PAPER

Quantized Gradient Descent Algorithm for Distributed Nonconvex Optimization

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SUMMARY This paper presents a quantized gradient descent algorithm for distributed nonconvex optimization in multiagent systems that takes into account the bandwidth limitation of communication channels. Each agent encodes its estimation variable using a zoom-in parameter and sends the quantized intermediate variable to the neighboring agents. Then, each agent updates the estimation by decoding the received information. In this paper, we show that all agents achieve consensus and their estimated variables converge to a critical point in the optimization problem. A numerical example of a nonconvex logistic regression shows that there is a trade-off between the convergence rate of the estimation and the communication bandwidth.

key words: *multiagent system, distributed nonconvex optimization, cooperative control*

1. Introduction

Recently, distributed optimization in multiagent systems has attracted tremendous attention in various engineering fields such as machine learning and sensor networks [1]. In multiagent systems, devices called agents attempt to find a global solution by cooperatively communicating with each other [2], [3]. In convex optimization, subgradient algorithms on various types of network topologies have been proposed [4]–[11]. For nonconvex problems, distributed algorithms have also been explored in many research articles [12]–[18].

These studies implicitly assume that the communication bandwidth is sufficient for successful application of their algorithms. In practice, however, the information between agents is transmitted with limited bandwidths [19]–[22]. For convex optimization, Yuan et al. proposed a distributed dual averaging method with quantized communication [23]. Yi and Hong proposed an encoding-decoding scheme using the zoom-in technique on time-varying undirected graphs [24]. Pu et al. investigated a distributed proximal-gradient method for multiagent systems whose communication channel has a finite data rate [25]. Li et al. presented a distributed algorithm on time-varying directed graphs with quantized communication [26]. Kajiyama et al. proposed a distributed optimization method with quantized communication to achieve linear convergence [27]. In [28]–[31], distributed optimization algorithms with quantized and event-triggered communication was proposed. Although the optimization method with quantized communication has been extensively considered for the convex case, the essentially different approach is required for the analysis of the algorithms for nonconvex optimization. The authors of [32]–[34] considered distributed stochastic quantization algorithms with a sign coding function. The authors of [35] proposed a distributed nonconvex optimization algorithm with compressed communication.

The main research contribution of this paper is to investigate a distributed quantized algorithm for a smooth nonconvex problem in an undirected time-invariant graph. The proposed method uses an encode-decode scheme and a zoom-in technique [24]. Each agent encodes the real-valued estimation to the closest integer and decodes the received quantized information to estimate the variables of other agents. After the quantized communication, each agent updates its estimation by a distributed gradient descent algorithm. We show that the estimations of all the agents converge to a critical point in the nonconvex optimization problem. We also consider a parameter setting that guarantees consensus between agents and convergence to a critical point even when communication is performed at a lower quantization level. To handle the nonconvexity of the local cost functions and the quantization error of the local communication between agents, we utilize the descent property of the cost functions in [36] under the assumption of their smoothness property. The proposed algorithm can be implemented in a distributed manner without using a global communication or a worker-server architecture in [32]–[34]. The quantized algorithms for distributed nonconvex optimization have been considered in [35]. Compared with this method, the proposed method clarified the relation between the step-size parameter and the possible quantization level of quantizers. In particular, we show that the proposed algorithm can be implemented with three-level quantizers.

This paper is organized as follows: Section 2 introduces the problem setting and the distributed quantized algorithm. Section 3 presents the convergence analysis of the proposed algorithm. Section 4 shows a numerical example of a nonconvex logistic regression. Section 5 concludes this paper.

2. Problem Formulation

Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}$ be the sets of real numbers, non-negative integers, and integers, respectively. $\lceil a \rceil$ represents the smallest integer greater than or equal to $a \in \mathbb{R}$. For a vector $x \in \mathbb{R}^p$, x_q or $[x]_q$ shows the q-th element of x. The Euclidean

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norm and the infinity norm of a vector $x \in \mathbb{R}^p$ are given by $\|x\| = \sqrt{x^\top x}$ and $\|x\|_\infty = \max_{1\le q\le p} |x_q|$, respectively. For a matrix $A \in \mathbb{R}^{n\times n}$, a_{ij} or $[A]_{ij}$ represents the (i, j)-th element of A.

2.1 Nonconvex Optimization

We consider the following nonconvex optimization problem with n agents:

$$\underset{w\in\mathbb{R}^p}{\text{minimize}}\, F(w) = \sum_{i=1}^{n} f_i(w), \tag{1}$$

where $f_i : \mathbb{R}^p \to \mathbb{R}$ is a local differentiable objection function that is not necessarily convex ($i \in \mathcal{V} = \{1,2,\ldots,n\}$).

Assumption 1: There exists a positive constant C_g such that $\|\nabla f_i(x)\| \le C_g$ for all $i \in \mathcal{V}$ and $x \in \mathbb{R}^p$. Moreover, f_i is L-smooth for all $i \in \mathcal{V}$.

Each agent communicates over a time-invariant undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of edges that represents communication between agents. The neighborhood of an agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} \mid \{i, j\} \in \mathcal{E}\} \cup \{i\}$. The maximum number of neighboring agents is represented by $\hat{\mathcal{N}} = \max_{i\in\mathcal{V}} |\mathcal{N}_i|$, where $|\mathcal{N}_i|$ is the number of elements in $\mathcal{N}_i$.

Assumption 2: The graph $\mathcal{G}$ is connected.

We consider a weight matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ whose elements represent the weights on the communication link.

Assumption 3: There is a positive scalar $\theta \in (0, 1)$ such that

$$a_{ij} = \begin{cases} a_{ji}\ (> \theta) & \text{if } j \in \mathcal{N}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Assumption 4: The elements of the weight matrix A satisfy that $\sum_{j=1}^{n} a_{ij} = 1$ for all $i \in \mathcal{V}$ and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j \in \mathcal{V}$.

Assumptions 3 and 4 show that the weight matrix is doubly stochastic. In this paper, the maximum weight is defined as $\hat{a} = \max_{i,j\in\mathcal{V}} a_{ij}$.

2.2 Distributed Quantized Algorithm

In this subsection, we consider a distributed quantized subgradient algorithm for the nonconvex optimization problem (1). At iteration $k \in \mathbb{N}$, each agent i converts the real-valued estimation $w_i[k] \in \mathbb{R}^p$ to the quantized data $y_i[k] \in \mathbb{Z}^p$ by its encoder Q. The quantized value with the encoder $Q = [q([x]_1), q([x]_2), \ldots, q([x]_p)]^\top$ is given by

$$y_i[k] = Q\left(\frac{w_i[k] - w_i^Q[k-1]}{h[k]}\right) \tag{2}$$

with

$$q([x]_q) = \begin{cases} 0 & \text{if } -\frac{1}{2} < [x]_q \le \frac{1}{2}, \\ M & \text{if } M - \frac{1}{2} < [x]_q \le M + \frac{1}{2}, \\ K & \text{if } K - \frac{1}{2} < [x]_q, \\ -q(-[x]_q) & \text{if } [x]_q \le -\frac{1}{2}, \end{cases}$$

where $w_i^Q[k]$ is the internal variable of agent i, $h[k]$ is the zoom-in parameter, and $M = 1, 2, \ldots, K-1$. Then, agent i sends the encoded data $y_i[k]$ to the neighboring agents.

At the same time, agent i receives $y_j[k]$ from the neighboring agent $j \in \mathcal{N}_i$ and decodes it with the zoom-in parameter as follows:

$$w_j^Q[k] = h[k] y_j[k] + w_j^Q[k-1]. \tag{3}$$

Let $e_i[k]$ be the quantization error of agent i at time k, that is,

$$\begin{aligned} &e_i[k] \\ &= Q\left(\frac{w_i[k] - w_i^Q[k-1]}{h[k]}\right) - \frac{w_i[k] - w_i^Q[k-1]}{h[k]}. \end{aligned}$$

Then, we have

$$w_i^Q[k] = w_i[k] + h[k]e_i[k]. \tag{4}$$

We note that, if $\|(w_i[k] - w_i^Q[k-1])/h[k]\|_\infty \le K + 1/2$, the encoder does not cause saturation, that is, $\|e_i[k]\|_\infty \le 1/2$ holds.

After the quantized communication, agent i updates the estimation by

$$\begin{aligned} w_i[k+1] = w_i[k] - d\sum_{j=1}^{n} a_{ij}(w_i^Q[k] - w_j^Q[k]) \\ - \alpha[k]\nabla f_i(w_i[k]), \end{aligned} \tag{5}$$

where $d > 0$ is a gain parameter. We make the following assumption about the step size $\alpha[k]$ and the zoom-in parameter $h[k]$.

Assumption 5: The step-size satisfies $\lim_{k\to\infty} \alpha[k] = 0$, $\sum_{k=0}^{\infty} \alpha[k] = \infty$, and $\sum_{k=0}^{\infty} \alpha^2[k] < \infty$. The zoom-in parameter $h[k]$ satisfies $\lim_{k\to\infty} h[k] = 0$ and $\sum_{k=0}^{\infty} h^2[k] < \infty$.

3. Convergence Analysis

First, we introduce the preliminary lemmas describing the convergence property of nonnegative sequences [6].

Lemma 1: Let $\{\mu[k]\}$ be a positive scalar sequence. If $\lim_{k\to\infty} \mu[k] = 0$, then we have $\lim_{k\to\infty} \sum_{\ell=1}^{k} \lambda^{k-\ell}\mu[\ell] = 0$, where $0 < \lambda < 1$. Moreover, if $\sum_{k=1}^{\infty} \mu[k] < \infty$, then $\sum_{\ell=1}^{\infty}\sum_{r=1}^{\ell} \lambda^{\ell-r}\mu[r] < \infty$.

Lemma 2: Suppose that $\{X[k]\}, \{Y[k]\}$, and $\{Z[k]\}$ are the sequences of the nonnegative scalars. Suppose also that $Y[k+1] \le Y[k] - X[k] + Z[k]$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} Z[k] < \infty$. Then, $\{Y[k]\}$ converges to a finite value and $\sum_{k=1}^{\infty} X[k] <$

∞.

The next lemma shows that the estimations of the agents converge to their average.

Lemma 3: Under Assumptions 1–5, if agents update their estimations by (5) and $\|e_i[k]\|_\infty \le 1/2$ for all $i \in \mathcal{V}$ and $k \in \mathbb{N}$, then we have

$$\lim_{k\to\infty} \|w_i[k] - \bar{w}[k]\| = 0, \tag{6}$$

where $\bar{w}[k] = (1/n)\sum_{i=1}^n w_i[k]$.

Lemma 3 can be proven in the same way as Lemma 8 in [4], and the proof is omitted in this paper.

The next theorem presents the convergence of the estimation of each agent to a critical point.

Theorem 1: Under Assumptions 1–5, if $\|e_i[k]\|_\infty \le 1/2$ for all $i \in \mathcal{V}$ and $k \in \mathbb{N}$, then $\liminf_{k\to\infty} \left\|\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right\| = 0$. Moreover, if each f_i is twice differentiable and $\|\nabla^2 f_i(\xi) - \nabla^2 f_i(\zeta)\| \le \gamma\|\xi - \zeta\|$ for all $\xi, \zeta \in \mathbb{R}^p$, we have $\lim_{k\to\infty} \left\|\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right\| = 0$, where γ is a positive constant.

Proof: We consider the weight matrix $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ such that

$$b_{ij} = \begin{cases} da_{ij} & \text{if } i \ne j, \\ 1 - d\sum_{\ell\in\mathcal{V}\setminus\{i\}} a_{i\ell} & \text{if } i = j. \end{cases}$$

We note that the weight matrix B is also doubly stochastic. Therefore, under Assumptions 2–4, for all $i, j \in \mathcal{V}$ and $k, r \in \mathbb{N}$ with $k \ge r$, we have

$$\left\| \left[B^{k-r+1}\right]_{ij} - \frac{1}{n} \right\| \le C\beta^{k-r}, \tag{7}$$

where $\beta = 1 - \theta/(4n^2)$ and $C = 1/\beta$ [4].

From (4) and (5), we have

$$\begin{aligned} &w_i[k+1] \\ &= w_i[k] - d\sum_{j=1}^n a_{ij}(w_i[k] - w_j[k]) \\ &\quad + dh[k]\sum_{j=1}^n a_{ij}(e_j[k] - e_i[k]) - \alpha[k]\nabla f_i(w_i[k]) \\ &= \sum_{j=1}^n b_{ij} w_j[k] + dh[k]\sum_{j=1}^n a_{ij}(e_j[k] - e_i[k]) \\ &\quad - \alpha[k]\nabla f_i(w_i[k]). \end{aligned} \tag{8}$$

If $\|e_i[k]\|_\infty \le 1/2$, we have $\|e_i[k]\| \le \sqrt{p}\|e_i[k]\|_\infty \le \sqrt{p}/2$. Thus, from Assumptions 1, 4, and 5, we have

$$\|\varepsilon_i[k]\| \le \sqrt{p}dh[k] + \sqrt{p}C_g\alpha[k], \ \forall i \in \mathcal{V}, \ \forall k \in \mathbb{N}, \tag{9}$$

$$\lim_{k\to\infty} \varepsilon_i[k] = 0, \ \sum_{k=1}^\infty \|\varepsilon_i[k]\|^2 < \infty, \quad \forall i \in \mathcal{V}, \tag{10}$$

where $\varepsilon_i[k] = dh[k]\sum_{j=1}^n a_{ij}(e_j[k]-e_i[k]) - \alpha[k]\nabla f_i(w_i[k])$. From (8), we have $\bar{w}[k+1] = \bar{w}[k] + (1/n)\sum_{i=1}^n \varepsilon_i[k]$ for all $k \in \mathbb{N}$. Thus, from (7), we obtain

$$\begin{aligned} &\|w_i[k+1] - \bar{w}[k+1]\| \\ &\le C\beta^k \sum_{j=1}^n \left\|w_j[0]\right\| + nC\sum_{r=0}^{k-1} \beta^{k-r-1} \max_{i\in\mathcal{V}} \|\varepsilon_i[r]\| \\ &\quad + 2\max_{i\in\mathcal{V}} \|\varepsilon_i[k]\|. \end{aligned} \tag{11}$$

Since $\sum_{i=1}^n \sum_{j=1}^n a_{ij}(e_j[k] - e_i[k]) = 0$ from Assumption 2, we have $\sum_{i=1}^n \varepsilon_i[k] = -\alpha[k]\sum_{i=1}^n \nabla f_i(w_i[k])$. Then, from Assumption 1 and the descent lemma (Lemma 2.1 in [36]), we have

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n f_i(\bar{w}[k+1]) \\ &\le \frac{1}{n}\sum_{i=1}^n f_i(\bar{w}[k]) \\ &\quad + \left(\frac{1}{n}\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right)^\top \left(\frac{1}{n}\sum_{i=1}^n \varepsilon_i[k]\right) \\ &\quad + \frac{1}{2}L\left\|\frac{1}{n}\sum_{i=1}^n \varepsilon_i[k]\right\|^2 \\ &\le \frac{1}{n}\sum_{i=1}^n f_i(\bar{w}[k]) \\ &\quad - \left(\frac{1}{n}\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right)^\top \left(\frac{\alpha[k]}{n}\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right) \\ &\quad - \left(\frac{1}{n}\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right)^\top \\ &\quad \left(\frac{\alpha[k]}{n}\sum_{i=1}^n (\nabla f_i(w_i[k]) - \nabla f_i(\bar{w}[k]))\right) + S[k], \end{aligned}$$

where $S[k] = c_1\alpha^2[k] + c_2 h^2[k]$ with positive constants c_1 and c_2. From Assumption 5, we have $\sum_{k=1}^\infty S[k] < \infty$. Then, from Assumption 1, we have

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n f_i(\bar{w}[k+1]) \\ &\le \frac{1}{n}\sum_{i=1}^n f_i(\bar{w}[k]) - \frac{\alpha[k]}{n^2}\left\|\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right\|^2 \\ &\quad + \frac{\alpha[k]LC_g}{n}\sum_{i=1}^n \|w_i[k] - \bar{w}[k]\| + S[k]. \end{aligned}$$

From Lemmas 1 and 2, and (11), $(1/n)\sum_{i=1}^n f_i(\bar{w}[k])$ converges to a finite value and

$$\sum_{k=1}^\infty \alpha[k]\left\|\sum_{i=1}^n \nabla f_i(\bar{w}[k])\right\| < \infty. \tag{12}$$

Then, it follows from Assumption 5 that

$$\liminf_{k\to\infty}\left\|\sum_{i=1}^{n}\nabla f_i(\bar{w}[k])\right\| = 0. \tag{13}$$

Moreover, from the assumption about the Lipschitz continuity of the Hessian of f_i, for $k \geq 2$, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\bar{w}[k+1]) \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\bar{w}[k]) \\
&\quad + \left(\frac{1}{n}\sum_{i=1}^{n}\nabla^2 f_i(\bar{w}[k])\right)\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[k]\right) \\
&\quad + \frac{1}{2}\gamma\left\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[k]\right\|^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\bar{w}[k]) + \frac{\alpha[k]L}{n}\left\|\sum_{i=1}^{n}\nabla f_i(w_i[k])\right\| \\
&\quad + \frac{1}{2}\gamma\left\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[k]\right\|^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\bar{w}[k]) \\
&\quad + \frac{\alpha[k]L}{n}\sum_{i=1}^{n}\|\nabla f_i(w_i[k]) - \nabla f_i(\bar{w}[k])\| \\
&\quad + \frac{\alpha[k]L}{n}\sum_{i=1}^{n}\|\nabla f_i(\bar{w}[k])\| + \frac{1}{2}\gamma\left\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[k]\right\|^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\bar{w}[k]) \\
&\quad + \frac{\alpha[k]L^2}{n}\sum_{i=1}^{n}\left\{C\beta^{k-1}\sum_{j=1}^{n}\|w_j[0]\|\right. \\
&\quad \left. + nC\sum_{r=0}^{k-2}\beta^{k-r-2}\max_{i\in\mathcal{V}}\|\varepsilon_i[r]\| + 2\max_{i\in\mathcal{V}}\|\varepsilon_i[k-1]\|\right\} \\
&\quad + \frac{\alpha[k]L}{n}\sum_{i=1}^{n}\|\nabla f_i(\bar{w}[k])\| + \frac{1}{2}\gamma\left\|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i[k]\right\|^2,
\end{aligned}$$

where the last inequality follows from Assumptions 1 and (11). Therefore, it follows from (12) and (13) and Lemmas 1 and 2 that $\lim_{k\to\infty}\left\|\sum_{i=1}^{n}\nabla f_i(\bar{w}[k])\right\| = 0$. □

In Theorem 1, it is assumed that the quantizer does not cause saturation. The next lemma shows a sufficient condition for avoiding saturation of the quantizer. To this end, we introduce a lemma that shows the boundedness of the sum of a series.

Lemma 4: For $G[k] = \sum_{r=0}^{k-2}\beta^{k-r-2}\left(\frac{k+2}{r+1}\right)^{\delta}$ with $\delta > 0$ and $k \geq 2$, we have

$$G[k] \leq \left\{\beta\left(\frac{5}{4}\right)^{\delta}\right\}^{k-2}\left(\frac{6^{\delta} - 20^{\delta}\beta}{4^{\delta} - 5^{\delta}\beta}\right) + \frac{10^{\delta}}{4^{\delta} - 5^{\delta}\beta}.$$

Lemma 4 can be proven in the same way as Lemma A.3 in [24], and the proof is omitted in this paper.

Theorem 2: Suppose that the zoom-in parameter $h[k]$ and the step-size $\alpha[k]$ are given by $h[k] = \frac{H}{(k+1)^{\delta_h}}$ and $\alpha[k] = \frac{A}{(k+1)^{\delta_\alpha}}$ for $k \in \mathbb{N}$, where H and A are positive constants, $1/2 < \delta_h \leq \delta_\alpha < 1$, and $\beta(5/4)^{\delta_h} \leq 1$. Under Assumptions 1–5 with $w_i[0] = 0$ for all $i \in \mathcal{V}$, if $K \geq \Omega$, we have $\|e_i[k]\|_\infty \leq 1/2$ for all $i \in \mathcal{V}$ and $k \in \mathbb{N}$, where

$$\begin{aligned}
\Omega = \max\left\{\left(d + \frac{1}{2} + \frac{AC_g}{H}\right)2^{\delta_h} - \frac{1}{2},\right.& \\
\frac{4\cdot 3^{\delta_h}\sqrt{p}d\hat{a}\hat{N}}{H}\left(AC_g + dH\right)& \\
+\left(d + \frac{1}{2} + \frac{AC_g}{H}\right)\left(\frac{3}{2}\right)^{\delta_h} - \frac{1}{2},& \\
\frac{2\sqrt{p}\left(4nC+2\right)d\hat{a}\hat{N}}{H}\left(AC_g + dH\right)& \\
\left.+\left(d + \frac{1}{2} + \frac{AC_g}{H}\right)\left(\frac{4}{3}\right)^{\delta_h} - \frac{1}{2}\right\}.&
\end{aligned} \tag{14}$$

Proof : We provide the proof by a mathematical induction on k. For $k = 1$, we have

$$\begin{aligned}
&\left\|\frac{w_i[1] - w_i^Q[0]}{h[1]}\right\|_\infty \\
&\leq \frac{2d\hat{a}\hat{N}}{h[1]}\max_{i\in\mathcal{V}}\{w_i[0] - \bar{w}[0]\} \\
&\quad + \left(d + \frac{1}{2}\right)\frac{h[0]}{h[1]} + \frac{\alpha[0]}{h[1]}\nabla f_i(w_i[0]) \\
&\leq \left(d + \frac{1}{2} + \frac{AC_g}{H}\right)2^{\delta_h}.
\end{aligned}$$

Similarly, for $k = 2$, we have

$$\begin{aligned}
&\left\|\frac{w_i[2] - w_i^Q[1]}{h[2]}\right\|_\infty \\
&\leq \frac{4\cdot 3^{\delta_h}\sqrt{p}d\hat{a}\hat{N}}{H}\left(AC_g + dH\right) \\
&\quad + \left(d + \frac{1}{2} + \frac{AC_g}{H}\right)\left(\frac{3}{2}\right)^{\delta_h}.
\end{aligned}$$

Thus, the statement holds for $k = 1, 2$.

Next, we assume that $\|e_i[s]\|_\infty \leq 1/2$ for $s \geq 2$. Then, we obtain

$$\begin{aligned}
&\left\|\frac{w_i[s+1] - w_i^Q[s]}{h[s+1]}\right\|_\infty \\
&= \left\|\frac{w_i[s+1] - w_i[s] - h[s]e_i[s]}{h[s+1]}\right\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\leq \left\| \frac{d \sum_{j=1}^{n} a_{ij}(w_i[s] - w_j[s])}{h[s+1]} \right\|_\infty + \left\| \frac{h[s]e_i[s]}{h[s+1]} \right\|_\infty \\
&\quad + \left\| \frac{\alpha[s]\nabla f_i(w_i[s])}{h[s+1]} \right\|_\infty \\
&\quad + \left\| \frac{dh[s] \sum_{j=1}^{n} a_{ij}(e_j[s] - e_i[s])}{h[s+1]} \right\|_\infty \\
&\leq \frac{2d\hat{a}\hat{\mathcal{N}}}{h[s+1]} \max_{i\in\mathcal{V}} \|w_i[s] - \bar{w}[s]\| \\
&\quad + \left(d + \frac{1}{2}\right) \frac{h[s]}{h[s+1]} + \frac{\alpha[s]C_g}{h[s+1]} \\
&\leq \frac{2d\hat{a}\hat{\mathcal{N}}}{h[s+1]} \left\{ nC \sum_{r=0}^{s-2} \beta^{s-r-2} (\sqrt{p}(C_g\alpha[r] + dh[r])) \right. \\
&\quad \left. +2\left(\sqrt{p}(C_g\alpha[s] + dh[s])\right) \right\} \\
&\quad + \left(d + \frac{1}{2} + \frac{AC_g}{H}\right) \left(\frac{4}{3}\right)^{\delta_h},
\end{aligned} \tag{15}$$

where the last inequality follows from (9) and (11).

From Lemma 4, the first term of the right-hand side of (15) is given by

$$\begin{aligned}
&\frac{\sum_{r=0}^{s-2} \beta^{s-r-2}(\alpha[r]C_g + dh[r])}{h[s+1]} \\
&\leq \frac{\sum_{r=0}^{s-2} \beta^{s-r-2}(AC_g + dH)}{H} \left(\frac{s+2}{r+1}\right)^{\delta_h} \\
&\leq \frac{AC_g + dH}{H} \left[\left\{ \beta \left(\frac{5}{4}\right)^{\delta_h} \right\}^{s-2} \left(\frac{6^{\delta_h} - 20^{\delta_h}\beta}{4^{\delta_h} - 5^{\delta_h}\beta} \right) \right. \\
&\quad \left. + \frac{10^{\delta_h}}{4^{\delta_h} - 5^{\delta_h}\beta} \right] \\
&\leq 4\frac{AC_g + dH}{H}.
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{\alpha[s]C_g + dh[s]}{h[s+1]} &= \frac{(s+1)^{\delta_h}}{H} \left\{ \frac{AC_g}{(s+1)^{\delta_\alpha}} + \frac{dH}{(s+1)^{\delta_h}} \right\} \\
&\leq \frac{AC_g + dH}{H}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left\| \frac{w_i[s+1] - w_i^Q[s]}{h[s+1]} \right\|_\infty \\
&\leq \frac{2\sqrt{p}\,(4nC + 2)\, d\hat{a}\hat{\mathcal{N}}}{H} (AC_g + dH) \\
&\quad + \left(d + \frac{1}{2} + \frac{AC_g}{H}\right) \left(\frac{4}{3}\right)^{\delta_h}.
\end{aligned}$$

This concludes the proof. □

Theorem 2 shows the relation between the step-size parameter A and the quantization level K. The smaller quantization level can be achieved for the smaller value of A. However, the smaller value of A results in the slower convergence. Therefore, there is a trade-off between the step-size parameter A and the quantization level K. Now, we consider the minimum quantization level. From (14), we have

$$\begin{aligned}
\lim_{d\to+0} \Omega &= \max \left\{ \left(\frac{1}{2} + \frac{AC_g}{H}\right) 2^{\delta_h} - \frac{1}{2}, \right. \\
&\qquad \left(\frac{1}{2} + \frac{AC_g}{H}\right) \left(\frac{3}{2}\right)^{\delta_h} - \frac{1}{2}, \\
&\qquad \left. \left(\frac{1}{2} + \frac{AC_g}{H}\right) \left(\frac{4}{3}\right)^{\delta_h} - \frac{1}{2} \right\} \\
&= \left(\frac{1}{2} + \frac{AC_g}{H}\right) 2^{\delta_h} - \frac{1}{2}.
\end{aligned}$$

It follows that, for sufficiently small d and A, $K = 1 \geq \Omega$ holds. Thus, with the appropriate parameter settings, agents can find a critical point of the nonconvex optimization problem by sending only the three integers -1, 0, and 1 for each element of the estimation value. In this case, the required communication bandwidth can be reduced to $\lceil p \log_2 3 \rceil$ bits.

Remark 1: In [14], [37], the authors established the global convergence of distributed optimization algorithms under the Polyak-Łojasiewicz condition, which guarantees that all critical points are global optimizers [38]. To show the global convergence of the proposed algorithm is a future direction of this paper.

Remark 2: The authors in [39], [40] have exploited the design of the optimal encoder for the stability of linear and nonlinear systems. The investigation of the design of the optimal quantizer is also a future direction.

4. Numerical Example

This section presents a numerical example of the proposed quantized algorithm over the multiagent system with four agents ($n = 4$). We consider a nonconvex logistic regression [14] for binary classification to divide the dataset into two classes through a one-layer neural network. Each agent has different partial data of the a9a dataset, which is a binary dataset with 32,561 observations and 123 features [41]. In this problem, the local objective function is given by

$$f_i(w) = \frac{1}{p} \sum_{q=1}^{p} \log\left(1 + e^{-([w]_q[\mu_i]_q)[\nu_i]_q}\right) + R(w),$$

where $p = 124$, μ_i is the feature vector, ν_i is the corresponding binary label, and $R(w) = \sum_{q=1}^{p} \frac{r_i[w]_q^2}{1+[w]_q^2}$ is a nonconvex regularizer with $r_i = 10^{-4}$.

We compare the convergence performance with different quantization levels $K = 1, 2, 3$. In all cases, we set $d = 1$, $h[k] = 1/(k+1)^{0.55}$, and $w_i[0] = 0$ for all $i \in \mathcal{V}$. The step-size is given by $\alpha[k] = 0.3/(k+1)^{0.55}$ for $K = 1$, $\alpha[k] = 1.5/(k+1)^{0.55}$ for $K = 2$, and $\alpha[k] = 2.5/(k+1)^{0.55}$

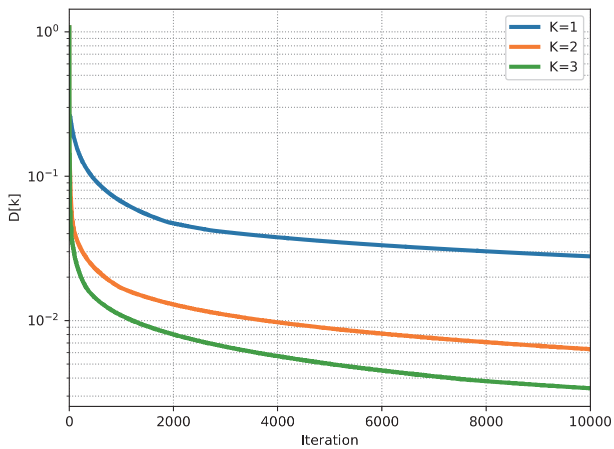

Fig. 1 $D[k]$ with different quantized levels.

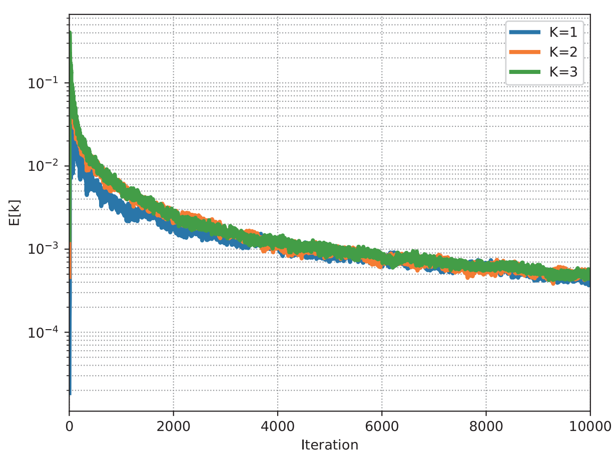

Fig. 2 $E[k]$ with different quantized levels.

for $K = 3$. The parameters of the step-sizes are set to satisfy the conditions of Theorem 2. In this example, we use the following two values to evaluate the convergence performance to a critical point and the degree of the consensus:

$$D[k] = \left\| \sum_{i=1}^{n} \nabla f_i(w_i[k]) \right\|_\infty ,$$

$$E[k] = \frac{1}{n} \sum_{i=1}^{n} \|w_i[k] - \bar{w}[k]\|^2.$$

Figure 1 shows the sum of the gradients of the local objective functions $D[k]$. We see that the gradient of the local objective function of each agent converges to 0 for all cases. The convergence rate, however, depends on the value of the quantization level K. A larger quantization level allows agents to convey more information. Thus, the convergence rate is better for a larger quantization level. Figure 2 shows the consensus error $E[k]$. It can be observed from Fig. 2 that the estimations of the agents achieve consensus. Moreover, there is no saturation in all cases. From these results, we see that the estimations of the agents converge to a critical point.

Finally, we compare the proposed subgradient-based algorithm with the compressed communication algorithm (Algorithm 1 in [35]). Figure 3 shows the sum of the gradients $D[k]$ for the subgradient algorithm without quantization, the proposed algorithm with the different values of the quantization level K, and the compressed communication algorithm (CCA) with the different values of the step-size η. From this figure, we see that the compressed communication algorithm can achieve faster convergence. However, for an appropriate quantization level, the proposed algorithm has a similar convergence performance with the compressed communication algorithm.

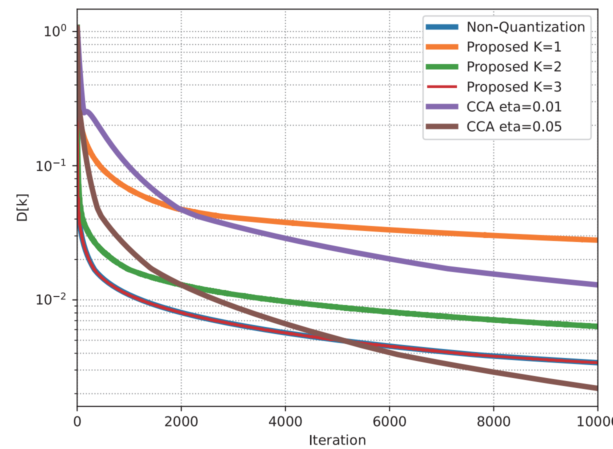

Fig. 3 Comparison with the compressed communication algorithm in [35].

5. Conclusions

In this paper, we presented a quantization scheme for nonconvex optimization on multiagent networks. We proposed a distributed gradient descent algorithm by which the estimation of every agent reaches consensus and the sum of the gradients of the local objective functions converges to 0. We also considered a sufficient condition for avoiding the saturation of the quantizer and showed that the quantization level can be set as $K = 1$ by appropriately setting the step-size and the zoom-in parameters. A numerical example of the application to a nonconvex logistic regression showed the validity of the proposed method. The communication network of the proposed method was assumed to be represented by an undirected graph. An extension to a more general network topology is one of our future research directions.

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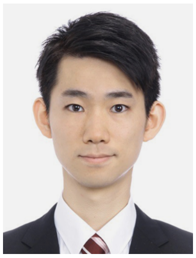

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