PAPER

Design and Analysis of Piecewise Nonlinear Oscillators with Circular-Type Limit Cycles

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SUMMARY This paper develops a design method and theoretical analysis for piecewise nonlinear oscillators that have desired circular limit cycles. Especially, the mathematical proof on existence, uniqueness, and stability of the limit cycle is shown for the piecewise nonlinear oscillator. In addition, the relationship between parameters in the oscillator and rotational directions and periods of the limit cycle trajectories is investigated. Then, some numerical simulations show that the piecewise nonlinear oscillator has a unique and stable limit cycle and the properties on rotational directions and periods hold.

key words: *nonlinear phenomena, limit cycles, piecewise nonlinear systems, nonlinear oscillators*

1. Introduction

A limit cycle is one of the most important phenomena in nonlinear systems along with chaos, fractals, and solitons, and has been actively studied in various research fields since old times. A limit cycle is defined as a 1-dimensional closed curve that attracts other trajectories in a phase space, and famous examples are as follows: stable gaits of humanoid robots [1] in robotics, periodic motions of machines [2] in mechanical engineering, oscillators [3] in electrical engineering, catalytic hypercycles [4] and the Belousov-Zhabotinsky reaction [5] in chemistry, circadian rhythms [6] and firefly flashing [7] in biology, boom-bust cycles [8] in economics. Systems that have stable limit cycles are called "oscillators," and various types of oscillators have been proposed: Duffing equation, Van del Pol equation, Lotka-Volterra equation, Hodgkin-Huxley equation, FitzHugh-Nagumo model, Kuramoto model, and so on [9]–[11]. Oscillators are applied for gait generation of robots as central pattern generators [12], [13], excitation of vibration for mechanical and electrical systems [14], and mathematical analysis on synchronization of multiple systems such as metronomes [15], fireflies [16], cardiocytes [17], and so on. In general, oscillators are represented by sets of nonlinear differential equations, and it is known that they have stable limit cycle when their internal parameters satisfy some conditions. However, shapes, rotational directions, and periods of limit cycles for these oscillators are fixed and cannot be designed. The authors' previous work [18]–[20] has developed "piecewise affine oscillators," which are represented by N-modal and 2-dimensional piecewise affine systems and generate desired polygonal-type limit cycles, and analyzed some mathematical properties for the oscillators. In the method, we can freely design shapes, rotational directions, and periods of limit cycles by tuning parameters of piecewise affine oscillators.

The main purpose of this paper is to derive a new type of piecewise nonlinear oscillators whose behaviors can be freely designed in order to extend shapes of limit cycles more than piecewise affine oscillators, and give mathematical analysis for the new oscillators. The contents of this paper are as follows. First, Sect. 2 gives a problem setting on circular-type closed curves and piecewise nonlinear systems. Next, Sect. 3 derives a piecewise nonlinear oscillator and proves existence, uniqueness, and stability of its limit cycle. In addition, some mathematical properties for the piecewise nonlinear oscillator are investigated. Then, some numerical simulations are illustrated in order to check the effectiveness of the proposed new oscillator in Sect. 4.

2. Problem Settings

This section shall give the problem settings of this study. Consider 2-dimensional Euclidean space $\mathbf{R}^2$ with the origin O, and denote its coordinate by $x = [x_1\ x_2]^\top \in \mathbf{R}^2$. We set N (≥ 2) points in $\mathbf{R}^2$: $P_i \neq O$ $(i = 1, \cdots, N)$, and denote the position vector of P_i by $p_i = [p_{i,1}\ p_{i,2}]^\top \in \mathbf{R}^2$. We also define the angle formed by the x^1-axis and the line segment $\overline{OP_i}$ as $\phi_i \in [0, 2\pi)$. Without loss of generality, it is assumed that the N points P_i $(i = 1, \cdots, N)$ are located while surrounding O in the counterclockwise rotation from the x_1-axis in a sequential order, that is to say, $0 \leq \phi_1 < \cdots < \phi_N < 2\pi$ holds. Next, we define the semi-infinite region D_i which is sandwiched by two half lines $\overline{OP_i}$ and $\overline{OP_{i+1}}$, where $P_{N+1} = P_1$. In addition, set a new point Q_i (its position vector: $q_i = [q_{i,1}\ q_{i,2}]^\top \in \mathbf{R}^2$) in D_i and denote an arc passing through three points P_i, Q_i, P_{i+1} by S_i. Now, we assume that $Q_i \in D_i$ is located in the outer region of $\overline{P_iP_{i+1}}$ (in the region such that O does not exist), and both of boundary lines of D_i are not tangent to S_i. *A circular-type closed curve* S is defined as a union of S_i:

$$S := \bigcup_{i=1}^{N} S_i. \tag{1}$$

Manuscript received September 29, 2022.
Manuscript revised February 10, 2023.
Manuscript publicized March 20, 2023.
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DOI: 10.1587/transfun.2022EAP1116

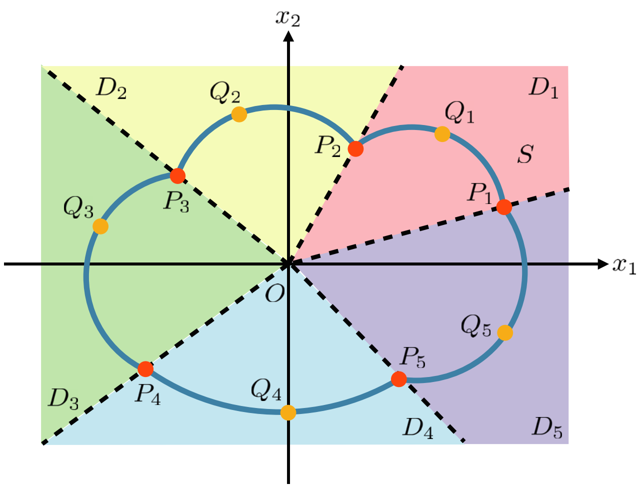

Fig. 1 Example of circular-type closed curve ($N = 5$).

An example for $N = 5$ is illustrated in Fig. 1.

Then, we consider the next nonlinear system:

$$\dot{x} = a_i + A_i x + f_i(x), \ \ x \in D_i, \tag{2}$$

where $a_i \in \mathbf{R}^2$, $A_i = \mathbf{R}^{2\times 2}$ are a vector of an nonlinear term and a matrix of a linear term, respectively, and $f_i(x) : \mathbf{R}^2 \to \mathbf{R}^2$ is a nonlinear function containing terms greater than or equal to squares of x. That is to say, we deal with the N-modal and 2-dimensional piecewise nonlinear system which is defined by N regions D_i $(i = 1, \cdots, N)$ and N nonlinear systems (2). Based on the above settings, a problem on limit cycles for (2) is stated as follows.

Problem 1: For the piecewise nonlinear system (2), design a_i, A_i, $f_i(x)$ $(i = 1, \cdots, N)$ such that a given circular-type closed curve S (1) is the unique and stable limit cycle for (2). ∎

Problem 1 above means that we design the piecewise nonlinear system (2) such that it has a unique and stable limit cycle, that is to say, it behaves like an oscillator. Hence, we call such a system *a piecewise nonlinear oscillator*.

3. Piecewise Nonlinear Oscillators

3.1 Design of Piecewise Nonlinear Oscillators

First, this subsection derives a piecewise nonlinear oscillator as a solution of Problem 1. In D_i, the equation of the circle going through three points P_i, Q_i, P_{i+1} can be represented by

$$x_1^2 + x_2^2 + l_i x_1 + m_i x_2 + n_i = 0, \tag{3}$$

where the coefficients l_i, m_i, n_i are calculated by

$$\begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} = -\begin{bmatrix} p_{i,1} & p_{i,2} & 1 \\ q_{i,1} & q_{i,2} & 1 \\ p_{i+1,1} & p_{i+1,2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} p_{i,1}^2 + p_{i,2}^2 \\ q_{i,1}^2 + q_{i,2}^2 \\ p_{i+1,1}^2 + p_{i+1,2}^2 \end{bmatrix}. \tag{4}$$

By using (3) (4), we now define *a limit cycle function* as

$$V_i(x) = x_1^2 + x_2^2 + l_i x_1 + m_i x_2 + n_i. \tag{5}$$

It must be noted that the regions such that $V(x) < 0$ and $V(x) > 0$ hold represents the inner and outer ones of S_i, respectively. We can see that if the value of $V(x)$ converges to 0 along a solution trajectory of the piecewise nonlinear oscillator (2), the solution trajectory also converges to S_i. Hence, we have to find a_i, A_i, $f_i(x)$ in (2) so that the above property is satisfied. We shall utilize a design method of systems such that a given function converges to 0, which is proposed in [21]. By applying the method to $V(x)$, we can obtain a system such that $V(x)$ converges to 0 along a solution trajectory of (2) as

$$\dot{x} = F_i(x) + G_i(x), \ \ x \in D_i,$$
$$\begin{cases} F_i(x) := \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \dfrac{\partial V_i}{\partial x}^{\mathsf{T}}, \\ G_i(x) := -V_i(x) \dfrac{\partial V_i}{\partial x}^{\mathsf{T}}, \end{cases} \tag{6}$$

where $\omega_i \neq 0$ are parameters to be set freely. Therefore, substituting (5) into (6) and summarizing terms, we find that a_i, A_i, $f_i(x)$ of (2) are given by

$$a_i = \begin{bmatrix} -\omega_i m_i - l_i n_i \\ \omega_i l_i - m_i n_i \end{bmatrix},$$
$$A_i = \begin{bmatrix} -l_i^2 - 2n_i & -2\omega_i - l_i m_i \\ 2\omega_i - l_i m_i & -m_i^2 - 2n_i \end{bmatrix},$$
$$f_i(x) = \begin{bmatrix} -3l_i x_1^2 - 2m_i x_1 x_2 - l_i x_2^2 - 2x_1^3 - 2x_1 x_2^2 \\ -m_i x_1^2 - 2l_i x_1 x_2 - 3m_i x_2^2 - 2x_1^2 x_2 - 2x_2^3 \end{bmatrix}. \tag{7}$$

Note that the nonlinear function $f_i(x)$ contains only terms on squares and cubes of x. However, it is only guaranteed that solution trajectories of the piecewise nonlinear oscillator (2), (7) from any initial states converge to S (1), and the existence of the unique and stable limit cycle has not been proven yet. The proof will be shown in the next subsection.

3.2 Analysis of Piecewise Nonlinear Oscillators

This subsection derives some mathematical properties of the piecewise nonlinear oscillator obtained in the previous subsection. First, we consider uniqueness and stability of the limit cycle. Clockwise and counterclockwise rotations of solution trajectories of the piecewise nonlinear oscillator are defined as follows (see also Fig. 2).

Definition 1: For the piecewise nonlinear oscillator (2), (7), if its solution trajectory from an initial state rotates in the clockwise/counterclockwise direction as viewed from the x_3-axis which forms the right-handed system with the x_1- and x_2-axes, it is called *in the clockwise/counterclockwise rotation*. ∎

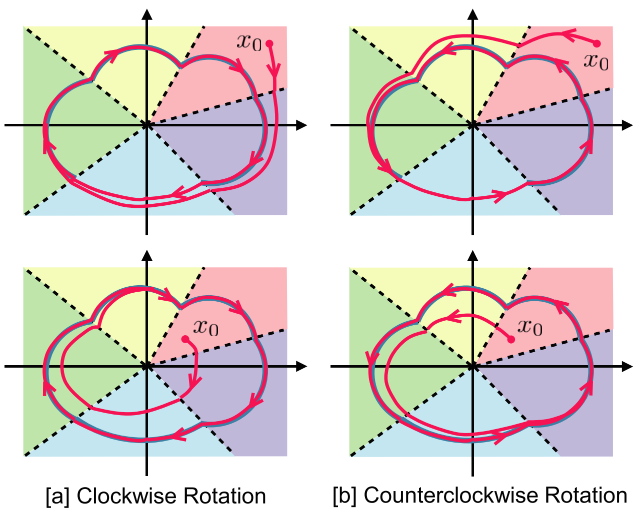

Fig. 2 Clockwise and counterclockwise rotations of solution trajectories of piecewise nonlinear oscillators.

By using parameters $\varepsilon_i^- < 0$, $\varepsilon_i^+ > 0\,(i = 1, \cdots, N)$, we define a subset in D_i as

$$M_i(\varepsilon_i) := \{\, x \in D_i \mid \varepsilon_i^- \le V_i \le \varepsilon_i^+ \,\}, \tag{8}$$

and a union of (8):

$$M(\varepsilon) := \bigcup_{i=1}^{N} M_i(\varepsilon_i), \tag{9}$$

where $\varepsilon_i := (\varepsilon_i^-, \varepsilon_i^+)$. It is assumed that parameters $\varepsilon_i^-, \varepsilon_i^+ (i = 1, \cdots, N)$ are determined such that two curves:

$$\begin{aligned} M^-(\varepsilon^-) &= \bigcup_{i=1}^{N} \{\, x \in \mathbf{R}^2 \mid V_i = \varepsilon_i^- \,\}, \\ M^+(\varepsilon^+) &= \bigcup_{i=1}^{N} \{\, x \in \mathbf{R}^2 \mid V_i = \varepsilon_i^+ \,\}, \end{aligned} \tag{10}$$

are closed (such an ε is said to be *admissible*), where $\varepsilon^- := (\varepsilon_1^-, \cdots, \varepsilon_N^-)$, $\varepsilon^+ := (\varepsilon_1^+, \cdots, \varepsilon_N^+)$, $\varepsilon := (\varepsilon^-, \varepsilon^+)$. If ε is admissible, $M(\varepsilon)$ is a bounded and closed set which surrounds S with a width as shown in Fig. 3. Now, the next lemma can be proven.

Lemma 1: For any admissible ε, $M(\varepsilon)$ is a positively invariant, bounded and closed set for the piecewise nonlinear oscillator (2), (7).

(Proof) Calculate time derivative of $V_i(x)$ along a solution trajectory of (2), (7), we have

$$\begin{aligned} \frac{d}{dt}(V_i^2) &= 2V_i\dot{V}_i = 2V_i\frac{\partial V_i}{\partial x}\dot{x} = 2V_i\frac{\partial V_i}{\partial x}(F_i + G_i) \\ &= 2V_i\frac{\partial V_i}{\partial x}\begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}\frac{\partial V_i}{\partial x}^{\mathsf{T}} - 2V_i^2\frac{\partial V_i}{\partial x}\frac{\partial V_i}{\partial x}^{\mathsf{T}} \\ &= -2V_i^2\frac{\partial V_i}{\partial x}\frac{\partial V_i}{\partial x}^{\mathsf{T}} < 0. \end{aligned}$$

Thus, the direction of the vector of (2), (7) is always inward of $M(\varepsilon)$. This fact shows that $M(\varepsilon)$ is a positively invariant, bounded and closed set. ∎

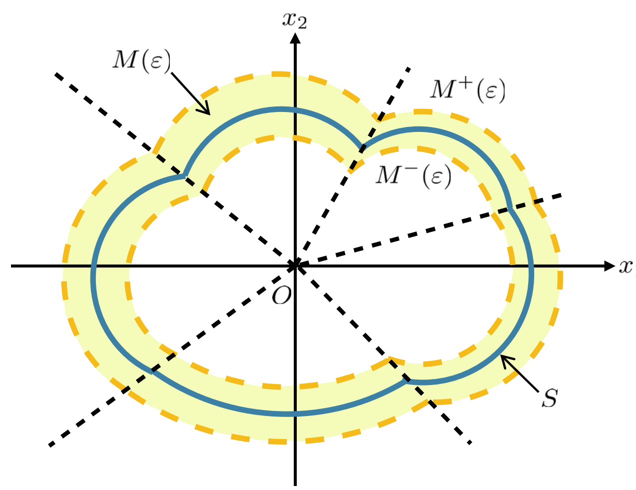

Fig. 3 $M(\varepsilon)$.

Next, we consider equilibrium points of the piecewise nonlinear oscillator (2), (7). The following lemma on equilibrium points of (2), (7) can be obtained.

Lemma 2: The equilibrium point of the piecewise nonlinear oscillator (2), (7) in D_i is only the central point of S_i:

$$x = \begin{bmatrix} -\dfrac{l_i}{2} \\ -\dfrac{m_i}{2} \end{bmatrix}. \tag{11}$$

(Proof) (2), (7) can be rewritten as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -\omega_i(2x_2 + m_i) \\ \omega_i(2x_1 + l_i) \end{bmatrix} \\ &\quad + \begin{bmatrix} -(2x_1 + l_i)(x_1^2 + x_2^2 + l_i x_1 + m_i x_2 + n_i) \\ -(2x_2 + m_i)(x_1^2 + x_2^2 + l_i x_1 + m_i x_2 + n_i) \end{bmatrix}. \end{aligned} \tag{12}$$

We first consider points not on S_i. In order to calculate equilibrium points of (2), (7), we set the right-hand side of (12) to 0, and divide the first equation by the second one:

$$\begin{aligned} &\frac{-\omega_i(2x_2 + m_i)}{\omega_i(2x_1 + l_i)} = \\ &\qquad \frac{(2x_1 + l_i)(x_1^2 + x_2^2 + l_i x_1 + m_i x_2 + n_i)}{(2x_2 + m_i)(x_1^2 + x_2^2 + l_i x_1 + m_i x_2 + n_i)}. \end{aligned} \tag{13}$$

So, we have $(2x_1 + l_i)^2 + (2x_2 + m_i)^2 = 0$ from (13), and thus the equilibrium point is given by (11). Next, consider points on S_i. Substituting (3) into (12) and calculating in the same way, we then obtain $\omega_i(2x_2 + m_i) = 0$, $\omega_i(2x_1 + l_i) = 0$. However, there exist no equilibrium point on S_i satisfying these two equations. Therefore, the only equilibrium point of (2), (7) is given by (11), and it is consistent with the central point of S_i from (3). ∎

Then, we consider an important property called “traver-

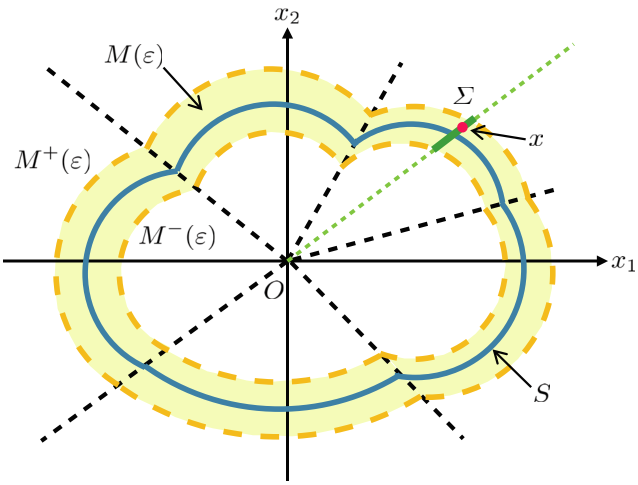

Fig. 4 Line segment Σ.

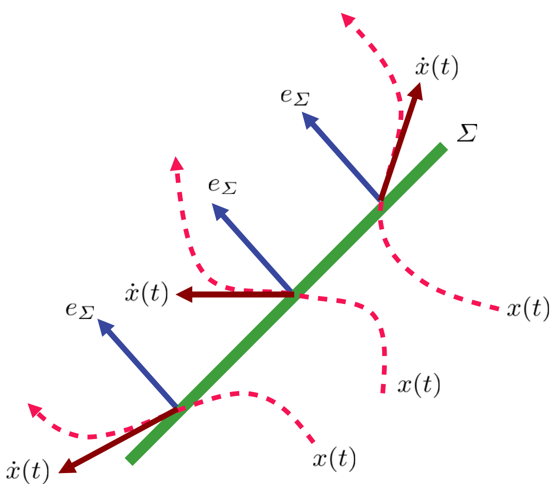

Fig. 5 Transversal line segment.

sal." for the piecewise nonlinear oscillator (2), (7). The concept of traversal is given by the following (see also Figs. 4 and 5).

Definition 2: Let Σ be a line segment in $M(\varepsilon)$. If the value of an inner product of the unit normal vector to Σ: e_Σ and the velocity vector of the piecewise nonlinear oscillator (2), (7) is not equal to 0, and its sign does not change at any point in Σ, then Σ is said to be *traversal* with respect to (2), (7). ∎

Now, a lemma on traversal for solution trajectories of the piecewise nonlinear oscillator (2), (7) can be derived.

Lemma 3: For the piecewise nonlinear oscillator (2), (7), assume that

$$\omega_i > 0, \ \forall i \in \{1, \cdots, N\} \tag{14}$$

or

$$\omega_i < 0, \ \forall i \in \{1, \cdots, N\} \tag{15}$$

holds. Then, there exists a traversal line segment $\Sigma \subset M(\varepsilon)$ such that $x \in \Sigma$ holds for any point $x \in M(\varepsilon)$ for any admissible ε, and it is satisfied that Σ infinitely intersects with any solution trajectory of (2), (7).

(Proof) We rewrite (12) into a polar coordinate expression. By using the polar coordinate (r, θ) based on the central point of S_i: (11), we set

$$x_1 = -\frac{l_i}{2} + r\cos\theta, \ \ x_2 = -\frac{m_i}{2} + r\sin\theta, \tag{16}$$

where r is the distance from (11) to x, and θ is the angle of x, which is measured from x_1-axis in the counterclockwise rotation. Then, (12) can be rewritten into the polar coordinate expression:

$$\begin{cases} \dot{r} = -2r\left(r^2 - \dfrac{l_i^2}{4} - \dfrac{m_i^2}{4} + n_i\right), & (17) \\ \dot{\theta} = 2\omega_i. & (18) \end{cases}$$

From (18), it is easily find that if (14) or (15) holds, a solution trajectory of (2), (7) always moves in the counterclockwise rotation or clockwise one with a constant velocity. Therefore, Σ is traversal and intersects with any solution trajectory of (2), (7) infinite times. ∎

Using Lemmas 1–3 shown above, we can prove the main theorem as follows.

Theorem 1: For the piecewise nonlinear oscillator (2), (7), assume that (14) or (15) holds and set an admissible ε such that $M(\varepsilon)$ does not contain the point (11). Then, the unique and stable limit cycle of (2), (7) in $M(\varepsilon)$ is equivalent to the circular-type closed curve S (1).

(Proof) According to [22], sufficient conditions on the existence of a limit cycle in $M(\varepsilon)$ are as follows; (i) $M(\varepsilon)$ is a positively invariant, bounded and closed set, (ii) there exists no equilibrium point inside and at the boundary of $M(\varepsilon)$, (iii) for any point $x \in M(\varepsilon)$, there exists a traversal line segment $x \in \Sigma \subset M(\varepsilon)$ such that a solution trajectory of (2), (7) infinitely intersects with Σ. The above conditions (i), (ii), and (iii) have been proven in Lemmas 1, 2, and 3, respectively. In addition, as the value of ε converges to 0, $M(\varepsilon)$ also converges to S. Consequently, it turns out that the unique and stable limit cycle of (2), (7) is S. ∎

Then, the following proposition on the relationship between rotational directions of solution trajectories and parameters is shown.

Proposition 1: For a solution trajectory of the piecewise nonlinear oscillator (2), (7) from an initial state, if (14) holds, it moves in the counterclockwise rotation, and conversely if (15) holds, it moves in the clockwise rotation.

(Proof) As shown in the proof of Lemma 3, if (14) / (15) holds, the angular velocity of θ is always positive/negative. Hence, a solution trajectory of (2), (7) from an initial state also moves in the counterclockwise/clockwise rotation. ∎

In addition, periods of solution trajectories for the piecewise nonlinear oscillator (2), (7) can be calculated by the next proposition.

Proposition 2: Denote the central angle of the arc S_i by ψ_i. When a solution trajectory of the piecewise nonlinear oscillator (2), (7) sufficiently converges to S, the period T in which it takes a lap around S is represented by

$$T = \sum_{i=1}^{N} \frac{\psi_i}{2|\omega_i|}. \tag{19}$$

(Proof) It can be confirmed that the magnitude of angler velocity for a solution trajectory of (2), (7) is equal to $2|\omega_i|$ from (18). So, the time in which it passes through S_i is given by $\psi_i/2|\omega_i|$, and hence we can obtain (19) by summing it for all the modes. ∎

From Propositions 1 and 2 shown above, it turns out that we can freely design behaviors such as rotational directions and periods for limit cycle trajectories of piecewise nonlinear oscillators (2), (7) by tuning the parameter ω_i $(i = 1, \cdots, N)$.

4. Simulations

This section performs some simulations in order to check the effectiveness of the proposed oscillator. Let us consider a circular-type closed curve S with $N = 4$ as shown in Fig. 6, and its data is given by $P_1 = (1, 0)$, $P_2 = (0, 2)$, $P_3 = (-3, 0)$, $P_4 = (0, -4)$, $Q_1 = (1, 2)$, $Q_2 = (-4, 3)$, $Q_3 = (-5, -2)$, $Q_4 = (2, -3)$.

First, as Simulation I, we set the parameter of the piecewise nonlinear oscillator (2), (7) as $\omega_i = 0.5$ $(i = 1, \cdots, 4)$, and the initial state is set as $x(0) = [5\ 4]^{\mathrm{T}}$, which is an exterior point of S. The simulation results are depicted in Figs. 7 and 8. Fig. 7 illustrates the solution trajectory of (2), (7) on the x_1x_2-space, and Fig. 8 shows the histories of x_1 and x_2. From these figures, it can be confirmed that the solution trajectory starting from the initial state converges to S, and then it behaves as a limit cycle trajectory on S in the counterclockwise rotation. The rotational direction (counterclockwise rotation) is coincident with Proposition 1 for the case where $\omega_i = 0.5 > 0$, $i \in \{1, \cdots, 4\}$ holds. In addition, the estimated period (19) in Proposition 2 is calculated as $T = 14.23$, and we can see that the value is equivalent to the simulation result from Fig. 8.

Next, as Simulation II, we set another value of the parameter for (2), (7) as $\omega_i = -1$ $(i = 1, \cdots, 4)$, and another initial state is set as $x(0) = [-2\ \ -3]^{\mathrm{T}}$, which is an interior point of S. The simulation results are shown in Figs. 9 and 10. From these figures, it turns out that the solution trajectory starting from the initial state converges to S, and then it behaves as a limit cycle trajectory on S in the clockwise rotation. The rotational direction (clockwise rotation) is coincident with Proposition 1 for the case where $\omega_i = -1 < 0$, $i \in \{1, \cdots, 4\}$ holds. Moreover, the estimated

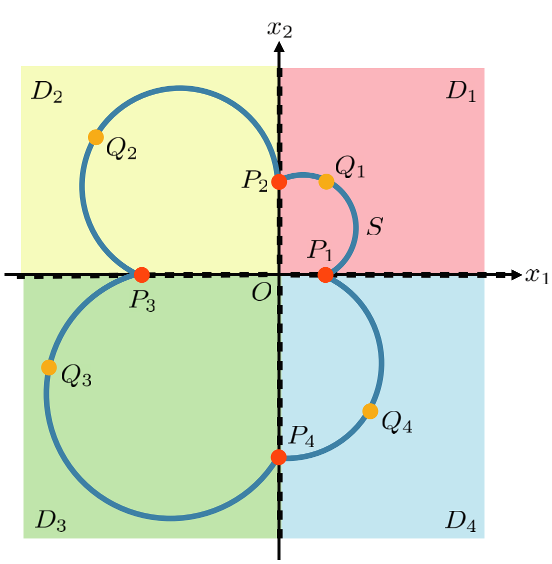

Fig. 6 Circular-type closed curve S in simulation.

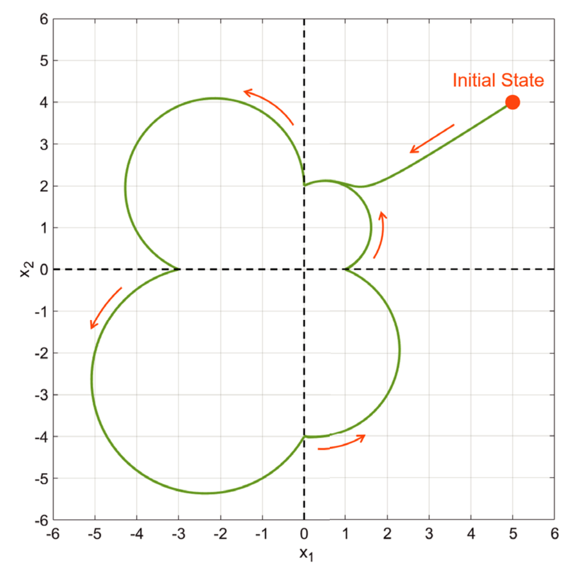

Fig. 7 Simulation I: Solution trajectory of piecewise nonlinear oscillator.

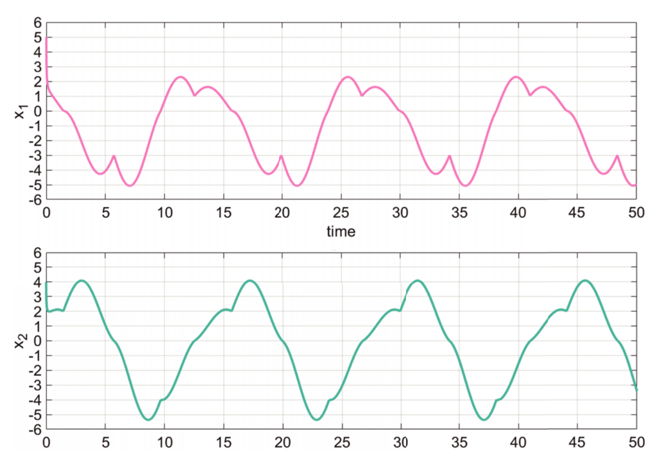

Fig. 8 Simulation I: Time history for x_1, x_2 of piecewise nonlinear oscillator.

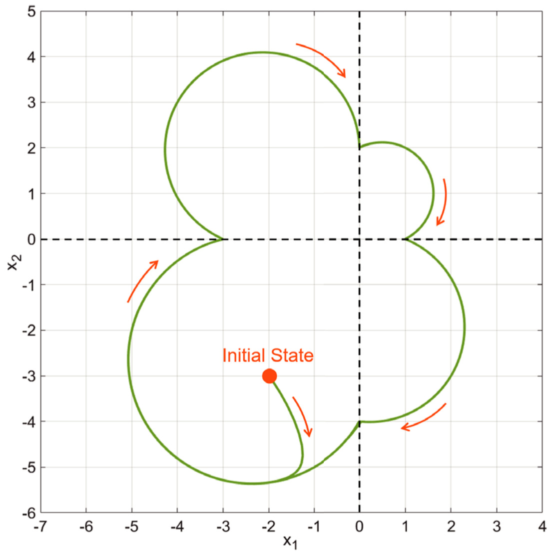

Fig. 9 Simulation result II: Solution trajectory of piecewise nonlinear oscillator.

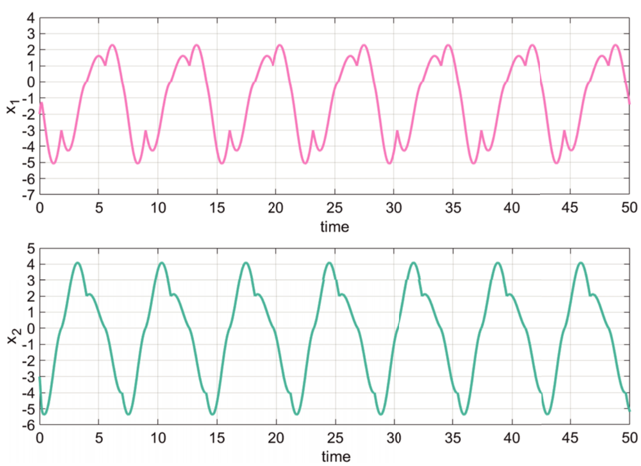

Fig. 10 Simulation result II: Time history for x_1, x_2 of piecewise nonlinear oscillator.

period (19) in Proposition 2 is calculated as $T = 7.11$, and we can see that the value is equivalent to the simulation result from Fig. 10.

Consequently, through the two types of simulations, we can see that the proposed oscillator can generate desired limit cycles trajectories, and hence it is effective.

5. Conclusions

This paper has developed a design method of piecewise nonlinear oscillators with desired circular-type limit cycles. The advantage of the proposed oscillators is that existence, uniqueness, and stability of the limit cycle is mathematically guaranteed, and the rotational direction and the period can be set by tuning a parameter in the system.

This paper has treated only a theoretical aspect of the piecewise nonlinear oscillators, however, various applications of the oscillator can be expected. First, we can consider an application to control theory for hybrid systems. For a given hybrid system, we transform it into a same form of the piecewise nonlinear oscillator by designing a state/output feedback law. Then, the closed-loop hybrid system has a unique and stable limit cycle, and a solution trajectory of the system behaves as a periodic motion. Some results for the case of the piecewise affine oscillators has been shown in [23], [24]. The second example is an application to vibrational excitation for mechanical systems. The aim is to excite periodic motions mechanical systems by determining specifications and physical parameters of the systems with the piecewise nonlinear oscillators. This method has the potential for power generation by wave power and wind power. Third, the piecewise nonlinear oscillators can be applied to robotics as "central pattern generators (CPG) [25]." The piecewise nonlinear oscillators installed at some joints of a robot are synchronized with suitable differences of phases, and then stable gaits for the robot are realized.

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