PAPER *Special Section on Signal Design and Its Applications in Communications*

Construction of Two Kinds of Optimal Wide-Gap Frequency-Hopping Sequence Sets

Ting WANG[†], **Xianhua NIU**[†a)], ***Nonmembers***, **Yaoxuan WANG**[†], ***Student Member***, **Jianhong ZHOU**[†], ***and*** **Ling XIONG**[†], ***Nonmembers***

SUMMARY The frequency hopping sequence plays a crucial role in determining the system's anti-jamming performance, in frequency hopping communication systems. If the adjacent frequency points of FHS can ensure wide-gap, it will better improve the anti-interference capability of the FH communication system. Moreover, if the period of the sequence is expanded, and each frequency point does not repeat in the same sequence, the system's ability to resist electromagnetic interference will be enhanced. And a one-coincidence frequency-hopping sequence set consists of FHSs with maximum Hamming autocorrelation 0 and cross-correlation 1. In this paper, we present two constructions of wide-gap frequency-hopping sequence sets. One construction is a new class of wide-gap one-coincidence FHS set, and the other is a WGFHS set with long period. These two WGFHS sets are optimal with respect to WG-Peng-Fan bound. And each sequence of these WGFHS sets is optimal with respect to WG-Lempel-Greenberger bound.

key words: *frequency-hopping sequence, wide-gap frequency-hopping sequence, one-coincidence frequency-hopping sequence, Hamming correlation*

1. Introduction

Frequency hopping (FH) spread spectrum, as one of the basic spread coding technologies in communication systems, has been widely used in mobile communication, UWB communication, radar system, Bluetooth, and so on, because of its good anti-interference ability, low interception rate, multiple access networking ability, and anti-fading ability [1]–[5].

FH is one of the most commonly used spread spectrum methods. Its working principle refers to the communication method in which the carrier frequency of the transmitted signals of both the transmitter and the receiver changes discretely according to a predetermined rule, that is, the carrier frequency used in communication changes randomly under the control of a pseudo-random change code [6]. This pseudo-random code is called frequency hopping sequence (FHS). At the same time, the performance of FHS has a decisive impact on the performance of the FH communication systems. In a system, each user is usually assigned an FHS. When two or more users transmit at the same time, it will cause signal interference. Hamming correlation is usually used to measure the degree of interference. In general, the smaller the number of FHS overlaps (the smaller the Hamming correlation), the better the FH Communication system's anti-multipath capability will be [7]–[9]. For instance, multiple access systems need FHS sets with low Hamming correlation and larger size, while radar system requires a single FHS with low Hamming autocorrelation. Therefore, constructing FHSs or FHS sets with good properties has attracted more attention. The study of the periodic Hamming autocorrelation of FHSs can be traced back to the seminal work of Lempel-Greenberger, which is used to measure the optimality of a single FHS [10]. Later, Peng and Fan made a contribution by developing the Peng-Fan bound [11] on FHS set which measures the optimality of FHS sets.

Since mutual interference is difficult to completely eliminate, it's important that we minimize collisions. During these decades, numerous research results of constructions of FHSs have been reported (See [12]–[15]). In particular, one-coincidence FHS (OC-FHS) set consists of FHSs which maximum Hamming cross-correlation of any two FHS is no more than 1 and the maximum autocorrelation of a sequence is 0. Firstly, the concept of the OC-FHS set was first proposed by Shaar [16] in the early 1980s which is exactly an OC-FHS set. then, Wang [17] et al. constructed a class of OC-FHS by Cartesian product for the first time. Later on, Lee [15] et al., proposed to construct an OC-FHS set by a primitive element of the prime field. Moreover, Niu [18] et al. used designed the direct product to obtain more new OC-FHS sets with flexible parameters.

In addition, FH communication is a kind of evasive anti-interference technology. If the radio station stays on a certain frequency for a long time, it will be vulnerable to all kinds of interference. This is especially true for slow frequency hopping systems. However, if the adjacent time slots of the FHS are greater than a given value. Such an FHS is called wide-gap FHS (WGFHS) [19], [20]. In this way, even if it is subject to strong single frequency interference, it will only lose information in one time slot, but not in several consecutive time slots. Information can be recovered through interleaving and error correction coding.

Generally speaking, there are two main types of methods for generating WGFHSs is based on frequency allocation, such as removing mid-band and dual-band [21], and the other is based on frequency point correction, taking adjacent or related frequency points as references, and calculating the current to meet the wide-gap require. Later, Ren [22] et al., proposed to construct sets of WGFHSs by way of combinatorial algebraic construction. But these WGFHSs

Manuscript received February 8, 2023.
Manuscript revised June 25, 2023.
Manuscript publicized August 16, 2023.
[†]The authors are with Xihua University, China.
a) E-mail: rurustef1212@gmail.com
DOI: 10.1587/transfun.2023SDP0008

periods are short period, which has certain limitations. In 2022, Li [23] et al., proposed two construction of single WGFHS with long period respectively. Compared with the construction of FHS, there are few construction of WGFHSs (refer to [21], [24], [25]), let alone a WGFHS set with a longer period or hava good properties. These motivate us to provide the optimal set of WGFHS with long period and the optimal set of one-coincidence WGFHS.

This paper is organized as follows. In Sect. 2, we will review some preliminaries. In Sect. 3, we will propose a construction of WGFHS set with long period which is optimal with respect to WG-Lempel-Greenberger bound. In Sect. 4, we will present a new class of optimal wide-gap one-coincidence Frequency-hopping sequence, Then, Sect. 5 gives two examples to prove our constructions. Finally, Sect. 6 concludes the paper.

2. Preliminaries

The following notations will be used throughout this paper.

- p is a prime and $p \geq 5$;
- $\langle x \rangle$ the least nonnegative residue of x module p for an integer x and a positive integer p;
- $\lfloor x \rfloor$ the largest integer less than or equal to x;
- $\lceil x \rceil$ the least integer greater than or equal to x.

2.1 Hamming Correlation Function

Throughout this paper, let F be the alphabet size $F = f_0, f_1, \ldots, f_{l-1}$ and l is a positive integer. Assume S is a set of length N which is defined over F. For any two FHSs $x = (x_0, x_1, \ldots, x_{N-1}), y = (y_0, y_1, \ldots, y_{N-1}) \in S$, the periodic Hamming correlation function $H_{x,y}(\tau)$ of x and y at time delay τ is defined as below:

$$H_{x,y}(\tau) = \sum_{k=0}^{N-1} h(x_k, y_{k+\tau}), 0 \leq \tau < N,$$

$h(x_k, y_{k+\tau}) = 1$ if $x_k = y_{k+\tau}$, $h(x_k = y_{k+\tau}) = 0$ otherwise, where all operations are performed module N among the position indices. If $x = y$, $H_{x,y}(\tau)$ is called the Hamming autocorrelation of x, denoted by $H_x(\tau)$ for simplicity. Let S be an FHS set with M sequences of length N over F. For any two distinct FHS $x, y \in S$, the following three measures are defined by

$$H_a(S) = \max_{1 \leq \tau < N} H_x(\tau),$$

$$H_c(S) = \max_{0 \leq \tau < N} H_{x,y}(\tau),$$

$$H(S) = max\{H_a(S), H_c(S)\}.$$

2.2 Wide-Gap Frequency-Hopping Sequences and One-Coincidence Frequency-Hopping Sequences

Definition 1 *For any FHS $x \in S$ and $x = (x_k)_{k=0}^{N-1}$ over a frequency set $F = \{0, 1, \ldots, l-1\}$, and a positive integer D, define*

$$|x_{k+1} - x_k| > D, k \geq 0,$$

Then x is called a wide-gap FHS set with minimum FH gap D.

Definition 2 *If an FHS set satisfies the two following simultaneously, the FHS set is called an OC-FHS set.*

(1) $H(x) = 0$, for all $x \in S$;
(2) $H(x, y) = 1$, for any two distinct FHS x and y in S at any time shift τ $(0 \leq \tau \leq N-1)$.

2.3 Optimality

An FHS set basically contains four parameters: alphabet size, sequence period, famliy size and the Hamming correlation. These parameters are not independent of each other, but mutually constrained. The relationship between these parameters can be used to measure whether the construction method is optimal. Throughout this paper, assume that an FHS with parameters $(N, l; \lambda)$ denoting the FHS of length N over F of alphabet size l with $H(x) = \lambda$ and an FHS set S with parameters $(N, M, l; \lambda)$ denoting the FHS set with M sequences of length N over F and $H_m(S) = \lambda$.

For every FHS y, Lempel and Greenberger established the following well-known lower bound [10] for $H(y)$

Lemma 1 *(Lempel-Greenberger Bound). For every FHS y with length N over F with $|F| = l$, we have*

$$H_c(y) \geq \left\lceil \frac{(N-\epsilon)(N+\epsilon-l)}{l(N-1)} \right\rceil, \tag{1}$$

where $\epsilon = N - \left\lfloor \frac{N}{l} \right\rfloor * l$.

Based on the above bound, if a FHS y satisfies Lemma 1, then y is called an optimal FHS with length N over F.

In 2004, Peng and Fan developed the following bound [11] is used to measure the optimality of a sequence set.

Lemma 2 *(Peng-Fan Bound). Let S be a set of M sequences with period N over F, we have*

$$H(S) \geq \left\lceil \frac{(NM-l)L}{(NM-1)l} \right\rceil, \tag{2}$$

An FHS set S is said to be optimal with respect to Peng-Fan bound if the Peng-Fan bound in Lemma 2 is met with equality. The sequence set S is called the optimal frequency hopping sequence set.

Later, in 2019, Li [26] et al. deduced WG-Lempel-Greenberger Bound of $H_c(y)$ for a WGFHS and the WG-Peng-Fan Bound of $H(S)$ for a WGFHS set as follows.

Lemma 3 *Let y be a WGFHS of period N over F with $|F| = l$. Then one has*

$$H_c(y) \geq \left\lceil \frac{(N-\epsilon)(N+\epsilon-l)}{l(N-3)} \right\rceil, \tag{3}$$

where ϵ is the least nonnegative residue of N modulo l.

Lemma 4 *Let S be a set of M WGFHSs of period N over F, then*

$$H(S) \geq \left\lceil \frac{(NM-l)L}{(NM-3)l} \right\rceil, \tag{4}$$

Based on the bound above, we define the optimality of a WGFHS y and a WGFHS set S in the following.

A WGFHS y is said to be an optimal frequency-hopping sequence if y achieves the Lemma 3. And a WGFHS set S is said to be an optimal frequency-hopping sequence set if the the bound in Lemma 4 is met with equality.

3. An Optimal WGFHS Set with New Parameters

In this section, we present a construction of WGFHS set with long period by algebraic means.

Construction 1 *For a positive integer w with $\frac{\sqrt{1+8p}+1}{2} \leq w \leq p-1$, the FHS set $C = \{\mathbf{c}^k = c_i^k, 1 \leq k \leq p-1, 0 \leq i \leq p-1\}$*

$$c_i^k = \begin{cases} \langle i*k \rangle, 0 \leq i \leq p-1 \\ \sum_{j=i-p}^{i+w-1-p} \langle j*k \rangle, p \leq i \leq 2p-1 \end{cases}$$

Theorem 1 *The set C in Construction 1 is an optimal set with parameters $(2p, p, w(p-w)+1+p; 2)$.*

Proof We are going to proof this Theorem:

Obviously, there are p FHSs of length $2p$ in FHS set C. Now, we consider the size of available frequency alphabet $|F|$.

Firstly, we need to ensure that the frequency points of $0 \leq i \leq p-1$ and $p \leq i \leq 2p-1$ are disjoint, so the maximum frequency point of $0 \leq i \leq p-1$ is smaller than the minimum frequency point of $p \leq i \leq 2p-1$. It can be seen that $\frac{\sqrt{1+8p}+1}{2} \leq w \leq p-1$. Thus, $|F|$ is composed of the sum of two frequency alphabet. When $0 \leq i \leq p-1$, we record the available frequency point as $|F_1|$, and when $p \leq i \leq 2p-1$, we record the available frequency point as $|F_2|$. So $|F| = |F_1| + |F_2|$.

For $0 \leq i \leq p-1$, $c_i^k = \langle i*k \rangle$, it is easily to know the frequency alphabet of $|F_1|$ is p. For $p \leq i_1 \neq i_2 \leq 2p-1$, the frequency alphabet of $c_i^k = \sum_{j=i-p}^{i+w-1-p} \langle j*k \rangle$ needs to be derived. When $p \leq i \leq 2p-1$, the element c_i^k satisfies the following inequation,

$$\sum_{j=0}^{w-1} j \leq \sum_{j=i-p}^{i+w-1-p} \langle jk \rangle \leq \sum_{j=p-w}^{p-1} j,$$

which implies that

$$\frac{w(w-1)}{2} \leq \sum_{j=i-p}^{i+w-1-p} \langle jk \rangle \leq wp - \frac{w(w+1)}{2}.$$

Obviously, one has

$$|F_2| = wp - \frac{w(w+1)}{2} - \frac{w(w-1)}{2} + 1 = w(p-w) + 1.$$

From this we can see $|F| = |F_1| + |F_2| = w(p-w) + 1 + p$.

Then, we will show that $H(C) = 2$.

Firstly, for $1 \leq k \leq p-1$, according to Lemma 1 [24] and Theorem 3.1 in [22], we know that every FHS in both $\langle i*k \rangle$ and $\sum_{j=i-p}^{i+w-1-p} \langle j*k \rangle$ are OC-FHSs. In addition, due to $\frac{\sqrt{1+8p}+1}{2} \leq w \leq p-1$, the smallest frequency point in $\sum_{j=i-p}^{i+w-1-p} \langle j*k \rangle$ is greater than the largest frequency point in $\langle i*k \rangle$. Therefore, we have $H_a(C) = 0$ for $1 \leq \tau \leq 2p-1$. As for Hamming crosscorrelation, it is sufficient to discuss the value $H_{k,y}(\tau)$ for $0 \leq \tau \leq 2p-1$ and $1 \leq k \neq y \leq p-1$. In the following, the discussion is divided into four cases:

- Case 1: When $\tau = 0$, the Hamming crosscorrelation of C is equal to

$$\begin{aligned} H_{k,y}(\tau) &= \sum_{i=0}^{2p-1} h[c_i^k, c_i^y] \\ &= \sum_{i=0}^{p-1} h[\langle i*k \rangle, \langle i*y \rangle] \\ &\quad + \sum_{i=p}^{2p-1} h[\sum_{j=i}^{i+w-1} \langle j*k \rangle, \sum_{j=i}^{i+w-1} \langle j*y \rangle] \\ &\leq 2 \end{aligned}$$

- Case 2: When $\tau = p$,

$$\begin{aligned} H_{k,y}(\tau) &= \sum_{i=0}^{p-1} h[\langle i*k \rangle, \sum_{j=i}^{i+w-1} \langle j*y \rangle] \\ &\quad + \sum_{i=p}^{2p-1} h[\sum_{j=i-p}^{i+w-1} \langle j*k \rangle, \langle i*y \rangle] \\ &= 0 \end{aligned}$$

- Case 3: When $1 \leq \tau \leq p-1$

$$\begin{aligned} H_{k,y}(\tau) &= \sum_{i=0}^{p-1-\tau} h[\langle i*k \rangle, \langle i*y \rangle] \\ &\quad + \sum_{i=p-1-\tau}^{p-1} h[\langle i*y \rangle, \sum_{j=i-p}^{i+w-1-p} \langle j*k \rangle] \\ &\quad + \sum_{i=p-1}^{2p-1-\tau} h[\sum_{j=i-p}^{i+w-1-p} \langle j*k \rangle, \sum_{j=i-p}^{i+w-1-p} \langle j*y \rangle] \\ &\quad + \sum_{i=2p-1-\tau}^{2p-1} h[\sum_{j=i-p}^{i+w-1-p} \langle j*y \rangle, \langle i*k \rangle] \\ &\leq 2 \end{aligned}$$

- Case 4: When $p < \tau \leq 2p-1$

$$H_{k,y}(\tau) = \sum_{i=0}^{\tau-p+1} h[\langle i*k \rangle, \langle i*y \rangle]$$

$$+\sum_{i=\tau-p+1}^{p-1} h[\langle i*y\rangle, \sum_{j=i}^{i+w-1}\langle j*k\rangle]$$
$$+\sum_{i=p-1}^{\tau} h[\sum_{j=i-p}^{i+w-1-p}\langle j*k\rangle, \sum_{j=i-p}^{i+w-1-p}\langle j*y\rangle]$$
$$+\sum_{i=\tau}^{2p-1} h[\sum_{j=i-p}^{i+w-1-p}\langle j*y\rangle, \langle i*k\rangle]$$
$$\leq 2$$

It is obvious that the maximum Hamming crosscorrelation of C is 2. That is $H_c(C) = 2$.

Therefore, the maximum period Hamming correlation $H(C) = 2$. Clearly, the set C with parameters $(2p, p, w(p-w)+1+p; 2)$.

Finally, we will show that C is an optimal FHS set with respect to Peng-Fan bound.

Substituting the parameters $(2p, p-1, w(p-w)+1+p)$ into Peng-Fan bound, the right side of Peng-Fan bound is

$$\left\lceil \frac{(2p(p-1)-(w(p-w)+1+p))2p}{(2p(p-1)-1)(w(p-w)+1+p)} \right\rceil = 2.$$

This implies the desired result.

Then, we need to prove that the FHS set C has the property of wide-gap. Before doing this, the definition of FHS distance $d(\boldsymbol{c}^k)$ of an FHS c^k is given as follows:

$$d(\boldsymbol{c}^k) = \min_{0\leq i\leq 2p-2}\{\left|\boldsymbol{c}_{i+1}^k - \boldsymbol{c}_i^k\right|\}$$

And then we get the following Theorem.

Theorem 2 *For any given minimum wide-gap D with $0 < D \leq \frac{p-1}{2}$ and $\frac{\sqrt{1+8p}+1}{2} \leq w \leq p-1$, the FHS set $(2p, M_W, w(p-w)+1+p; 2)$ is an optimal WGFHS set.*

Proof For any given minimum wide-gap $0 < D \leq \frac{p-1}{2}$, the number of sequence with the minimum wide-gap greater than D is recorded as M_W in the FHS set.

From Lemma 1 in [24], we can know $|\langle (i+1)k\rangle - \langle ik\rangle| = min\{k, p-k\}$, And for a given integer D with $0 < D \leq \frac{p-1}{2}$. There are a total of $p-2D-1$ frequency hopping sequences with period $l = p$ and minimum gap is greater than D. From Theorem 4.2 in [22], we know that $\left|\sum_{j=i-p}^{i+w-1-p}\langle (j+1)*k\rangle - \sum_{j=i-p}^{i+w-1-p}\langle j*k\rangle\right| = min\{\langle wk\rangle, p-\langle wk\rangle\}$. and according to Theorem 4.3 in [22], for a given integer D with $0 < D \leq \frac{p-1}{2}$, there are $p-2D-1$ FHS with gap D. So, for a given integer D, the number of sequences with the minimum wide-gap greater than D in the sequence set C is at least $p-4D-1$. From this we can see that $M_W \geq p-4D-1$.

Obviously, for $\frac{\sqrt{1+8p}+1}{2} \leq w \leq p-1$, when $0 < D \leq \frac{p-1}{2}$, the WGFHS set satisfies the WG-Lempel-Greenberger bound and WG-Peng-Fan bound optimality. Based on above situation, we can obtain a WGFHS set C_D is a $(2p, M_W, w(p-w)+1+p)$ FHS set for $0 < D \leq \frac{p-1}{2}$.

Finally, we will show that C_D is an optimal FHS set with respect to WG-Peng-Fan bound and WG-Lempel-Greenberger bound. Substituting $L = 2p, l = w(p-w)+1+p, (resp. L = p, M_W, l = w(p-w)+1+p)$ into the right side of the bound, one gets

$$\left\lceil \frac{(2p-\epsilon)(2p+\epsilon-(w(p-w)+1+p))}{(p-3)(w(p-w)+1+p)} \right\rceil = 0$$

and

$$\left\lceil \frac{(2p(p-4D-1)-(w(p-w)+1+p))2p}{(2p(p-4D-1)-3)(w(p-w)+1+p)} \right\rceil = 2.$$

This finish the proof.

4. A New Class of Optimal WG-OC-FHS Sets

In this section, we present a class of OC-FHS sets with the optimal Hamming correlation by algebraic means. Then, given a range of D, we can obtain a new class of WG-OC-FHS set, which satisfies the optimality of both WG-Peng-Fan Bound and WG-Lempel-Greenberger Bound.

Construction 2 *For a positive integer w with $2 \leq w \leq p-2$, where α is an primitive element of group $GF(p)$ and where the FHS set $B = \{\boldsymbol{b}^k = b_i^k, 0 \leq i \leq p-2, 0 \leq k \leq p-1\}$*

$$b_i^k = \sum_{j=k}^{k+w-1}\langle \alpha^i + j\rangle + k, 0 \leq i \leq p-2$$

Theorem 3 *The set B in Construction 2 is an optimal $(p-1, p, wp-w^2+p-1; 1)$ OC-FHS set.*

Proof We are going to prove this theorem:
Firstly, we will show that the elements of each FHS are all distinct. Suppose not, then there exist two different integer $0 \leq i_1 \neq i_2 \leq p-2$ such that

$$\sum_{j=k}^{k+w-1}\langle \alpha^{i_1} + j\rangle + k = \sum_{j=k}^{k+w-1}\langle \alpha^{i_2} + j\rangle + k. \tag{5}$$

Obviously, Eq. (5) also holds for module p, then we have

$$w\alpha^{i_1} \equiv w\alpha^{i_2} \bmod p,$$

which deduces that $i_1 \equiv i_2 \bmod p$.

Since $0 \leq i_1, i_2 \leq p-2$, it is easy to obtain that $i_1 = i_2$ which contradicts the hypothesis. Therefore, every FHS $\boldsymbol{b}^k$ is non-repeating that is $H_a(B) = 0$.

Then, we will show that $H_c(B) = 1$.

For $0 \leq k \leq p-1$, $\boldsymbol{b}^k$ is non-repeating. On the other hand, for $0 \leq k \neq l \leq p-1$, assume that $\boldsymbol{b}_i^k = \boldsymbol{b}_{i+\tau}^l$, we can easily obtain

$$\sum_{j=k}^{k+w-1} \langle \alpha^i + j \rangle + k = \sum_{j=l}^{l+w-1} \langle \alpha^{i+\tau} + j \rangle + l.$$

Performing module p and simplifying the above equation, it yields

$$k - l \equiv (\alpha^{i+\tau} - \alpha^i) = \alpha^i(\alpha^\tau - 1). \tag{6}$$

Note that $0 \le k \neq l \le p-1$, and by the knowledge of linear congruence, Eq. (6) has only one solution. Therefore, $H_{k,l}(X, Y) = 1$ for all $0 \le \tau \le p-2$. Prove from the above two steps that we can know that B is an OC-FHS set.

Next, we will illustrate that B is a $(p-1, p, wp - w^2 + p - 1)$ FHS set.

Obviously, B has p FHSs of length $p-1$. Now, we consider the size of available frequency alphabet $|F|$. For $0 \le i_1 \neq i_2 \le p-2$ and $0 \le j \le p-1$, it is clear that $\langle \alpha^{i_1} + j \rangle \neq \langle \alpha^{i_2} + j \rangle$. Then, for $0 \le j \le p-1$, the elements of $\boldsymbol{b}_k(i)$ satisfie the following inequation,

$$\sum_{j=0}^{w-1} j + 1 \le \sum_{j=k}^{k+w-1} \langle \alpha^i + j \rangle + k \le \sum_{j=p-w}^{p-1} \langle j \rangle + p - 1,$$

which implies that

$$\frac{w(w-1)}{2} + 1 \le \sum_{j=k}^{k+w-1} \langle \alpha^i + j \rangle + k \le wp - \frac{w(w+1)}{2} + p - 1.$$

Obviously, we can get

$$|F| = wp - \frac{w(w+1)}{2} + p - 1 - \frac{w(w-1)}{2} - 1 + 1 = wp - w^2 + p - 1.$$

Finally, we will show that B is an optimal OC-FHS set with respect to Peng-Fan Bound.

Substituting the parameters $(p-1, p, wp - w^2 + p - 1)$ into Peng-Fan Bound, the right side of Peng-Fan Bound is

$$\left\lceil \frac{(p(p-1) - (wp - w^2 + p - 1))(p-1)}{(p(p-1) - 1)(wp - w^2 + p - 1)} \right\rceil = 1. \tag{7}$$

This implies the desired result. □

Remark 1 The range of values for w in the FHS set B

- When $w = 1$, B is the prime sequence set.
- When $w = p$, do not satisfy the properties of the sequence.
- When $w = p-1$, the One-coincidence property is not satisfied.
- When $2 \le w \le p-2$, this is what we discussed.

Then, we need to prove that the OC-FHS set has the property of wide-gap. Before doing this, the definition of FHS distance $d(\boldsymbol{b}^k)$ of an FHS $\boldsymbol{b}^k$ is given as follows:

$$d(\boldsymbol{b}^k) = \min_{0 \le i \le p-3} \left\{ \left| \boldsymbol{b}_{i+1}^k - \boldsymbol{b}_i^k \right| \right\}.$$

And then we get the following Lemma.

Lemma 5 *For $0 \le \xi_1, \xi_2 \le p-1$, and $2 \le w \le p-2$, d_i represent the gap between adjacent frequency points, then the minimum wide-gap of each sequence can be expressed as $d_\xi = min\{d_i\}$.*

$$d_i = \begin{cases} |w(\xi_1 - \xi_2)|, 0 \le \xi_1, \xi_2 \le p - w \\ |(p-w)(\xi_1 - \xi_2)|, p - w < \xi_1, \xi_2 \le p - 1 \\ |(p-w)(p-\xi_1) - w\xi_2|, 0 \le \xi_2 \le p - w < \xi_1 \le p - 1 \end{cases}$$

Proof According to the Construction2, we can easily see that $\left\langle \alpha^{i_1} + j \right\rangle + k \neq \left\langle \alpha^{i_2} + j \right\rangle + k$, for $0 \le i_1 \neq i_2 \le p-2$, and $0 \le k \le p-1$. Therefore, we define the distance between consecutive element $\boldsymbol{b}_{i+1}^k$ and $\boldsymbol{b}_i^k$ of FHS $\boldsymbol{b}^k$. According to the property of the structure, it can be known that the wide-gap can be expressed in the form of arithmetic progression. Hence, we have

$$\begin{aligned} d(\boldsymbol{b}^k) &= \min_{0 \le i \le p-2} \{\boldsymbol{b}_{i+1}^k - \boldsymbol{b}_i^k\} \\ &= (\sum_{j=k}^{k+w-1} \left\langle \alpha^{i+1} + j \right\rangle + k) - (\sum_{j=k}^{k+w-1} \left\langle \alpha^i + j \right\rangle + k) \\ &= \sum_{j=k}^{k+w-1} \left\langle \alpha^{i+1} + j \right\rangle - \sum_{j=k}^{k+w-1} \left\langle \alpha^i + j \right\rangle \end{aligned} \tag{8}$$

Let $\left\langle \alpha^i + j \right\rangle = \xi_1$, $\left\langle \alpha^{i+1} + j \right\rangle = \xi_2$. We can know that ξ_1, ξ_2 can traverse the number of $0, 1, 2, \ldots, p-1$. Then we should discuss Eq. (8) in the following cases. It can be seen from the structure that each $\boldsymbol{b}_i^k$ can be expressed in the form of arithmetic progression

Case 1): $0 \le \xi_1, \xi_2 \le p - w$,

$$\begin{aligned} D_1 &= \left| \frac{w(w-1+2\xi_1)}{2} - \frac{w(w-1+2\xi_2)}{2} \right| \\ &= \left| w(\xi_1 - \xi_2) \right| \end{aligned}$$

Easily to know that the minimize value of D_1 is w.

Case 2): $p - w < \xi_1, \xi_2 \le p - 1$, can be seen

$$\begin{aligned} D_2 &= \left| \frac{(p-\xi_1)(p+\xi_1-1) + (\xi_1-p+w)(\xi_1-p+w-1)}{2} \right. \\ &\quad \left. - \frac{(p-\xi_2)(p+\xi_2-1) + (\xi_2-p+w)(\xi_2-p+w-1)}{2} \right| \\ &= \left| (\xi_1 - \xi_2)(w - p) \right| \end{aligned}$$

we can know that the minimize value of D_2 is $p - w$.

Case 3): $0 \le \xi_2 \le p - w < \xi_1 \le p - 1$, the result is:

$$\begin{aligned} D3 &= \left| \frac{(p-\xi_1)(p+\xi_1-1) + (\xi_1-p+w)(\xi_1-p+w-1)}{2} \right. \\ &\quad \left. - \frac{w(w-1+2\xi_2)}{2} \right| \\ &= \left| (p-w)(p-\xi_1) - w\xi_2 \right| \end{aligned}$$

Case 4): $0 \le \xi_1 \le p - w < \xi_2 \le p - 1$,

$$\begin{aligned} D4 &= \left| \frac{(p-\xi_2)(p+\xi_2-1) + (\xi_2-p+w)(\xi_2-p+w-1)}{2} \right. \\ &\quad \left. - \frac{w(w-1+2\xi_1)}{2} \right| \\ &= \left| (p-w)(p-\xi_2) - w\xi_1 \right| \end{aligned}$$

From the result of Case 4, it can be observed that when $0 \le \xi_1 \le p-w < \xi_2 \le p-1$ or $0 \le \xi_2 \le p-w < \xi_1 \le p-1$, the obtained result is the same. Since ξ_1 and ξ_2 represent two in the group p, we can combine Case 3 and Case 4 into one case. Summarizing the above conclusion, we can easily get

$$d_i = \begin{cases} |w(\xi_1 - \xi_2)|, 0 \le \xi_1, \xi_2 \le p-w \\ |(p-w)(\xi_1 - \xi_2)|, p-w < \xi_1, \xi_2 \le p-1 \\ |(p-w)(p-\xi_1) - w\xi_2|, 0 \le \xi_2 \le p-w < \xi_1 \le p-1 \end{cases}$$

□

Theorem 4 *For any given minimum wide-gap D with $2 \le D \le \left\lfloor \frac{p}{4} \right\rfloor$ and $D \ne min\{w, p-w\}$, the set $(p-1, M_{WG}, wp - w^2 + p - 1; 1)$ is an optimal WG-OC-FHS set.*

Proof For any given minimum wide-gap ($2 \le D \le \left\lfloor \frac{p}{4} \right\rfloor$), the number of sequences with the minimum wide-gap greater than D is recorded as M_{WG} in the OC-FHS set.

According to Lemma 5, it can be observed that when $0 \le \xi_1, \xi_2 \le p-w$, the minimum value of d_i occurs at $|\xi_1 - \xi_2| = 1$, and in this case, the minimum value is w. When $p-w < \xi_1, \xi_2 \le p-1$, the minimum value of d_i is achieved at $|\xi_1 - \xi_2| = 1$, resulting in a minimum value of $p-w$. When $D = w$ and $D = p-w$, there are very few frequency hopping sequences with a minimum gap greater than D. Therefore, we restrict the values of $D \ne min\{w, p-w\}$.

Since the range of w is $2 \le w \le p-2$, the situation where the minimum interval is 1 only exist in case 3 and case 4. According to mathematical knowledge, it is known that there can be at most four sets of solutions that satisfy the minimum gap of 1. By following the same reasoning, there can be at most four sets of solutions that satisfy the minimum gap of 2. Based on the maximum occurrence, we can conclude that when the minimum gap is D, there can be at most 4D sequences in the sequence set that have a minimum gap less than or equal to D. Furthermore, there must be at least p-4D sequences that satisfy the wide gap property. It can be concluded that when $2 \le D \le \left\lfloor \frac{p}{4} \right\rfloor$ satisfies $M_{WG} \ge (p-4D)$.

Based on above situations, we can obtain a WG-OC-FHS set C is a $(p-1, M_{WG}, 1; wp - w^2 + p - 1)$ FHS set for $2 \le D \le \left\lfloor \frac{p}{4} \right\rfloor$.

At last we will show that C is optimal with respect to bounds (3), (4). Substituting $L = p-1$, $l = wp - w^2 + p - 1$ into the right side of the bound (3), we get

$$\left\lceil \frac{(p-1-\epsilon)(p-1+\epsilon-(wp-w^2+p-1))}{(p-4)(wp-w^2+p-1)} \right\rceil = 0,$$

Similarly, substituting $L = p-1$, M_{WG}, $l = wp - w^2 + p - 1$ into the right side of bounds (4). If the boundary value of M_{WG} can make the equation hold, then all the value of M_{WG} can make it true.

$$\left\lceil \frac{((p-1)(p-4D)-(wp-w^2+p-1))(p-1)}{((p-1)(p-4D)-3)(wp-w^2+p-1)} \right\rceil = 1,$$

The proof is finished. □

5. Example and Discussion

We will use two examples to illustrate the two constructions in this paper. Example 1 is derived based on Sect. 3, while Example 2 is obtained from Sect. 4.

Example 1. According to Construction 1, let $p = 17$, $D = 2$ and $w = 7$, the FHS set $C = \boldsymbol{c}_k$, $1 \le k \le p-1$ with parameters $(34, 16, 88)$ is given in Table 1. And we know that C is optimal with respect to Peng-Fan bound. (Lemma 2).

In addition, we take FHS gap $D = 2$, the sequences with $D > 2$ are $\{\boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6, \boldsymbol{c}_8, \boldsymbol{c}_9, \boldsymbol{c}_{11}, \boldsymbol{c}_{13}, \boldsymbol{c}_{14}\}$, we get a WGFHS set C_D as shown in the following Table 2. It is clear that $C_D = \{\boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_6, \boldsymbol{c}_8, \boldsymbol{c}_9, \boldsymbol{c}_{11}, \boldsymbol{c}_{13}, \boldsymbol{c}_{14}\}$ is a $(34, 8, 88)$ WGFHS set with FHS gap $D > 2$. And we can verify to know that it is optimal with respect to WG-Lempel-Greenberger bound (Lemma 3) and WG-Peng-Fan bound (Lemma 4) as well.

Example 2. Based on Construction 2, let $p = 19$, $\alpha = 3$ and $w = 7$, the OC-FHS set $B = \boldsymbol{b}_k, 0 \le k \le 18$ with

Table 1 The optimal set C.

$\boldsymbol{c}_k$	Frequencies	$d(\boldsymbol{c}_k)$
$\boldsymbol{c}_1$	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 81, 71, 61, 51, 41, 31)	1
$\boldsymbol{c}_2$	(0, 2, 4, 6, 8, 10, 12, 14, 16, 1, 3, 5, 7, 9, 11, 13, 15, 42, 56, 70, 67, 64, 61, 58, 55, 52, 49, 63, 60, 57, 54, 51, 48, 45)	2
$\boldsymbol{c}_3$	(0, 3, 6, 9, 12, 15, 1, 4, 7, 10, 13, 16, 2, 5, 8, 11, 14, 46, 50, 54, 58, 62, 66, 53, 57, 61, 65, 69, 56, 43, 47, 51, 55, 59)	3
$\boldsymbol{c}_4$	(0, 4, 8, 12, 16, 3, 7, 11, 15, 2, 6, 10, 14, 1, 5, 9, 13, 50, 61, 72, 66, 60, 54, 65, 59, 53, 47, 58, 52, 46, 40, 51, 62, 56)	4
$\boldsymbol{c}_5$	(0, 5, 10, 15, 3, 8, 13, 1, 6, 11, 16, 4, 9, 14, 2, 7, 12, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 48, 49, 50, 51, 52, 53)	1
$\boldsymbol{c}_6$	(0, 6, 12, 1, 7, 13, 2, 8, 14, 3, 9, 15, 4, 10, 16, 5, 11, 41, 49, 57, 48, 56, 64, 55, 63, 71, 62, 70, 61, 52, 60, 51, 42, 50)	6
$\boldsymbol{c}_7$	(0, 7, 14, 4, 11, 1, 8, 15, 5, 12, 2, 9, 16, 6, 13, 3, 10, 45, 60, 58, 56, 54, 52, 67, 65, 63, 61, 59, 57, 55, 53, 51, 49, 47)	2
$\boldsymbol{c}_8$	(0, 8, 16, 7, 15, 6, 14, 5, 13, 4, 12, 3, 11, 2, 10, 1, 9, 66, 71, 76, 64, 69, 57, 62, 50, 55, 43, 48, 36, 41, 46, 51, 56, 61)	5
$\boldsymbol{c}_9$	(0, 9, 1, 10, 2, 11, 3, 12, 4, 13, 5, 14, 6, 15, 7, 16, 8, 36, 48, 43, 55, 50, 62, 57, 69, 64, 76, 71, 66, 61, 56, 51, 46, 41)	5
$\boldsymbol{c}_{10}$	(0, 10, 3, 13, 6, 16, 9, 2, 12, 5, 15, 8, 1, 11, 4, 14, 7, 57, 59, 61, 63, 65, 67, 52, 54, 56, 58, 60, 45, 47, 49, 51, 53, 55)	2
$\boldsymbol{c}_{11}$	(0, 11, 5, 16, 10, 4, 15, 9, 3, 14, 8, 2, 13, 7, 1, 12, 6, 61, 70, 62, 71, 63, 55, 64, 56, 48, 57, 49, 41, 50, 42, 51, 60, 52)	6
$\boldsymbol{c}_{12}$	(0, 12, 7, 2, 14, 9, 4, 16, 11, 6, 1, 13, 8, 3, 15, 10, 5, 48, 64, 63, 62, 61, 60, 59, 58, 57, 56, 55, 54, 53, 52, 51, 50, 49)	1
$\boldsymbol{c}_{13}$	(0, 13, 9, 5, 1, 14, 10, 6, 2, 15, 11, 7, 3, 16, 12, 8, 4, 52, 58, 47, 53, 59, 65, 54, 60, 66, 72, 61, 50, 56, 62, 51, 40, 46)	4
$\boldsymbol{c}_{14}$	(0, 14, 11, 8, 5, 2, 16, 13, 10, 7, 4, 1, 15, 12, 9, 6, 3, 56, 69, 65, 61, 57, 53, 66, 62, 58, 54, 50, 46, 59, 55, 51, 47, 43)	3
$\boldsymbol{c}_{15}$	(0, 15, 13, 11, 9, 7, 5, 3, 1, 16, 14, 12, 10, 8, 6, 4, 2, 60, 63, 49, 52, 55, 58, 61, 64, 67, 70, 56, 42, 45, 48, 51, 54, 57)	2
$\boldsymbol{c}_{16}$	(0, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 81, 91, 84, 77, 70, 63, 56, 49, 42, 35, 28, 21, 31, 41, 51, 61, 71)	1

Table 2 The optimal WGFHS set C_D with FHS gap $D > 2$.

$\boldsymbol{c}_k$	Frequencies	$d(\boldsymbol{c}_k)$
$\boldsymbol{c}_3$	(0, 3, 6, 9, 12, 15, 1, 4, 7, 10, 13, 16, 2, 5, 8, 11, 14, 46, 50, 54, 58, 62, 66, 53, 57, 61, 65, 69, 56, 43, 47, 51, 55, 59)	3
$\boldsymbol{c}_4$	(0, 4, 8, 12, 16, 3, 7, 11, 15, 2, 6, 10, 14, 1, 5, 9, 13, 50, 61, 72, 66, 60, 54, 65, 59, 53, 47, 58, 52, 46, 40, 51, 62, 56)	4
$\boldsymbol{c}_6$	(0, 6, 12, 1, 7, 13, 2, 8, 14, 3, 9, 15, 4, 10, 16, 5, 11, 41, 49, 57, 48, 56, 64, 55, 63, 71, 62, 70, 61, 52, 60, 51, 42, 50)	6
$\boldsymbol{c}_8$	(0, 8, 16, 7, 15, 6, 14, 5, 13, 4, 12, 3, 11, 2, 10, 1, 9, 66, 71, 76, 64, 69, 57, 62, 50, 55, 43, 48, 36, 41, 46, 51, 56, 61)	5
$\boldsymbol{c}_9$	(0, 9, 1, 10, 2, 11, 3, 12, 4, 13, 5, 14, 6, 15, 7, 16, 8, 36, 48, 43, 55, 50, 62, 57, 69, 64, 76, 71, 66, 61, 56, 51, 46, 41)	5
$\boldsymbol{c}_{11}$	(0, 11, 5, 16, 10, 4, 15, 9, 3, 14, 8, 2, 13, 7, 1, 12, 6, 61, 70, 62, 71, 63, 55, 64, 56, 48, 57, 49, 41, 50, 42, 51, 60, 52)	6
$\boldsymbol{c}_{13}$	(0, 13, 9, 5, 1, 14, 10, 6, 2, 15, 11, 7, 3, 16, 12, 8, 4, 52, 58, 47, 53, 59, 65, 54, 60, 66, 72, 61, 50, 56, 62, 51, 40, 46)	4
$\boldsymbol{c}_{14}$	(0, 14, 11, 8, 5, 2, 16, 13, 10, 7, 4, 1, 15, 12, 9, 6, 3, 56, 69, 65, 61, 57, 53, 66, 62, 58, 54, 50, 46, 59, 55, 51, 47, 43)	3

Table 3 The optimal OC-FHS set.

b_k	Frequencies	$d(\boldsymbol{b}_k)$
b_0	(28, 42, 84, 77, 56, 69, 70, 35, 63, 33, 57, 91, 98, 81, 49, 105, 45, 93)	1
b_1	(36, 50, 92, 85, 64, 58, 78, 43, 71, 22, 46, 99, 106, 70, 57, 94, 34, 82)	6
b_2	(44, 58, 100, 93, 72, 47, 86, 51, 79, 30, 35, 107, 95, 59, 65, 83, 23, 71)	5
b_3	(52, 66, 108, 101, 80, 36, 94, 59, 87, 38, 24, 96, 84, 48, 73, 72, 31, 60)	1
b_4	(60, 74, 97, 109, 88, 25, 102, 67, 95, 46, 32, 85, 73, 37, 81, 61, 39, 49)	10
b_5	(68, 82, 86, 98, 96, 33, 110, 75, 103, 54, 40, 74, 62, 26, 89, 50, 47, 38)	2
b_6	(76, 90, 75, 87, 104, 41, 99, 83, 111, 62, 48, 63, 51, 34, 97, 39, 55, 27)	12
b_7	(84, 98, 64, 76, 112, 49, 88, 91, 100, 70, 56, 52, 40, 42, 105, 28, 63, 35)	2
b_8	(92, 106, 53, 65, 101, 57, 77, 99, 89, 78, 64, 41, 29, 50, 113, 36, 71, 43)	10
b_9	(100, 114, 42, 54, 90, 65, 66, 107, 78, 86, 72, 30, 37, 58, 102, 44, 79, 51)	1
b_{10}	(108, 103, 31, 43, 79, 73, 55, 115, 67, 94, 80, 38, 45, 66, 91, 52, 87, 59)	5
b_{11}	(116, 92, 39, 32, 68, 81, 44, 104, 56, 102, 88, 46, 53, 74, 80, 60, 95, 67)	6
b_{12}	(105, 81, 47, 40, 57, 89, 33, 93, 45, 110, 96, 54, 61, 82, 69, 68, 103, 75)	1
b_{13}	(94, 70, 55, 48, 46, 97, 41, 82, 34, 118, 104, 62, 69, 90, 58, 76, 111, 83)	2
b_{14}	(83, 59, 63, 56, 35, 105, 49, 71, 42, 107, 112, 70, 77, 98, 47, 84, 119, 91)	4
b_{15}	(72, 48, 71, 64, 43, 113, 57, 60, 50, 96, 120, 78, 85, 106, 36, 92, 108, 99)	3
b_{16}	(61, 37, 79, 72, 51, 121, 65, 49, 58, 85, 109, 86, 93, 114, 44, 100, 97, 107)	3
b_{17}	(50, 45, 87, 80, 59, 110, 73, 38, 66, 74, 98, 94, 101, 122, 52, 108, 86, 115)	4
b_{18}	(39, 53, 95, 88, 67, 99, 81, 46, 74, 63, 87, 102, 109, 111, 60, 116, 75, 123)	2

Table 4 The optimal WG-OC-FHS set C with gap $D > 3$.

b_k	Frequencies	$d(\boldsymbol{b}_k))$
b_1	(35, 49, 91, 84, 63, 57, 77, 42, 70, 21, 45, 98, 105, 69, 56, 93, 33, 81)	6
b_2	(42, 56, 98, 91, 70, 45, 84, 49, 77, 28, 33, 105, 93, 57, 63, 81, 21, 69)	5
b_4	(56, 70, 93, 105, 84, 21, 98, 63, 91, 42, 28, 81, 69, 33, 77, 57, 35, 45)	10
b_6	(70, 84, 69, 81, 98, 35, 93, 77, 105, 56, 42, 57, 45, 28, 91, 33, 49, 21)	12
b_8	(84, 98, 45, 57, 93, 49, 69, 91, 81, 70, 56, 33, 21, 42, 105, 28, 63, 35)	10
b_{10}	(98, 93, 21, 33, 69, 63, 45, 105, 57, 84, 70, 28, 35, 56, 81, 42, 77, 49)	5
b_{11}	(105, 81, 28, 21, 57, 70, 33, 93, 45, 91, 77, 35, 42, 63, 69, 49, 84, 56)	6
b_{14}	(83, 59, 63, 56, 35, 105, 49, 71, 42, 107, 112, 70, 77, 98, 47, 84, 119, 91)	4
b_{17}	(50, 45, 87, 80, 59, 110, 73, 38, 66, 74, 98, 94, 101, 122, 52, 108, 86, 115)	4

Table 5 Comparison of parameters with other result.

Parameters		**Optimality**				***Reference***
(L,M,l;λ)	***Constraints***	***L-G Bound***	***P-F Bound***	***WG-L-G Bound***	***WG-P-F Bound***	
$(p-1, p, p; 1)$	p is a prime	optimal	optimal	not	not	[27]
$(p, p-1, p; 1)$		optimal	optimal	not	not	[16], [27]
$(p^2-p, p, p^2; 1)$		optimal	optimal	not	not	[15]
$(u, \eta, l; 1)$	$n = \prod_{i=1}^{r}(q_i - 1), l = \prod_{i=1}^{r}(q_i), \eta = min\{q_i : 1 \leq i \leq r\}$	optimal	optimal	not	not	[18]
$(p, p-1-2D, w(p-w)+1;1)$	$1 \leq w \leq p-1$ and p is a odd prime, $0 < D \leq \frac{p-1}{2}$	optimal	optimal	optimal	optimal	[22]
$(p-1, M_{WG}, wp-w^2+p-1; 1)$	$2 \leq w \leq p-2$, p is a prime and $p > 4$	optimal	optimal	optimal	optimal	Theorem 4
$(2p, 1, p; 2)$	p is a positive integer	optimal	-	optimal	-	[23]
$(3p, 1, p; 3)$		optimal	-	optimal	-	
$(2p, M_W, w(p-w)+1+p;2)$	$\frac{\sqrt{1+8p}+1}{2} \leq w \leq p-1$, p is a prime and $p > 5$	optimal	optimal	optimal	optimal	Theorem 2

parameters $(18, 19, 102)$ is given in Table 3.

Form Theorem 3 we know that $H_a(B) = 0$, and $H_c(B) = 1$. Therefore, B is optimal with respect to Peng-Fan Bound. (Lemma 2) and Lempel-Greenberger Bound (Lemma 1). In addition, we take FHS gap $D = 3$, the sequences with $D > 3$ are $\{b_1, b_2, b_4, b_6, b_8, b_{10}, b_{11}, b_{14}, b_{17}\}$, we get a WG-OC-FHS set C as shown in the following Table 4.

It is clear that $C = \{b_1, b_2, b_4, b_6, b_8, b_{10}, b_{11}, b_{14}, b_{17}\}$ is a $(18, 9, 102)$ WG-OC-FHS set with FHS gap $D > 3$. And we can verify to know that it is optimal with respect to WG-Lempel-Greenberger Bound (Lemma 3) and WG-Peng-Fan Bound (Lemma 4) as well.

We compare the parameters obtained from the two construction methods with the existing parameters and present the results in Table 5. Compared to references [15], [16], [18], [27], the Construction 2 exhibit the wide-gap property and are optimal with respect to the WG-Lempel-Greenberger bound and WG-Peng-Fan bound. In contrast to reference [22], this construction methods obtain a sequence set with different parameters that satisfies optimality. Additionally, the resulting sequence set demonstrates randomness. The method in Construction 1 can generate a class of long-period hopping sequence sets, which better satisfy the requirement of having a larger number of sequences in communication systems.

6. Conclusion

This paper mainly constructs two types of wide-gap frequency hopping sequence sets. The first type is the one-coincidence wide-gap frequency hopping sequence set, which exhibits new parameters, good randomness, and optimality with respect to the WG-Peng-Fan bound. Within this sequence set, each sequence is optimal with respect to the WG-Lempel-Greenberger bound. The another type is the long-period wide-gap frequency hopping sequence set. Using the same parameters, multiple long-period wide-gap frequency hopping sequences are generated using this construction method. Each sequence within this set is optimal with respect to the WG-Lempel-Greenberger bound, and the set of these sequences is optimal with respect to the WG-Peng-Fan bound.

Acknowledgments

The work of this paper was supported by the National Science of China (No.62171387), the China Postdoctoral Science Foundation (No.2019M663475) and Xihua University Graduate Innovation Foundation (No.YCJJ2020019).

References

[1] W. Pan, S.D. Zhou, and Y. Yao, “Performance analysis of differential frequency hopping communication system,” ACTA ELECTRONICA SINICA, 1999.

[2] L.M. Surhone, M.T. Tennoe, and S.F. Henssonow, Bluetooth Special Interest Group, Bluetooth Special Interest Group, 2010.

[3] J. Lansford, A. Stephens, and R. Nevo, “Wi-Fi (802.11b) and bluetooth: Enabling coexistence,” IEEE Netw., vol.15, no.5, pp.20–27, 2001.

[4] K.W. Gurgel, H.H. Essen, and S.P. Kingsley, “High-frequency radars: Physical limitations and recent developments,” Coastal Engineering, vol.37, no.3–4, pp.201–218, 1999.

[5] H. Olofsson, J. Naslund, and J. Skold, “Interference diversity gain in frequency hopping GSM,” 1995 IEEE 45th Vehicular Technology Conference. Countdown to the Wireless Twenty-First Century, pp.102–106 1995.

[6] M. Simon, J. Omura, R. Scholtz, and B. Levitt, Spread Spectrum Communications Handbook, McGraw-Hill, 2002.

[7] D.V. Sarwate, “Reed-Solomon codes and the design of sequences for spread-spectrum multiple-access communications,” Reed-Solomon Codes and Their Applications, Wiley-IEEE Press, 1994.

[8] W. Mei and X. Chen, “Families of frequency-hopping sequences with optimal hamming correlation properties,” Journal of National University of Defense Technology, 1988.

[9] G. Ge, Y. Miao, and Z. Yao, “Optimal frequency hopping sequences: Auto- and cross-correlation properties,” IEEE Trans. Inf. Theory, vol.55, no.2, pp.867–879, 2009.

[10] A. Lempel and H. Greenberger, “Families of sequences with optimal hamming-correlation properties,” IEEE Trans. Inf. Theory, vol.20, no.1, pp.90–94, 1974.

[11] D. Peng and P. Fan, “Lower bounds on the hamming auto- and cross correlations of frequency-hopping sequences,” IEEE Trans. Inf. Theory, vol.50, no.9, pp.2149–2154, 2004.

[12] J.H. Chung and K. Yang, “k-fold cyclotomy and its application to frequency-hopping sequences,” IEEE Trans. Inf. Theory, vol.57, no.4, pp.2306–2317, 2011.

[13] Y. Yang, X. Tang, U. Parampalli, and D. Peng, “New bound on frequency hopping sequence sets and its optimal constructions,” IEEE Trans. Inf. Theory, vol.57, no.11, pp.7605–7613, 2011.

[14] X. Niu, C. Xing, Y. Liu, and L. Zhou, “A construction of optimal frequency hopping sequence set via combination of multiplicative and additive groups of finite fields,” IEEE Trans. Inf. Theory, vol.66, no.8, pp.5310–5315, 2020.

[15] T.H. Lee, H.H. Jung, and J.H. Chung, “A new one-coincidence frequency-hopping sequence set of length p2 - p,” 2018 IEEE Information Theory Workshop (ITW), 2018.

[16] A. Shaar and P. Davies, “Prime sequences: Quasi-optimal sequences for or channel code division multiplexing,” Electron. Lett., vol.19, no.21, pp.888–890, 1983.

[17] W. Hong and H. Ping, “Construction of a one-coincidence frequency-hopping sequence set with optimal performance,” Proc. Second International Conference on Mechatronics and Automatic Control, Lecture Notes in Electrical Engineering, vol.334, pp.915–923, 2015.

[18] X. Niu and C. Xing, “New extension constructions of optimal frequency-hopping sequence sets,” IEEE Trans. Inf. Theory, vol.65, no.9, pp.5846–5855, 2019.

[19] L. Guan, Z. Li, J. Si, and R. Gao, “Generation and characteristics analysis of cognitive-based high-performance wide-gap FH sequences,” IEEE Trans. Veh. Technol., vol.64, no.11, pp.5056–5069, 2015.

[20] H. Zhang, “Design and performance analysis of frequency hopping sequences with given minimum gap,” 2010 International Conference on Microwave and Millimeter Wave Technology, 2010.

[21] F. Hong and S. Zhang, “A study of hopping patterns with broad intervals,” J. Chengdu Inst. Telecommun. Eng., vol.2, pp.10–16, 1985.

[22] W. Ren and F. Wang, “A new class of optimal wide-gap one-coincidence frequency-hopping sequence sets,” Advances in Mathematics of Communications, vol.17, no.2, pp.342–352, 2023.

[23] P. Li, C. Fan, S. Mesnager, Y. Yang, and Z. Zhou, “Constructions of optimal uniform wide-gap frequency-hopping sequences,” IEEE Trans. Inf. Theory, vol.68, no.1, pp.692–700, 2021.

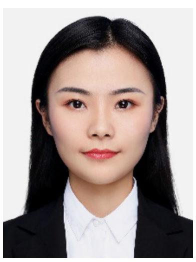
[24] W. Mei and Y. Yang, “Families of frequency hopping sequences with

given minimum gap," Journal of China Institute of Communications, vol.18, no.5, pp.37–44, 1997.

[25] Z. Huaqing, "Frequency hopping sequences with given minimum gap," 2010 2nd International Conference on Future Computer and Communication, pp.V3-611–V3-614, 2010.

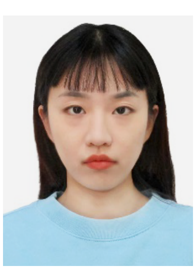
[26] P. Li, C. Fan, and Y. Yang, "New bounds on the partial hamming correlation of wide-gap frequency-hopping sequence sets with low hit zone," IEEE Access, vol.7, pp.167409–167419, 2019.

[27] A. Shaar and P. Davies, "A survey of one-coincidence sequences for frequency-hopped spread-spectrum systems," IEE Proc. F (Communications, Radar and Signal Processing), vol.131, no.7, pp.719–724, 1984.

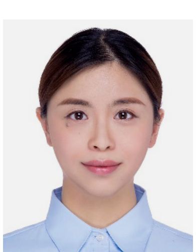


Ting Wang received the B.Eng. degree from Xihua University, Chengdu, China, in 2020. She is currently pursuing the master's degree with the School of Computer and Software Engineering, Xihua University, Chengdu. Her research interests include sequence design and coding theory.

Xianhua Niu received the B.S. degree in communication engineering and the Ph.D. degree in information security from Southwest Jiaotong University, Chengdu, China, in 2006 and 2012, respectively. She is currently a Professor with the School of Computer and Software Engineering, Xihua University, and a Post-Doctoral Member with the National Key Laboratory of Science and Technology on Communications, University of Electronic Science and Technology of China, Chengdu. Her research interests include sequence design and coding theory.



Yaoxuan Wang received the B.Eng. degree from Chengdu College of University of Electronic Science and Technology of China, in 2020. She is currently pursuing the master's degree with the School of Computer and Software Engineering, Xihua University, Chengdu. Her research interests include structural sequence design, data link.



Jianhong Zhou is currently an Associate Professor with the School of Computer and Software Engineering, Xihua University, China. She also holds a postdoctoral position with the National Key Laboratory of Science and Technology on Communications, UESTC. Her research interests include next generation cellular networks, blockchain in IoT, edge intelligence and so on.

Ling Xiong received the M.S. and Ph.D. degrees in Information security from the School of Information Science and Technology of Southwest Jiaotong University (SWJTU), Chengdu, PR China. She is an associate professor fellow in school of computer and software engineering, Xihua university. She is currently pursuing the postdoctoral research in the the School of Computer Science and Engineering, University of Electronic Science and Technology of China (UESTC), Chengdu, PR China. Her research interests include the security and privacy in Internet of Things and blockchain.