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Logic Functions of Polyphase Complementary Sets

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SUMMARY A set consisting of K subsets of M sequences of length L is called a complementary sequence set expressed by $A(L, K, M)$, if the sum of the out-of-phase aperiodic autocorrelation functions of the sequences within a subset and the sum of the cross-correlation functions between the corresponding sequences in any two subsets are zero at any phase shift. Suehiro et al. first proposed complementary set $A(N^n, N, N)$ where N and n are positive integers greater than or equal to 2. Recently, several complementary sets related to Suehiro's construction, such as N being a power of a prime number, have been proposed. However, there is no discussion about their inclusion relation and properties of sequences. This paper rigorously formulates and investigates the (generalized) logic functions of the complementary sets by Suehiro et al. in order to understand its construction method and the properties of sequences. As a result, it is shown that there exists a case where the logic function is bent when n is even. This means that each series can be guaranteed to have pseudo-random properties to some extent. In other words, it means that the complementary set can be successfully applied to communication on fluctuating channels. The logic functions also allow simplification of sequence generators and their matched filters.

key words: *complementary sequence, aperiodic correlation function, sequence design, logic function, bent function*

1. Introduction

Golay proposed a complementary pair, which is a pair of bi-phase sequences with orthogonality such that the sum of the aperiodic autocorrelation functions of each sequence at the same phase shifts is zero except for the zero-phase shift [1]. There have been discussions on their extensions and related sequences [2]–[6]. The concept was extended to the complementary set of two binary complementary pairs, such that the sum of the aperiodic crosscorrelation functions between the corresponding sequences in the pairs is zero for any shift [7], and also to polyphase ones [8].

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Furthermore these were generalized by Suehiro et al. on the concept of the complete complementary codes, expressed as $A(L, K, M)$, where L denotes sequence length, K the number of subsets, M the number of sequences in each subset, and they gave a systematic construction method of $A(N^n, N, N)$ with $N \geq 2$ and $n \geq 2$ [9], [10]. In this paper, the construction method will be called Suehiro's method. The complementary set can be given by multiplication or interleaving with each row or each element in complex Hadamard matrices including binary Hadamard matrices and the Discrete Fourier Transform (DFT) matrices, in which each row has orthogonal elements with an absolute value of one [11].

In general, many kinds of sequence sets with orthogonal properties are basically constructed using complex Hadamard matrices, so the construction methods of complementary sets, such as ZCZ codes [12]–[14] and the Reed-Muller code [15], [16], are closely related. In particular, the construction methods of complementary sequence sets were discussed by using generalized Boolean functions, which can uniquely express the generated sequences, so that the difference between the sequence sets can be clarified and their characteristics and relevance to ZCZ codes and Reed-Muller codes were clarified [17], [19]. Previously, the authors also derived the logic functions of some ZCZ codes and showed their differences [20].

It was also shown that when $A(L_0, K_0, M_0)$ and $A(L_1, K_1, M_1)$ are complementary sets, a new complementary set $A(L_0L_1, K_0K_1, M_0M_1)$ is given by using the Kronecker Product, and the application of the DFT matrices of order $L_0 = L_1$ including the binary Hadamard matrix of order 2 can generate $A(N, N, N)$ with $N = L_0^2$ and generally $K = L$ is suitable for any application [21]. Recently complementary sequence sets $A(L, N, N)$ by applying the above extension method [22] were discussed, and several methods for $L = N^n$ have been compared, where N is mostly primes including 2. Not limited to this discussion, differences from Suehiro's method, inclusion relationships and the properties of sequence sets have not been discussed. Deriving the generalized logic function of the target sequence set not only clarifies the difference from the traditional construction method [20], but also determines the number of sequences that can be generated and the compactness of its generator and matched filter bank [24].

In this paper, based on the above, the logic function of the complementary sets including biphase and quadriphase ones, which are given by Suehiro's method, is derived and carefully observed. We previously discussed basics complementary sets by Suehiro's method. [23]. The logic function is the same as the generalized Boolean function except that it allows mapping to the real numbers modulo integer q, since even if the logic function includes real-valued functions and irrational numbers, it can be handled well [25]–[28] This discussion will be closely related to references [17]–[19].

In Sect. 2, the basic matters required in this paper such as vectors and the (complex) Hadamard matrix are explained, and complementary sets are defined. In Sect. 3, it is shown that the sequence elements in the complementary set can be represented by the product of the elements of the Hadamard matrices. In Sect. 4, the logic function from the above representation in Sect. 3 and characteristic examples are described concretely. In Sect. 5, it is investigated whether the logic functions of the above complementary sets given by the Kronecker product [21], [22] and Suehiro's method is bent, which can generate pseudo-random sequences with good properties Finally, the results of the discussion are summarized.

2. Basic Matters

This section describes the basics of complex Hadamard matrices including the Discrete Fourier transform (DFT) matrix and defines complementary sets.

2.1 Complex Hadamard Matrices

Let $\boldsymbol{x}$ be an m-dimensional vector consisting of elements x_i modulo q defined by

$$\boldsymbol{x} = (x_0, x_1, \cdots, x_{m-1}) \in V_q^m \tag{1}$$

whose elements are the coefficients of q-ary expansion of an integer $x (0 \le x \le q^m - 1)$ expressed by

$$x = x_0 q^0 + x_1 q^1 + \cdots + x_{m-1} q^{m-1}. \tag{2}$$

Let

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= \boldsymbol{x}\boldsymbol{y}^t \\ &= x_0 y_0 + x_1 y_1 + \cdots + x_{m-1} y_{m-1} \pmod{q}, \end{aligned} \tag{3}$$

where t is the transpose of a vector. The complex Sylvester-type Hadamard matrix of order $N = q^m$ is given as

$$H = [h_{y,x}]_{0 \le x, y < N} = [\omega_q^{\boldsymbol{y} \cdot \boldsymbol{x}}]_{\boldsymbol{y}, \boldsymbol{x} \in V_q^m}, \tag{4}$$

where $\omega_q = \exp(2\pi\sqrt{-1}/q)$. It satisfy

$$HH^* = H^*H = NI, \tag{5}$$

since

$$\begin{aligned} &\sum_{\boldsymbol{x} \in V_q^m} \omega_q^{\boldsymbol{y} \cdot \boldsymbol{x} - \boldsymbol{y}' \cdot \boldsymbol{x}} = \sum_{\boldsymbol{x} \in V_q^m} \omega_q^{(\boldsymbol{y} - \boldsymbol{y}')\boldsymbol{x}^t} \\ &= \prod_{i=0}^{m-1} [\sum_{x_i \in V_q} \omega_q^{(y_i - y_i')x_i}] = \begin{cases} N & (\boldsymbol{y} = \boldsymbol{y}'), \\ 0 & (\boldsymbol{y} \ne \boldsymbol{y}'), \end{cases} \end{aligned} \tag{6}$$

where I denotes the unit matrix of order N, and $*$ the complex conjugate transpose of a matrix. If $m = 1$, that is, $x = \boldsymbol{x} = x_0$ and $y = \boldsymbol{y} = y_0$, Eq. (4) denotes the well-known DFT (Discrete Fourier Transform) matrix of order $q = N$.

In general, an Hadamard matrix can be given by swapping any rows or columns of the original Hadamard matrix or multiplying any rows or columns by an element with a magnitude of 1 (unit magnitude). Hereinafter, in order to distinguish it from the Sylvester-type Hadamard matrix H, an arbitrary Hadamard matrix of order N will be expressed as

$$B^s = [b_{y,x}^s]_{0 \le x, y < N}, \tag{7}$$

where $s (\ge 0)$ indicates an identifier for any Hadamard matrix, and $B^s (B^s)^* = (B^s)^* B^s = NI$.

2.2 Complementary Sets

Let A be a set of K subsets A_z of M polyphase sequences $\boldsymbol{a}_{z,y}$ of length L defined by

$$\left.\begin{aligned} A &= \{A_0, \cdots, A_z, \cdots, A_{K-1}\} \\ A_z &= \{\boldsymbol{a}_{z,0}, \cdots, \boldsymbol{a}_{z,y}, \cdots, \boldsymbol{a}_{z,M-1}\} \\ \boldsymbol{a}_{z,y} &= (a_{z,y,0}, \cdots, a_{z,y,x}, \cdots, a_{z,y,L-1}) \end{aligned}\right\}, \tag{8}$$

where $a_{z,y,x}$ denotes an complex element with a unit magnitude where $|a_{z,y,x}| = 1$. The aperiodic correlation function between $\boldsymbol{a}_{z,y}$ and $\boldsymbol{a}_{z',y}$ is defined by

$$R_{\boldsymbol{a}}(z, z', y, \tau) = \begin{cases} \displaystyle\sum_{x=0}^{L-1-\tau} a_{z',y,x} (a_{z,y,x+\tau})^* & (0 \le \tau < L), \\ \displaystyle\sum_{x=0}^{L-1+\tau} a_{z',y,x-\tau} (a_{z,y,x})^* & (-L < \tau < 0), \\ 0 & (|\tau| \ge L). \end{cases} \tag{9}$$

If the size of the subset is 1, i.e. only $y = 0$, then y will be ignored.

Consider the aperiodic correlation function between the subsets of A_z and $A_{z'}$ defined by

$$\hat{R}_A(z, z', \tau) = \sum_{y=0}^{M-1} R_{\boldsymbol{a}}(z, z', y, \tau). \tag{10}$$

If the set A possesses the correlation properties

$$\hat{R}_A(z, z', \tau) = \begin{cases} ML & (\tau = 0, z = z'), \\ 0 & (\text{otherwise}), \end{cases} \tag{11}$$

it is called a complementary sequence set which is expressed by $A(L, K, M)$.

Note that in general, $A(2^n, 2, 2)$ is well-known as a complete binary complementary pair [7], [9].

3. Construction of Complementary Sets

In this section, the construction method of generalized complementary sequence sets $A(N^n, N, N)$ presented by Suehiro

[9], [10] is clarified.

3.1 Suehiro's Construction Method

For the clarity of deriving the logic function of $A(N^n, N, N)$, the construction method is summarized as follows. Let x be an integer variable that takes a value from 0 to $N^n - 1$ determined by n, written as

$$x = x_{n-1}N^{n-1} + \cdots + x_k N^k + \cdots + x_0, \tag{12}$$

where $0 \le x_k < N$. The complementary set $A(N^n, N, N)$ with $n \ge 2$ is rewritten as

$$\left.\begin{array}{lcl} A^n & = & \{A^n_0, \cdots, A^n_z, \cdots, A^n_{N-1}\} \\ A^n_z & = & \{\boldsymbol{a}^n_{z,0}, \cdots, \boldsymbol{a}^n_{z,y}, \cdots, \boldsymbol{a}^n_{z,N-1}\} \\ \boldsymbol{a}^n_{z,y} & = & (a^n_{z,y,0}, \cdots, a^n_{z,y,x}, \cdots, a^n_{z,y,N^n-1}) \end{array}\right\}. \tag{13}$$

Let $\boldsymbol{b}^0_y$ be the yth row of the Hadamard matrix B^0 with $s = 0$, which is defined by Eq. (7), expressed by

$$\boldsymbol{b}^0_y = (b^0_{y,0}, \cdots, b^0_{y,x}, \cdots, b^0_{y,N-1}). \tag{14}$$

Consider a set of N sequences of length N^2 with $n = 2$ defined by

$$\left.\begin{array}{lcl} C & = & \{\boldsymbol{c}_0, \cdots, \boldsymbol{c}_z, \cdots, \boldsymbol{c}_{N-1}\} \\ \boldsymbol{c}_z & = & (c_{z,0}, \cdots, c_{z,x}, \cdots, c_{z,N^2-1}) \end{array}\right\}, \tag{15}$$

where $0 \le x < N^2$. When

$$\boldsymbol{c}_z = (b^1_{z,0}\boldsymbol{b}^0_0, b^1_{z,1}\boldsymbol{b}^0_1, \cdots, b^1_{z,N-1}\boldsymbol{b}^0_{N-1}), \tag{16}$$

where $b^1_{z,k}$ is an element of the Hadamard matrix B^1, the set C is referred to as the mate of N-shift orthogonal sequences with each other, since these aperiodic auto/cross-correlation functions takes zero at N shifts, that is,

$$R_{\boldsymbol{C}}(z, z', kN) = \begin{cases} N & (k = 0, z = z'), \\ 0 & (\text{otherwise}). \end{cases} \tag{17}$$

Use of the mate C of Eq. (15) and the Hadamard matrix B^2 gives a complementary set $A(N^2, N, N)$ defined by

$$\begin{aligned} \boldsymbol{a}^2_{z,y,x} = (&b^2_{y,0}c_{z,0}, b^2_{y,1}c_{z,1}, \cdots, b^2_{y,N-1}c_{z,N-1}, \\ &b^2_{y,0}c_{z,N}, b^2_{y,1}c_{z,N+1}, \cdots, b^2_{y,N-1}c_{z,2N-1}, \cdots, \\ &b^2_{y,0}c_{z,N^2-N}, b^2_{y,1}c_{z,N^2-N+1}, \cdots, b^2_{y,N-1}c_{z,N^2-1}), \end{aligned} \tag{18}$$

where $0 \le x < N^2$.

Let D^n be a set of N sequences of length $N^n (n \ge 3)$ written as

$$\left.\begin{array}{lcl} D^n & = & \{\boldsymbol{d}^n_0, \cdots, \boldsymbol{d}^n_z, \cdots, \boldsymbol{d}^n_{N-1}\} \\ \boldsymbol{d}^n_z & = & (d^n_{z,0}, \cdots, d^n_{z,x}, \cdots, d^n_{z,N^n-1}) \end{array}\right\}. \tag{19}$$

The set D^n of N-shift orthogonal sequences of length N^n can be given by interleaving the sequences in A^{n-1}_z in order, as shown in

$$\begin{aligned} \boldsymbol{d}^n_z = (&a^{n-1}_{z,0,0}, a^{n-1}_{z,1,0}, \cdots, a^{n-1}_{z,N-1,0}, \\ &a^{n-1}_{z,0,1}, a^{n-1}_{z,1,1}, \cdots, a^{n-1}_{z,N-1,1}, \cdots\cdots, \\ &a^{n-1}_{z,0,N-1}, a^{n-1}_{z,1,N-1}, \cdots, a^{n-1}_{z,N-1,N-1}). \end{aligned} \tag{20}$$

The use of D^3 and B^3 gives a complementary set $A(N^3, N, N)$ defined by

$$\begin{aligned} \boldsymbol{a}^3_{zy} = (&b^3_{y,0}d_{z,0}, b^3_{y,1}d_{z,1}, \cdots, b^3_{y,N-1}d_{z,N-1}, \\ &b^3_{y,0}d_{z,N}, b^3_{y,1}d_{z,N+1}, \cdots, b^3_{y,N-1}d_{z,2N-1}, \cdots\cdots, \\ &b^3_{y,0}d_{z,N^2-N}, b^3_{y,1}d_{z,N^2-N+1}, \cdots, b^3_{y,N-1}d_{z,N^2-1}). \end{aligned} \tag{21}$$

Similarly, a complementary set $A(N^n, N, N)$ whose length is extended N times can be easily generated by using the Hadamard set $A(N^{n-1}, N, N)$, the mate of N-shift orthogonal sequences D^n and the Hadamard matrix B^n.

3.2 Hadamard Matrix Representation

Consider expressing the elements $a^n_{z,y,x}$ in A^n of Eq. (13). From Eqs. (15) and (16), the elements $c_{z,x}$ of $\boldsymbol{c}_z$ in C can be written as

$$c_{z,x} = b^1_{z,x_1} b^0_{x_1,x_0}, \tag{22}$$

where $x = x_0 + x_1 N$ $(0 \le x < N^2)$. The complementary set $A(N^2, N, N)$ can be expressed by the mate C and the Hadamard matrix B^2, which are expressed by

$$a^2_{z,y,x} = b^2_{y,x_0} c_{z,x} = b^2_{y,x_0} b^1_{z,x_1} b^0_{x_1,x_0}. \tag{23}$$

The elements in $\boldsymbol{d}_z$ of Eq. (20) can be given as

$$d^3_{z,x} = b^2_{x_0,x_1} b^1_{z x_2} b^0_{x_2,x_1}, \tag{24}$$

where $x = x_0 + x_1 N + x_2 N^2$ $(0 \le x < N^3)$.

Generally speaking, interleaving the sequences $\boldsymbol{a}^n_{z,y} (0 \le y < N)$ in A^{n-1}_z means replacing y and $(x_0, x_1, \cdots, x_{n-2})$ with x_0 and $(x_1, x_2, \cdots, x_{n-1})$, respectively. Therefore the complementary set $A(N^3, N, N)$ can be obtained as

$$a^3_{z,y,x} = b^3_{y,x_0} d_{z,x} = b^3_{y,x_0} b^2_{z,x_1} b^1_{x_0,x_2} b^0_{x_2,x_1}. \tag{25}$$

From the above discussion, using D^4, B^4 and $A(N^4, N, N)$ gives

$$a^4_{z,y,x} = b^4_{y,x_0} d_{z,x} = b^4_{y,x_0} b^3_{x_0,x_1} b^2_{z,x_2} b^1_{x_1,x_3} b^0_{x_3,x_2}. \tag{26}$$

Similarly, $a^n_{z,y,x}$ can be derived using the same logic. Note that the above argument is equivalent to induction. Therefore, the complementary sets by Suehiro can be summarized as the following theorem.

Theorem 1 The complementary sets $A(N^n, N, N)$ defined by Eq. (13) can be given as

$$a^n_{z,y,x} = \begin{cases} b^2_{y,x_0} b^1_{z,x_1} b^0_{x_1,x_0} & (n = 2), \\ b^n_{y,x_0} b^{n-1}_{x_0,x_1} \cdots b^3_{x_{n-4},x_{n-3}} b^2_{z,x_{n-2}} \\ \quad b^1_{x_{n-3},x_{n-1}} b^0_{x_{n-1},x_{n-2}} & (n \ge 3), \end{cases} \tag{27}$$

where $b^s_{y',x'}$ denotes the $N \times N$ Hadamard matrix with $0 \le y', x' < N$ in Eq. (7).

4. Logic Function of Complementary Sets

This section considers the (generalized) logic functions to represent a complementary sequence set $A(N^n, N, N)$ with $n \geq 2$ and $N = q^m (m \geq 1)$, which is an operation modulo q and allows real-valued output values. The logic functions mapping from V_q^n to Z_q or R_q are formulated, where R_q denotes the set of real numbers modulo q.

4.1 Formalization of Logic Function

Let $\boldsymbol{x}$ be a vector related to $x(0 \leq x < N^n)$ of Eq. (12), which is defined by

$$\left.\begin{aligned} \boldsymbol{x} &= (\boldsymbol{x}_0, \cdots, \boldsymbol{x}_k, \cdots, \boldsymbol{x}_{n-1}) \in V_q^{mn} \\ \boldsymbol{x}_k &= (x_{k,0}, \cdots, x_{k,i}, \cdots, x_{k,m-1}) \in V_q^m \end{aligned}\right\}, \tag{28}$$

where $x_{k,i} = x_{km+i}$. Let $\boldsymbol{y}$ be a vector that also includes $\boldsymbol{x}_k$ and $\boldsymbol{z}$. These vectors are corresponding to y, x_k and z with $0 \leq y, x_k, z < N$), respectively.

Theorem 2 Let $f_m^n(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x})$ be the logic functions of the complementary set $A(N^n, N, N)$ with $N = q^m$ in Theorem 1, written as

$$a_{z,y,x}^n = \omega_q^{f_m^n(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x})}, \tag{29}$$

where $\boldsymbol{z}, \boldsymbol{y} \in V_q^m$. Let $h^s(\boldsymbol{y}, \boldsymbol{x}_k)$ be the logic function of the Hadamard matrix of Eq. (7) is expressed as

$$B^s = [b_{y,x_k}^s = \omega_q^{h^s(\boldsymbol{y},\boldsymbol{x}_k)}]_{0 \leq y, x_k < N}. \tag{30}$$

The complementary set can be expressed as

$$f_m^2(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) = h^2(\boldsymbol{y}, \boldsymbol{x}_0) + h^1(\boldsymbol{z}, \boldsymbol{x}_1) + h^0(\boldsymbol{x}_1, \boldsymbol{x}_0) \tag{31}$$

for $n = 2$, and

$$\begin{aligned} f_m^n(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) = &\ h^n(\boldsymbol{y}, \boldsymbol{x}_0) + h^{n-1}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \cdots \\ &+ h^3(\boldsymbol{x}_{n-4}, \boldsymbol{x}_{n-3}) + h^2(\boldsymbol{z}, \boldsymbol{x}_{n-2}) \\ &+ h^1(\boldsymbol{x}_{n-3}, \boldsymbol{x}_{n-1}) + h^0(\boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-2}) \end{aligned} \tag{32}$$

for $n \geq 3$.

4.2 Representation for Different Hadamard Matrices

This section investigates the logic function of a complementary set $A(N^n, N, N)$, which is generated by the different Hadamard matrices $B^s (0 \leq s \leq n)$ of order N, as shown in Eq. (7).

Let P^s and Q^s be non-singular matrices of order N consisting $N^2 - N$ zero elements and N elements with an absolute value of 1 (unit magnitude), written as

$$P^s = [p_{y,x}^s]_{0 \leq y, x \leq N-1},\ Q^s = [q_{y,x}^s]_{0 \leq y, x \leq N-1}, \tag{33}$$

which play the role of swapping any rows or columns, respectively. Let $\hat{G}^s$ and G^s be the diagonal complex matrices of order N whose diagonal elements take any complex numbers with an absolute value of one. Any Hadamard matrix of order N can be expressed by

$$B^s = [b_{y,x}^s]_{0 \leq y, x < N-1} = \hat{G}^s P^s H Q^s G^s, \tag{34}$$

where H denotes the Hadamard matrix of Eq. (4).

Let p_{y,w_y}^s represent the N elements that take the value 1 in the matrix P^s, where $w_y \neq w_{y'}$ for $0 \leq y \neq y' \leq N-1$ and let $q_{v_x,x}^s$ represent the N elements that take the value 1 in the matrix Q^s, where $v_x \neq v_{x'}$ for $0 \leq x \neq x' \leq N-1$. The element p_{y,w_y}^s and q_{x,v_x}^s represent replacing the y-th row with the w_y-th row and the x-th column with the v_x-th column, respectively, that is, w_x and v_x are the sorts of $y, x \in \{0, 1, \cdots, N-1\}$, respectively. Since the matrices P^s and Q^s represent the operations of swapping rows and columns respectively, they can be expressed as $P^s H Q^s = [h_{w_y,v_x}^s]$. From the above discussion, the following lemma can be given.

[Lemma 1] Let w and v be rearrangements of y and x with $0 \leq x, y \leq N-1$ in the Hadamard matrix B^s of Eq. (30) written as

$$\left.\begin{aligned} \begin{pmatrix} y \\ w^s \end{pmatrix} &= \begin{pmatrix} 0 & \cdots & y & \cdots & N-1 \\ w_0^s & \cdots & w_y^s & \cdots & w_{N-1}^s \end{pmatrix} \\ \begin{pmatrix} x \\ v^s \end{pmatrix} &= \begin{pmatrix} 0 & \cdots & x & \cdots & N-1 \\ v_0^s & \cdots & v_x^s & \cdots & v_{N-1}^s \end{pmatrix} \end{aligned}\right\}, \tag{35}$$

where $w_y^s \neq w_{y'}^s$ for $y \neq y'$ and $v_x^s \neq v_{x'}^s$ for $x \neq x'$. If s is obvious as shown below, it will be omitted because the notation is confusing. Any Hadamard matrix of Eq. (34) can be expressed as

$$\begin{aligned} B^s = &\ \underbrace{\mathrm{diag}[p_{0,w_0}^s, p_{1,w_1}^s, \cdots, p_{N-1,w_{N-1}}^s]}_{\hat{G}^s} \underbrace{[h_{w_y,v_x}^s]}_{P^s H Q^s} \\ &\ \underbrace{\mathrm{diag}[q_{x_0,0}^s, q_{x_1,1}^s, \cdots, q_{x_{N-1},N-1}^s]}_{G^s}, \end{aligned} \tag{36}$$

where $0 \leq y, x \leq N-1$.

As shown in Lemma 1, if P^s and Q^s transpose the rows or columns of B^s, then the vectors $\boldsymbol{y}$, let us discuss the vectors $\boldsymbol{x}_k$ and $\boldsymbol{y}$ with the superscript s for convenience.

[Theorem 3] The logic function of the Hadamard matrix B^s in Theorem 2 can be written as

$$h^s(\boldsymbol{y}, \boldsymbol{x}_k) = \boldsymbol{y}^s \cdot \boldsymbol{x}_k^s + \hat{g}^s(\boldsymbol{y}) + g^s(\boldsymbol{x}_k), \tag{37}$$

where $\boldsymbol{y}$ also includes $\boldsymbol{z}$ and $\boldsymbol{x}_{k'}$, and $\hat{g}^s(\cdot)$ and $g^s(\cdot)$ denotes logic functions giving the diagonal elements of diagonal matrices $\hat{G}^s$ and G^s of Eq. (30), respectively. Note that $\hat{g}^s(\cdot)$ and $g^s(\cdot)$ are allowed to map to R_q.

4.3 Examples

The logic functions of complementary sequence sets in Theorem 1 are illustrated as the following examples.

Example 1 Consider a three-phase complementary set $A^2 = A(3^2, 3, 3)$ with $N = 3$ generated by the same Hadamard matrices of order 3 corresponding to the DFT matrix, $B^s(0 \leq s \leq 2)$, which is expressed as

$$B^s = H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \tag{38}$$

where $0, 1, 2$ denote respectively sequence elements $\omega^0 = 1, \omega_3^1, \omega_3^2$. The mate of N-shift orthogonal sequences of length N^2 defined by Eq. (15) is written as

$$C = \begin{Bmatrix} \boldsymbol{c}_0 = (000012021) \\ \boldsymbol{c}_1 = (000120210) \\ \boldsymbol{c}_2 = (000201102) \end{Bmatrix}. \tag{39}$$

Therefore $A^2 = A(3^2, 3, 3)$ can be derived from Eq. (18) as

$$\begin{aligned} A^2 &= \{A_0^2, A_1^2, A_2^2\}, \\ A_0^2 &= \begin{Bmatrix} \boldsymbol{a}_{0,0}^2 = (000012021) \\ \boldsymbol{a}_{0,1}^2 = (012021000) \\ \boldsymbol{a}_{0,2}^2 = (021000012) \end{Bmatrix}, \\ A_1^2 &= \begin{Bmatrix} \boldsymbol{a}_{1,0}^2 = (000120210) \\ \boldsymbol{a}_{1,1}^2 = (012102222) \\ \boldsymbol{a}_{1,2}^2 = (021111201) \end{Bmatrix}, \\ A_2^2 &= \begin{Bmatrix} \boldsymbol{a}_{2,0}^2 = (000201102) \\ \boldsymbol{a}_{2,1}^2 = (012210111) \\ \boldsymbol{a}_{2,2}^2 = (021222120) \end{Bmatrix}. \end{aligned} \tag{40}$$

Theorem 2 gives the logic function of A^2 as

$$\begin{aligned} f_1^2(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) &= h^2(y, x_0) + h^1(z, x_1) + h^0(x_1, x_0) \\ &= zx_1 + yx_0 + x_1x_0, \end{aligned} \tag{41}$$

where $\boldsymbol{z} = z \in V_3^1$, $\boldsymbol{y} = y \in V_3^1$ with $0 \le y, z \le 2$, $\boldsymbol{x} = (x_0, x_1) \in V_3^2$ with $0 \le x \le 8$, and $\hat{g}^s(\cdot) = g^s(\cdot) = 0\ (0 \le s \le 2)$.

Table 1 shows the truth table of $\boldsymbol{a}_{z,y}^2$ for $y = 0, 1, 2$ on $z = 0, 1$ (omitted when $z = 2$) where $\boldsymbol{a}_{0,0}^2 = x_1x_0 = \boldsymbol{c}_0$, and can confirm that the logic function of Eq. (41) is correct.

Example 2 Consider a three-phase complementary set $A^2 = A(3^2, 3, 3)$ with $N = 3$ generated by using three different Hadamard matrices, which are written as

$$\begin{aligned} B^0 &= \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}, B^1 = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \\ B^2 &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \end{aligned} \tag{42}$$

Table 1 Truth table for Eq. (41).

	$\boldsymbol{x}$		$A_0^2 = \{\boldsymbol{a}_{0,y}^2\}$			$A_1^2 = \{\boldsymbol{a}_{1,y}^2\}$		
x	x_1	x_0	$\boldsymbol{a}_{0,0}^2$	$\boldsymbol{a}_{0,1}^2$	$\boldsymbol{a}_{0,2}^2$	$\boldsymbol{a}_{1,0}^2$	$\boldsymbol{a}_{1,1}^2$	$\boldsymbol{a}_{1,2}^2$
0	0	0	0	0	0	0	0	0
1	0	1	0	1	2	0	1	2
2	0	2	0	2	1	0	2	1
3	1	0	0	0	0	1	1	1
4	1	1	1	2	0	2	0	1
5	1	2	2	1	0	0	2	1
6	2	0	0	0	0	2	2	2
7	2	1	2	0	1	1	2	0
8	2	2	1	0	2	0	2	1

where 0, 1 and 2 denote sequence elements $\omega^0 = 1, \omega_3^1$ and ω_3^2, respectively. The mate of N-shift orthogonal sequences of length N^2 defined by Eq. (15) is written as

$$C = \begin{Bmatrix} \boldsymbol{c}_0 = (111120102) \\ \boldsymbol{c}_1 = (111012210) \\ \boldsymbol{c}_2 = (111201021) \end{Bmatrix}. \tag{43}$$

Therefore $A^2 = A(3^2, 3, 3)$ can be derived from Eq. (18) as

$$\begin{aligned} A^2 &= \{A_0^2, A_1^2, A_2^2\}, \\ A_0^2 &= \begin{Bmatrix} \boldsymbol{a}_{0,0}^2 = (102111120) \\ \boldsymbol{a}_{0,1}^2 = (000012021) \\ \boldsymbol{a}_{0,2}^2 = (201210222) \end{Bmatrix}, \\ A_1^2 &= \begin{Bmatrix} \boldsymbol{a}_{1,0}^2 = (102111120) \\ \boldsymbol{a}_{1,1}^2 = (000012021) \\ \boldsymbol{a}_{1,2}^2 = (201102000) \end{Bmatrix}, \\ A_2^2 &= \begin{Bmatrix} \boldsymbol{a}_{2,0}^2 = (102222012) \\ \boldsymbol{a}_{2,1}^2 = (000120210) \\ \boldsymbol{a}_{2,2}^2 = (201021111) \end{Bmatrix}. \end{aligned} \tag{44}$$

As shown in Lemma 1, these can be represented by use of DFT matrix H. The Hadamard matrix B^0 can be written as

$$\begin{aligned} B^0 &= \begin{bmatrix} & 1 & \\ & & 0 \\ 2 & & \end{bmatrix} H \begin{bmatrix} & 1 & \\ & & 0 \\ 2 & & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 2 \end{bmatrix} \begin{bmatrix} & 0 & \\ & & 0 \\ 0 & & \end{bmatrix} H \begin{bmatrix} & 0 & \\ & & 0 \\ 0 & & \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & \\ & 0 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \end{aligned} \tag{45}$$

where blank indicates 0. Similarly, B^1 and B^2 can be respectively written as

$$\begin{aligned} B^1 &= H \begin{bmatrix} 2 & & \\ & & 0 \\ & 1 & \end{bmatrix} = H \begin{bmatrix} & 0 & \\ & & 0 \\ 0 & & \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \\ B^2 &= \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix} H \begin{bmatrix} 1 & & \\ & 0 & \\ & & 2 \end{bmatrix}. \end{aligned} \tag{46}$$

Lemma 1 and Eqs. (45) and (46) give

$$\left.\begin{aligned} \begin{pmatrix} y \\ w^0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} x \\ v^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} y \\ w^1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} x \\ v^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \\ \begin{pmatrix} y \\ w^2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} x \\ v^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{aligned}\right\}, \tag{47}$$

and

$$\left.\begin{array}{llll}\hat{g}^0(y) & = (1,0,2), & g^0(x_0) = (2,1,0) \\ \hat{g}^1(y) & = 0, & g^1(x_0) = (2,1,0) \\ \hat{g}^2(y) & = (2,1,0), & g^2(x_0) = (1,0,2)\end{array}\right\}, \tag{48}$$

where $y = \boldsymbol{y} \in V_q^1$, $x_0 = \boldsymbol{x}_0 \in V_q^1$, and w^s and v^s respectively indicate the sort of y and x_0 in $h^s(\ ,\ )$ as shown in Lemma 1.

Theorems 2 and 3 give the logic function of A^2 written as

$$\begin{aligned} f_1^2(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) &= h^2(y, x_0) + h^1(z, x_1) + h^0(x_1, x_0) \\ &= z v_{x_1}^1 + u_{x_1,x_0} + g_{y,x_0}, \end{aligned} \tag{49}$$

and

$$\left.\begin{array}{lll} u_{x_1,x_0} & = & w_{x_1}^0 v_{x_0}^0 + g_{x_0}^2 + g_{x_1}^1 + g_{x_1}^0 + g_{x_0}^0 \\ g_{y,x_0} & = & y x_0 + \hat{g}_y^2 \end{array}\right\}, \tag{50}$$

where $g_{x_k}^s$ and $\hat{g}_y^s$ indicate $g^s(x_k)$ and $\hat{g}^s(y)$, respectively, so that they fit in Table 2 derived from Eqs. (49) and (50).

In Table 2, (0), (1), (2) and $(0_0), (0_1), (0_2)$ denote respectively u_{x_1,x_0} of Eq. (50), $(0) + v_{x_1}$, $(0) + 2v_{x_1}$, and g_{y,x_0} for $0 \le y \le 2$, and $w^0 v^0 = w_{x_1}^0 v_{x_0}^0$. Since $\boldsymbol{a}_{z,y}^2 = (z) + (1_y)$, the complementary set of Eq. (44) is obtained.

Note that if even one of $g_{x_k}^s$'s in Eq. (50) take a real value modulo q, the complementary set has more phases. For example, if even one of them has a term that is a multiple 0.5, it is a $2q$-phase complementary set.

Example 3 Consider a biphase complementary set $A^4 = A(2^4, 2, 2)$ generated by a logic function given as

$$\begin{aligned} f_1^4(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) &= h^4(y, x_0) + h^3(x_0, x_1) + h^2(z, x_2) \\ &\quad + h^1(x_1, x_3) + h^0(x_3, x_2) \\ &= x_2 x_3 + x_3 x_1 + x_1 x_0 + y x_0 + z x_2, \end{aligned} \tag{51}$$

where $g^s(\cdot) = 0$ $(0 \le s \le 4)$ and $0 \le y \le 3$. According to the truth table of $f_1^4(\cdot)$, the set is given as

Table 2 Truth table of A^2 of Eq. (41).

x	x_1	x_0	$w_{x_1}^0$	$v_{x_0}^0$	$g_{x_0}^2$	$g_{x_1}^1$	$g_{x_1}^0$	$g_{x_0}^0$
0	0	0	1	2	1	2	1	2
1	0	1	1	0	0	2	1	1
2	0	2	1	1	2	2	1	0
3	1	0	2	2	1	1	0	2
4	1	1	2	0	0	1	0	1
5	1	2	2	1	2	1	0	0
6	2	0	0	2	1	0	2	2
7	2	1	0	0	0	0	2	1
8	2	2	0	1	2	0	2	0
x	w^0v^0	(0)	(0_0)	(0_1)	(0_2)	v_{x_1}	(1)	(2)
0	2	2	2	1	0	0	2	2
1	0	1	2	2	2	0	1	1
2	1	0	2	0	1	0	0	0
3	1	2	2	1	0	2	1	0
4	0	2	2	2	2	2	1	0
5	2	2	2	0	1	2	1	0
6	0	2	2	1	0	1	0	1
7	0	0	2	2	2	1	1	2
8	0	1	2	0	1	1	2	0

$$\begin{array}{lll} A^4 & = & \{A_0^4, A_1^4\}, \\ A_0^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{0,0}^4 = (0001000100101101) \\ \boldsymbol{a}_{0,1}^4 = (0100010001111000)\end{array}\right\}, \\ A_1^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{1,0}^4 = (0001111000100010) \\ \boldsymbol{a}_{1.1}^4 = (0100101101110111)\end{array}\right\}, \end{array} \tag{52}$$

where 0 and 1 denote sequence elements 1 and −1, respectively. This set denotes well-discussed binary complementary pairs [1].

When $g^s(x_1) = x_1/2$ and $g^0(x_0) = x_0/2$, it is a quadriphase complementary set $A(2^4, 2, 2)$ which is written as

$$\begin{array}{lll} A^4 & = & \{A_0^4, A_1^4\}, \\ A_0^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{0,0}^4 = (0110011001322310) \\ \boldsymbol{a}_{0,1}^4 = (0312031203302112)\end{array}\right\}, \\ A_1^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{1,0}^4 = (0110233201320132) \\ \boldsymbol{a}_{1.1}^4 = (0312213003300330)\end{array}\right\}, \end{array} \tag{53}$$

where $0, 1, 2$ and 3 denote sequence elements $1, j, -1$ and j, respectively.

Example 4 Consider a biphase complementary set $A(2^4, 2^2, 2^2)$. Let $\boldsymbol{x}_0^0 = \boldsymbol{x}_0 = (x_0, x_1)$, $\boldsymbol{x}_1^1 = \boldsymbol{x}_1 = (x_2, x_3)$, $\boldsymbol{z} = (z_0, z_1)$, $\boldsymbol{y} = (y_0, y_1)$ and $g^s(\cdot) = \hat{g}^s(\cdot) = 0$. Substituting them into Eq. (32) gives the logic function

$$\begin{aligned} f_2^4(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) &= x_0 x_2 + x_1 x_3 \\ &\quad + y_1 x_3 + y_0 x_2 + z_1 x_1 + z_0 x_0. \end{aligned} \tag{54}$$

Therefore the biphase complementary set A is given as

$$\begin{array}{lll} A^4 & = & \{A_0^4, A_1^4, A_2^4, A_3^4\}, \\ A_0^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{0,0}^4 = (0000010100110110) \\ \boldsymbol{a}_{0,1}^4 = (0000101011001001) \\ \boldsymbol{a}_{0,2}^4 = (0000010111001001) \\ \boldsymbol{a}_{0,3}^4 = (0000101011000110)\end{array}\right\}, \\ A_1^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{1,0}^4 = (0101000001100011) \\ \boldsymbol{a}_{1,1}^4 = (0101111101101100) \\ \boldsymbol{a}_{1,2}^4 = (0101000010011100) \\ \boldsymbol{a}_{1,3}^4 = (0101111110010011)\end{array}\right\}, \\ A_2^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{2,0}^4 = (0011011000000101) \\ \boldsymbol{a}_{2,1}^4 = (0011100100001010) \\ \boldsymbol{a}_{2,2}^4 = (0011011011111010) \\ \boldsymbol{a}_{2,3}^4 = (0011100111110101)\end{array}\right\}, \\ A_3^4 & = & \left\{\begin{array}{l}\boldsymbol{a}_{3,0}^4 = (0110001101010000) \\ \boldsymbol{a}_{3,1}^4 = (0110110001011111) \\ \boldsymbol{a}_{3,2}^4 = (0110001110101111) \\ \boldsymbol{a}_{3,3}^4 = (0110110010100000)\end{array}\right\}, \end{array} \tag{55}$$

where 0 and 1 denote 1 and −1, respectively. As shown in Example 2, it is possible to generate a large number of complementary sets by swapping the rows and columns of the Hadamard matrices.

Similarly, a large number of biphase or quadriphase complementary sets $A(2^{nm}, 2^m, 2^m)$, and quadriphase complementary sets $A(4^{mn}, 4^m, 4^m)$ can be generated.

5. Consideration

In this section, the complementary sets discussed in the previous sections are considered from these logic functions. First, in order to check whether each sequence in the complementary sequence set has a high randomness property, it is checked whether its logic function is bent.

The Fourier transform of Eq. (29) is defined by

$$F(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{\lambda}) = \frac{1}{\sqrt{q^m}} \sum_{\boldsymbol{x} \in V_q^n} \omega^{f_m^n(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}) - \boldsymbol{\lambda} \boldsymbol{x}^t}, \tag{56}$$

where $\boldsymbol{\lambda} \in V_q^n$. If the Fourier transform has the unit magnitude for all $\boldsymbol{\lambda}$, i.e.,

$$|F(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{\lambda})| = 1 \text{ for all } \boldsymbol{\lambda}, \tag{57}$$

the function $f_m^n(\cdot)$ is called bent for all $\boldsymbol{z}$ and $\boldsymbol{y}$.

[Theorem 4] Assume that the Hadamard matrices in Theorem 1 are all the same. If n is even, Eqs. (31) and (32) in Theorem 2 are bent.

(Proof) To simplify the expression without loss of generality, s is negligible, and $\hat{g}^s(\cdot) = g^s(\cdot) = 0$. the vectors $\boldsymbol{z}$, $\boldsymbol{y}$, $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ can be replaced by $z, y, (x_0, x_1)$ and (λ_0, λ_1). The Fourier transform of Eq. (31) can be simply written as

$$\begin{aligned} F(z, y, \boldsymbol{\lambda}) &= \frac{1}{\sqrt{q^2}} \sum_{x_1 \in V_q} \sum_{x_1 \in V_q} \omega_q^{z x_1 + y x_0 + x_0 x_1 - \lambda_0 x_0 - \lambda_1 x_1} \\ &= \frac{1}{q} \sum_{x_0 \in V_q} \omega_q^{x_0 (y - \lambda_0)} \sum_{x_1 \in V_q} \omega_q^{x_1 (x_0 + z - \lambda_1)} \\ &= \omega_q^{(\lambda_1 - z)(y - \lambda_0)}, \end{aligned} \tag{58}$$

since the second term in the above equation takes the value q only when $x_0 = \lambda_1 - z$. Similarly the Fourier transform of Eq. (32) for $n = 2m$ can be calculated sequentially as

$$\begin{aligned} & F(z, y, \boldsymbol{\lambda}) \\ &= \frac{1}{q^m} \sum_{x_{n-1} \in V_q} \omega_q^{-x_{n-1}\lambda_{n-1}} \sum_{x_{n-2} \in V_q} \omega_q^{x_{n-2}(x_{n-1} + z - \lambda_{n-2})} \\ & \sum_{x_{n-3} \in V_q} \omega_q^{x_{n-3}(x_{n-1} - \lambda_{n-3})} \sum_{x_{n-4} \in V_q} \omega_q^{x_{n-4}(x_{n-3} - \lambda_{n-4})} \\ & \quad \cdots \sum_{x_1 \in V_q} \omega_q^{x_1 (x_2 - \lambda_1)} \sum_{x_0 \in V_q} \omega_q^{x_0 (x_1 + y - \lambda_{n-4})} \\ &= \frac{1}{q^{m-1}} \omega^{-\hat{\lambda}_0 \lambda_1} \sum_{x_{n-1} \in V_q} \omega_q^{-x_{n-1}\lambda_{n-1}} \\ & \quad \cdots \sum_{x_3 \in V_q} \omega_q^{x_3 (x_5 - \lambda_3)} \sum_{x_2 \in V_q} \omega_q^{x_2 (x_3 - \lambda_2)} \\ &= \frac{1}{q^{m-2}} \omega^{-\hat{\lambda}_0 \lambda_1 - \hat{\lambda}_2 \lambda_3} \\ & \quad \sum_{x_{n-1} \in V_q} \omega_q^{-x_{n-1}\lambda_{n-1}} \cdots \sum_{x_4 \in V_q} \omega_q^{x_4 (x_3 - \lambda_4)} \\ &= \omega^{-\hat{\lambda}_0 \lambda_1 - \hat{\lambda}_2 \lambda_3 - \cdots - \hat{\lambda}_{n-2} \lambda_{n-1}}, \end{aligned} \tag{59}$$

where $\hat{\lambda}_0 = \lambda_0 - y$, $\hat{\lambda}_{n-2} = \lambda_{n-2} - \hat{\lambda}_{n-4} - z$, and $\hat{\lambda}_{2k} = \lambda_{2k} - \hat{\lambda}_{2k-2}$ for $1 \le k \le \frac{n}{2} - 2$. Since the above equation satisfies Eq. (57), the function is bent.

All of the sequences generated by the bent function will have good randomness. Since the sum of the generated sequence elements can be written as

$$|\sum_{\boldsymbol{x} \in V_q^n} \omega^{f_m^n(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x})}| = \sqrt{q^n} |F(z, \boldsymbol{y}, \boldsymbol{0})| = \sqrt{q^n}, \tag{60}$$

all sequences in all subsets are considered approximately balanced, that is, the elements of each sequence will be generated approximately evenly. It can be seen in Eqs. (40), (52) and (55) in the examples in Sect. 4. In fact, as shown in Eq. (44) of Example 2, each sequence seems to be nearly balanced even if different Hadamard matrices are used.

On the other hand, when constructing a complementary sequence by using the Kronecker product of matrices [21], [22], the logic function seems not to be the bent function as shown below. For ease of discussion, let $A^0 = [a^0_{y_0, x_0}]$ and $A^1 = [a^1_{y_1, x_1}]$ be Hadamard matrices of order L_0 and L_1 respectively, where $0 \le y_0, x_0 < L_0$ and $0 \le y_1, x_1 < L_0$. Let $A = [a_{y,x}]$ be an Hadamard matrix of order $L = L_0 L_1$ by using the Kronecker product of the matrices A^1 and A^0, that is, $A = [a_{y,x}] = A^0 \otimes A^1$ for $0 \le y, x < L$. Let $f^0(y_0, x_0)$ and $f^1(y_1, x_1)$ be the logic functions of A^0 and A^1 respectively, which are.expressed by $a^0_{y_0, x_0} = \omega_{q_0}^{f^0(y_0, x_0)}$ and $a^1_{y_1, x_1} = \omega_{q_1}^{f^1(y_1, x_1)}$. Let $f(y, x)$ be a logic function of the Hadamard matrix A of order $L = L_1 L_2$. The elements of A can be written as

$$a_{y,x} = \omega_{q_0 q_1}^{f(y,x)} = \omega_{q_0}^{f^0(y_0, x_0)} \omega_{q_1}^{f^1(y_1, x_1)}, \tag{61}$$

where $y = y_1 L_0 + y_0$ and $x = x_1 L_0 + x_0$. Therefore the logic function is written as

$$f(y, x) = q_1 f^0(y_0, x_0) + q_0 f^1(y_1, x_1). \tag{62}$$

Since the product of x_0 and x_1 does not appear, a bent function cannot be derived.

6. Conclusions

The logic function of complementary sets of polyphase sequences proposed by Suehiro et al. have been rigorously formulated, which includes binary and quadriphase ones. It has been shown that there exists a case where the logic function is a bent function which can generate good pseudo-random sequences. It seems that the logic functions of complementary sets generated by using the Kronecker product of matrices do not include bent functions. New complementary sets and other related sequence sets may be constructed by further deriving generalized logic functions of several complementary sets. By clarifying their differences and considering them, additional insights and developments will be achieved.

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