfies $wt(\sum_{j=1}^{t-2}\lambda_j\mathbf{h}_j)\geq t(t-2)-\sum_{j=1}^{t-3}2j>1$, which is a contradiction.

2) If the support of $\mathbf{h}_{t+2}$ belongs to the rows of B or C. We can see $\lambda_1=\cdots=\lambda_{t+1}=0$, cause $|A|\neq 0$, thus $\lambda_{t+2}=0$.

Therefore, any $k+1$ columns of H are linearly independent. Thus we have the minimum distance $d\geq k+2$. On the other hand, from (1.2), we have

$$d\leq n-k-\left\lceil\frac{kt}{r}\right\rceil+t+1=k+2.$$

Therefore, an q-ary optimal single-parity $(\frac{k(k-1)}{2}+k+2,k,2,k-1)$-LRC with minimum distance $d=k+2$. □

Example 3.6. *Let $q=5$, $k=4$, and $H=\left[\begin{array}{c|c} T_4 & I_8 \\ M & \end{array}\right]$, where $M=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$. Then the resulting code C with parity-check matrix H is an optimal single-parity* $(12,4,2,3)$-*LRC with minimum distance* $d=6$.

Example 3.7. *Let $q=7, k=5$, and $H=\left[\begin{array}{c|c} T_5 & I_{12} \\ M & \end{array}\right]$, where $M=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$. Then the resulting code C with parity check matric H is an optimal single-parity* $(17,5,2,4)$-*LRC with minimum distance* $d=7$.

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