LETTER

A Computer-Aided Solution to Find All Feasible Schemes of Cyclic Interference Alignment for Propagation-Delay Based X Channels

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SUMMARY To obtain a full picture of potential applications for propagation-delay based X channels, it is important to obtain all feasible schemes of cyclic interference alignment including the encoder, channel instance, and decoder. However, when the dimension goes larger, theoretical analysis about this issue will become tedious and even impossible. In this letter, we propose a computer-aided solution by searching the channel space and the scheduling space, which can find all feasible schemes in details. Examples are given for some typical X channels. Computational complexity is further analyzed.
key words: *computer-aided solution, all feasible schemes, cyclic interference alignment, propagation delay based X channel*

1. Introduction

X channel [1] can achieve a higher multiplexing gain measured in degrees of freedom (DoF) by providing full-connected message transmission between transmitters and receivers. Propagation-delay (PD) based X channel can be constructed in the fundamental temporal domain, when there is no support from other domains such as the spatial domain by multiple antennas at each node. By exploiting the PD differences among links, cyclic interference alignment (IA) plays the key role of DoF study, which overlaps undesired messages (i.e., interference) into the same time-slot(s).

Existing works about PD-based X channels mainly focus on the DoF achievability, which is the most important theoretical problem. It is shown that the DoF upper bound of 4/3 is achievable for 2×2 X channel in [2]. For the $K \times 2$ X channel with two receivers and arbitrary K transmitters, its DoF upper bound of $2K/(K+1)$ had been shown to be achievable by one feasible scheme in [3], [4] and K schemes in [5]. For the $2 \times K$ X channel, the same DoF is also achievable by one feasible scheme as shown in [6]. However, not all scenarios can achieve their upper bounds, e.g., the DoF suboptimal feasible schemes for 3×3 X channels in [7] and for $2 \times K$ multicast X channels in [8].

On the other hand, it is also important to find all feasible schemes for the related scenarios, which can give a full picture of potential applications for the PD based X channels. To the authors' best knowledge, there are few works about this issue. In this letter, we propose a computer-aided solution for the above problem. By searching the channel space and the scheduling space, all feasible schemes can be found. We further analyze the computational complexity.

2. System Model

Assuming a $M \times N$ X channel, there are M transmitters and N receivers, denoted by S_j and D_i respectively, $j \in \{1, \cdots, M\}, i \in \{1, \cdots, N\}$. Each transmitter S_j sends an independent and different unicast message $W_{i,j}$ to each receiver D_i. For the PD based X channel, we can adopt the circular right-shift polynomial model, where the channel is partitioned into $n \in \mathbb{N}$ equally sized dimensions and normalized. The cycle period for transmission is n, among which a single time-slot is addressed by offsets $x^0, \cdots, x^{n-1}$, from 0 to $n-1$.

The PD coefficients of all links between transmitters and receivers are organized into a $N \times M$ polynomial matrix:

$$\boldsymbol{D} = \begin{bmatrix} x^{\tau_{1,1}} & \cdots & x^{\tau_{1,M}} \\ \vdots & \vdots & \vdots \\ x^{\tau_{N,1}} & \cdots & x^{\tau_{N,M}} \end{bmatrix} \tag{1}$$

where $\tau_{i,j}$ denotes the PD between receiver D_i and transmitter S_j. Although the practical link PD can be any positive real number, we need to map it to discrete integers. This can be done by methods such as the least common multiple. Detailed processing is out of the scope of this letter. However, further assumptions will be made in the next section.

S_j encodes its messages into the polynomial $v_j(x)$ by

$$v_j(x) = \sum_{i=1}^{N} W_{i,j} x^{p_{i,j}} \mod (x^n - 1) \tag{2}$$

where mod denotes the modulo operation, and $p_{i,j}$ indicates the scheduled time-slot for $W_{i,j}$.

By omitting the channel noise, receiver D_i obtains

$$r_i(x) = \boldsymbol{D}(i,:) \begin{bmatrix} v_1(x) \\ \vdots \\ v_M(x) \end{bmatrix} \mod (x^n - 1) \tag{3}$$

where $\boldsymbol{D}(i,:)$ indicates the ith row of $\boldsymbol{D}$.

An feasible scheme will give a specified PD channel instance (i.e., $\{\tau_{i,j}\}$) and specified encoder (i.e., the scheduling parameters $\{p_{i,j}\}$), based on which the receiver can correctly obtain interference free messages which it desires and discard the time-slots occupied by the cyclic IA.

Manuscript received September 10, 2022.
Manuscript publicized November 2, 2022.
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DOI: 10.1587/transfun.2022EAL2078

3. Proposed Solution

To find all feasible schemes, we firstly analyze the required conditions. Then we propose a computer-aided solution.

3.1 Required Conditions

Generally, there are three common conditions which should be satisfied for all PD based X channels:

- **intra-user interference condition**: the messages sent from the same transmitter S_j must be separable, i.e.: $\forall j \in \{1, \cdots, M\}$, $\forall i, k \in \{1, \cdots, N\}$, and $i \neq k$

$$x^{p_{i,j}} \neq x^{p_{k,j}} \mod (x^n - 1) \tag{4}$$

- **multiple-access interference condition**: the messages desired by each receiver must be separable, i.e.: $\forall j, l \in \{1, \cdots, M\}$, $\forall i \in \{1, \cdots, N\}$, and $j \neq l$

$$x^{\tau_{i,j}+p_{i,j}} \neq x^{\tau_{i,l}+p_{i,l}} \mod (x^n - 1) \tag{5}$$

- **inter-user interference condition**: the messages desired by each receiver must be interference free, i.e.: $\forall j, l \in \{1, \cdots, M\}$, $\forall i, k \in \{1, \cdots, N\}$, and $i \neq k$

$$x^{\tau_{i,j}+p_{i,j}} \neq x^{\tau_{k,l}+p_{k,l}} \mod (x^n - 1) \tag{6}$$

The above conditions are enough to check the feasibility of specified parameter sets $\{\tau_{i,j}\}$ and $\{p_{i,j}\}$, while the condition of cyclic IA is not necessarily explicit.

3.2 A Computer-Aided Solution

For a given $M \times N$ X channel, we are interested in finding all feasible schemes which can achieve the largest possible DoF. Thanks to the circular right-shift polynomial model and the property of cyclic transmission, all involved parameters are discrete and constrained. Thus the parameter space is limited and enumerable. Hence, searching method via computer is applicable. From this observation, we propose the following computer-aided solution.

The first step is to determine the searching space, including both $\{\tau_{i,j}\}$ and $\{p_{i,j}\}$:

- **the channel space**: i.e., the possible PD matrices composed by $\{\tau_{i,j}\}$. Due to cyclic transmission, a PD τ is equivalent to its n-fold extensions. For simplicity, we can assume that the minimum PD is set to be 0 as the reference origin, the maximum PD is integer no larger than $n - 1$, and the PD gap is also normalized. So we have $\tau_{i,j} \in \{0, 1, \cdots, n-1\}$. Without loss of generality, we can assume that the shortest link PD is between S_1 and D_1, i.e., $\tau_{1,1} = 0$. So $MN - 1$ parameters need to be considered for the channel space.
- **the scheduling space**: i.e., the possible scheduling parameters composed by $\{p_{i,j}\}$. Since the messages are repeatedly transmitted with a period of n, each cycle follows the same scheduling pattern. So one cycle will be enough to demonstrate the message scheduling. The offset from 0 to $n - 1$ represent the possible values of scheduled time-slot, i.e., $p_{i,j} \in \{0, 1, \cdots, n - 1\}$. Moreover, according to the circular transmission characteristics, an feasible scheduling scheme can be whole right-shifted without affecting the receiver. Thus, we can always set $p_{1,1} = 0$. Now there are also $MN - 1$ parameters to be determined for the scheduling space.

The searching ordering of the two spaces has different effects. If the channel space is searched first, we can obtain all potential feasible scheduling schemes for a specified channel instance. On the other hand, searching the scheduling space first will results in finding all possible feasible PD channel instances for it. Since we are often interested in studying the encoder for a given channel instance, the former will be chosen for demonstration in the following content.

Next, the required conditions (4)(5)(6) can be checked for each of the candidate point in the searching space. If any of the conditions is violated, the searching process just continues, otherwise the feasible candidate that satisfies all these conditions will be recorded. By enumerating all possible candidate points in the searching space, we can collect all feasible schemes of $\{\tau_{i,j}\}$ and $\{p_{i,j}\}$. In particular, (4) only involve the scheduling space, which can be checked at the corresponding positions to speed up the process.

Finally, we can output all feasible schemes that satisfy the above three conditions and do some statistical work. The most interesting questions include: 1) how many channel instances are feasible? 2) how many scheduling schemes are feasible for each feasible channel instance? In fact, We can

Algorithm 1 Proposed solution to find all feasible schemes

Input: M, N, $\{W_{i,j}\}$, n
Initialization: $\{\tau_{i,j}\}$, $\{p_{i,j}\}$, set $\tau_{1,1} = 0$, $p_{1,1} = 0$, $counter_1 = 0$
repeat
  Searching in the channel space
  if Explicit IA condition is violated **then**
    Continue
  else
    Set $counter_2 = 0$
    repeat
      Searching in the scheduling space
      if Condition (4) is violated **then**
        Continue
      else
        if Either (5) or (6) is violated **then**
          Continue
        else
          Record the current $\{\tau_{i,j}\}$ and $\{p_{i,j}\}$
          Add $counter_2$ by 1
          if $counter_2 == 1$ **then**
            Add $counter_1$ by 1
          end if
        end if
      end if
    until All candidates in the scheduling space have been checked
  end if
until All candidates in the channel space have been checked
Output: all recorded $\{\tau_{i,j}\}$ and $\{p_{i,j}\}$ and the corresponding counters

set two counters $counter_1$ and $counter_2$ in the code for these two questions. If at least one feasible scheduling scheme is found for a candidate channel instance, we increase $counter_1$ by 1. If one or more feasible scheduling schemes are found for the same channel instance, we increase $count_2$ by 1 for each of them. For a new channel instance, $counter_2$ will be reset to 0. All feasible $\{\tau_{i,j}\}$, $\{p_{i,j}\}$, and the corresponding counters are output to a file.

We should remark that if an explicit IA condition (which is only related with $\{\tau_{i,j}\}$) exists, we can check it at a proper position of searching phase in the channel space for speedup. The above procedure is summarized by Algorithm 1.

4. Results and Analysis

4.1 Results for Typical X Channels

4.1.1 2×2 X Channel

For this basic X channel, its DoF upper bound $4/3$ is achievable. So we have $n = 3$. The proposed algorithm outputs total 18 feasible channel instances and each of them has only one feasible scheduling scheme.

4.1.2 2×3 X Channel

This simple X channel has an achievable DoF upper bound $3/2$ by $n = 4$. There are total 384 feasible channel instances and each of them has only one feasible scheduling scheme.

4.1.3 2×4 X Channel

With $n = 5$, we find that 15000 feasible channel instances can sent 8 messages based on cyclic IA and achieve its DoF upper bound $8/5$.

4.1.4 2×5 X Channel

$N = 5$ will leads to a larger searching space. The DoF upper bound $5/3$ can be reached with $n = 6$. Total 933120 feasible channel instances are collected by our proposed solution.

4.1.5 3×3 X Channel

Now we show the result of 3×3 X channel, whose DoF upper bound is not achievable. According to [7], we can sent 10 messages with $n = 6$. By sending one extra message $W_{3,1}^{\text{ext}}$ and assuming that it is behind $W_{3,1}$ (i.e., $p_{3,1} < p_{3,1}^{\text{ext}} \le n-1$), the proposed algorithm shows that total 77760 channel instances are feasible. Unlike the above DoF optimal cases, there are 5180 channel instances that each has two different feasible scheduling schemes.

4.2 Computational Complexity

From the above results, we can see the number of feasible schemes increases significantly. On the other hand, the required time for getting them also increases fast. Here we provide a theoretical analysis.

Since exhaust enumeration method is used, the computational complexity is determined by the searching space. Both channel space and scheduling space have n^{MN-1} candidates. Thus the total complexity is $n^{2(MN-1)}$, which looks polynomial in n. However, $MN > n$ generally holds. For example, the $2\times N$ X channel has a complexity of $(N+1)^{2(2N-1)}$, which is in fact a $O(n^n)$ level. Thus if the dimension goes large, the solution becomes inefficient.

Even though, the proposed algorithm can be efficiently decoupled due to the independent relationship among $\{\tau_{i,j}\}$ and $\{p_{i,j}\}$. Thus parallel computation can be used to reduce the running time. For a given X channel with proper dimension, the searching time is finite and could be short enough at the cost of distributed resource.

5. Conclusions

A computer-aided solution was provided to find all feasible schemes of the cyclic IA for X channels. The proposed algorithm works well for small dimensions and can be implemented in parallel. The number of feasible schemes is reported for typical X channels.

Acknowledgments

This research was funded by the Innovation Program of Shanghai Municipal Education Commission of China under Grant 2021-01-07-00-10-E00121 and the Natural Science Foundation of Shanghai under Grant 20ZR1423200.

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