LETTER

On the Optimality of Gabidulin-Based LRCs as Codes with Multiple Local Erasure Correction

Geonu KIM[†*a)], ***Nonmember and*** **Jungwoo LEE**[†b)], ***Member***

SUMMARY The Gabidulin-based locally repairable code (LRC) construction by Silberstein et al. is an important example of distance optimal (r, δ)-LRCs. Its distance optimality has been further shown to cover the case of multiple (r, δ)-locality, where the (r, δ)-locality constraints are different among different symbols. However, the optimality only holds under the ordered (r, δ) condition, where the parameters of the multiple (r, δ)-locality satisfy a specific ordering condition. In this letter, we show that Gabidulin-based LRCs are still distance optimal even without the ordered (r, δ) condition.

key words: *locally repairable codes, multiple locality, local erasure correction, Gabidulin code*

1. Introduction

Locally repairable codes (LRCs) have been devised to mitigate the poor *repair* efficiency of conventional erasure codes in distributed storage systems [1]. LRCs have been first introduced in [2] by constraining the number of symbols required for the repair of a symbol, i.e., correction of the symbol erasure, to be at most the *locality* r. The notion of (r, δ)-locality [3], [4] further extends the conventional r-locality by imposing a more general constraint $\delta \geq 2$ on the minimum distance of the punctured local codes.

Recently, interests have arisen in having different locality constraints on different symbols [5]–[8]. In particular, the notion of multiple r-locality has been introduced in [5], and further extended to *multiple* (r, δ)*-locality* [6]. Especially, the LRC construction based on Gabidulin codes, originally proposed in [9] and extended in [7], [8], has been shown to be distance optimal even under a slightly more general problem setting referred as *unequal* (r, δ)*-locality*, given that a certain order in the parameters of the multiple (r, δ)-locality is satisfied [8].

1.1 Contribution and Organization

Our contribution is given by the following theorem. The proof will be discussed in Sect. 3.

Theorem 1: Gabidulin-based (r, δ)-LRCs are distance optimal LRCs with multiple (r, δ)-locality.

The distance optimality of Gabidulin-based (r, δ)-LRCs for unequal (r, δ)-locality under the ordered (r, δ) condition [8] is also valid for multiple (r, δ)-locality, since multiple (r, δ)-locality is a special case of unequal (r, δ)-locality with an additional *disjointness* constraint such that both the symbol to be repaired and the symbols used in the repair are specified with the same (r, δ) parameter**, and the Gabidulin-based (r, δ)-LRCs satisfy that disjointness constraint. Theorem 1 generalizes the distance optimality of Gabidulin-based (r, δ)-LRCs beyond the case of the ordered (r, δ) condition***. It can be useful for heterogeneous distributed storage systems, where some storage clusters can tolerate higher repair bandwidth. Using longer local codes with larger locality in such clusters will reduce overall storage overhead, even if local distance is also increased accordingly in order to preserve local failure protection capability.

The remainder of this letter is organized as follows. In Sect. 2, some important preliminaries are provided. Section 3 presents the detailed proof of Theorem 1.

2. Background

2.1 Notation

The following notation is used throughout this letter.

- For an integer i, $[i] = \{1, \ldots, i\}$.
- For the sets $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \sqcup \mathcal{Y}$ denotes the disjoint union. In other words, the usage of $\mathcal{A} \sqcup \mathcal{B}$ implies $\mathcal{A} \cap \mathcal{B} = \emptyset$.
- For a code $\mathscr{C}$ of length n, the punctured code with support $\mathcal{T} \subset [n]$ and the corresponding generator matrix are denoted as $\mathscr{C}|_{\mathcal{T}}$ and $G|_{\mathcal{T}}$, respectively. Furthermore, $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) = \mathrm{rank}(G|_{\mathcal{T}})$.
- For a polynomial evaluation code $\mathscr{C}$ of length n, where the evaluation points lie in an extension field, $\mathrm{rank}_{\mathrm{E}}(\mathcal{T})$ denotes the rank of the evaluation points indexed by $\mathcal{T} \subset [n]$ over the base field.

2.2 LRCs with Multiple Local Erasure Correction

Let us begin with the following definition on LRCs with multiple (r, δ)-locality. (See also [6].)

Definition 1: Let $[n] = \bigsqcup_{j=1}^{s^*} \mathcal{N}_j$ and $|\mathcal{N}_j| = n_j$, $j \in [s^*]$.

Manuscript received December 29, 2018.
Manuscript revised April 29, 2019.

†The authors are with INMAC, Department of Electrical & Computer Engineering, Seoul National University, Seoul, Korea.
*Presently, with SK Hynix, Icheon 17336, South Korea.
a) E-mail: geonu.kim@sk.com
b) E-mail: junglee@snu.ac.kr

DOI: 10.1587/transfun.E102.A.1326

**The disjointness constraint is denoted by the condition $\mathcal{S}_i \subset \mathcal{N}_j$ in Definition 1.

***The work in [6] is also restricted to the ordered (r, δ) condition.

Copyright © 2019 The Institute of Electronics, Information and Communication Engineers

A linear $[n, k]$ code $\mathscr{C}$ is said to have multiple (r, δ)-locality with parameters $\{(n_j, r_j, \delta_j)\}_{j\in[s^*]}$, if for every symbol with index $i \in \mathcal{N}_j$, $j \in [s^*]$, there exists a symbol index set $\mathcal{S}_i \subset \mathcal{N}_j$ such that

- $i \in \mathcal{S}_i$,
- $|\mathcal{S}_i| \le r_j + \delta_j - 1$,
- $d(\mathscr{C}|_{\mathcal{S}_i}) \ge \delta_j$.

Furthermore, define

- integers p_j, q_j such that $n_j = p_j(r_j + \delta_j - 1) + q_j$ and $0 \le q_j \le r_j + \delta_j - 2$,
- $m_j \triangleq \dfrac{n_j}{r_j + \delta_j - 1} = p_j + \dfrac{q_j}{r_j + \delta_j - 1}$,
- $k_j \triangleq \begin{cases} \lfloor m_j \rfloor r_j & \text{if } 0 \le q_j \le \delta_j - 2, \\ n_j - \lceil m_j \rceil(\delta_j - 1) & \text{if } \delta_j - 1 \le q_j \le r_j + \delta_j - 2. \end{cases}$

We also have the following remark.

Remark 1: In Definition 1, applying the Singleton bound to $\mathscr{C}|_{\mathcal{S}_i}$ gives $\mathrm{rank}_{\mathrm{G}}(\mathcal{S}_i) \le r_j$.

2.3 Gabidulin-Based LRC Construction

The LRC construction with multiple (r, δ)-locality based on Gabidulin codes is given below.

Construction 1 (Const. 1 in [8]): For integers $m_j \ge 1$, $r_j \ge 1$, $\delta_j \ge 2$, $j \in [s^*]$, and $k \le \sum_{j=1}^{s^*} m_j r_j \le t$, let $n_j = m_j(r_j + \delta_j - 1)$ and $n = \sum_{j=1}^{s^*} n_j$. A Linear $[n, k]_{q^t}$ code is constructed by the following steps.

1. Encode k information symbols by a $[\sum_{j=1}^{s^*} m_j r_j, k]_{q^t}$ Gabidulin code.
2. Partition the Gabidulin codeword symbols into $\sum_{j=1}^{s^*} m_j$ local groups, where m_j local groups are of size r_j, $j \in [s^*]$.
3. Encode each local group of size r_j by multiplying the generator matrix of an $[r_j + \delta_j - 1, r_j, \delta_j]_q$ maximum distance separable (MDS) code.†

In the proof of Theorem 1, we use some important properties of Gabidulin-based LRCs collected from previous work. The following lemma states that we can use $\mathrm{rank}_{\mathrm{E}}(\cdot)$ instead of $\mathrm{rank}_{\mathrm{G}}(\cdot)$ as long as either rank is less than k.

Lemma 1 (Lem. 9 in [8]): Let $\mathcal{T} \subset [n]$ be an index set of some code symbols in Construction 1. If either $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) < k$ or $\mathrm{rank}_{\mathrm{E}}(\mathcal{T}) < k$, we have

$$\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) = \mathrm{rank}_{\mathrm{E}}(\mathcal{T}).$$

The remark and lemma below are very useful in handling the computation of $\mathrm{rank}_{\mathrm{E}}(\cdot)$. In particular, $\mathrm{rank}_{\mathrm{E}}(\cdot)$ of certain symbols can be computed by first partitioning the symbols with mutually exclusive local groups (the union of the groups covers the entire symbols) of Construction 1, counting the number of symbols in each group with limits, and then simply adding them up.

†The scalar multiplications are over $\mathbb{F}_{q^t}$.

Remark 2 (Rem. 5 in [8]): The subspace generated by the evaluation points of Construction 1 is a direct sum of the subspaces each generated by the evaluation points of each local group. Therefore, $\mathrm{rank}_{\mathrm{E}}(\mathcal{T})$, $\mathcal{T} \subset [n]$, is the sum of each $\mathrm{rank}_{\mathrm{E}}(\cdot)$ computed separately on each local group.

Lemma 2 (Special case of Lem. 9 in [10]): Let $\mathcal{U}$ be the encoded symbol index set of a local group in Construction 1, encoded by an $[r_j + \delta_j - 1, r_j, \delta_j]_q$ MDS code. For an arbitrary set $\mathcal{T} \subset \mathcal{U}$, we have

$$\mathrm{rank}_{\mathrm{E}}(\mathcal{T}) = \min(|\mathcal{T}|, r_j).$$

Within the m_j encoded local groups by a certain $[r_j + \delta_j - 1, r_j, \delta_j]_q$ MDS code, a greedily selected symbol set is a worst case set in terms of $\mathrm{rank}_{\mathrm{E}}(\cdot)$. Such a greedy selection is formally described by the set $\mathcal{T}'$ in the following remark,

Remark 3 (Special case of Lem. 8 in [8]): Let $\mathcal{N}_j$, $j \in [s^*]$, be the index set of the n_j encoded symbols in Construction 1, that correspond to the m_j local groups encoded by the $[r_j + \delta_j - 1, r_j, \delta_j]_q$ MDS code. For an index set $\mathcal{T}' \subset \mathcal{N}_j$ corresponding to the entire symbols of some $p_j \le m_j$ local groups and some $0 \le q_j \le r_j + \delta_j - 2$ symbols from another local group, we have

$$\mathrm{rank}_{\mathrm{E}}(\mathcal{T}) \ge \mathrm{rank}_{\mathrm{E}}(\mathcal{T}'),$$

for any symbol index set $\mathcal{T} \subset \mathcal{N}_j$ such that $|\mathcal{T}| = |\mathcal{T}'|$.

3. General Optimality of Gabidulin-Based LRCs

In this section we provide the proof of Theorem 1. The outline is given first, followed by the details of the proof.

3.1 Outline

We require the following two lemmas in order to show the outline of the proof for Theorem 1. Note that Lemma 4 does not result from simple substitution.

Lemma 3 (Lem. A.1 in [4]): For a symbol index set $\mathcal{T} \subset [n]$ of a linear $[n, k, d]$ code such that $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) \le k - 1$, we have

$$d \le n - |\mathcal{T}|,$$

with equality if and only if $\mathcal{T}$ is of largest cardinality such that $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) = k - 1$.

Lemma 4 (Lem. 2 in [8]): For a symbol index set $\mathcal{T} \subset [n]$ of a linear $[n, k, d]$ code such that $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) \le k - 1$, let γ be the number of redundant symbols indexed by $\mathcal{T}$, i.e., $\gamma = |\mathcal{T}| - \mathrm{rank}_{\mathrm{G}}(\mathcal{T})$. We have

$$d \le n - k + 1 - \gamma.$$

For a Gabidulin-based LRC $\mathscr{C}^*$ having multiple (r, δ)-locality (Construction 1), let $\mathcal{T}^* \subset [n]$ be its *distance defining set* by Lemma 3, i.e., a symbol index set of largest cardinality such that $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}^*) = k - 1$. Accordingly, we have

$$|\mathcal{T}^*| = n - d^*,$$

where d^* denotes the minimum distance of $\mathscr{C}^*$. The number of redundant symbols in $\mathcal{T}^*$ can be written as

$$\gamma^* = |\mathcal{T}^*| - \mathrm{rank}_{\mathrm{G}}(\mathcal{T}^*) = n - d^* - k + 1. \tag{1}$$

We claim the distance optimality of $\mathscr{C}^*$ by showing that the minimum distance d of $\mathscr{C}$ is upper bounded by $d \le d^*$, where $\mathscr{C}$ is an arbitrary LRC having multiple (r, δ)-locality (Definition 1) with length, dimension and (r, δ)-locality parameters identical to $\mathscr{C}^*$. The required upper bound can be derived by constructing an *upper bound defining* set $\mathcal{T}$ for $\mathscr{C}$, such that

$$\mathrm{rank}_{\mathrm{G}}(\mathcal{T}) \le k - 1, \tag{2}$$

and

$$\gamma = |\mathcal{T}| - \mathrm{rank}_{\mathrm{G}}(\mathcal{T}) \ge \gamma^*. \tag{3}$$

Given such a set $\mathcal{T}$ and applying Lemma 4, we can get

$$\begin{aligned} d &\le n - k + 1 - \gamma \\ &\le n - k + 1 - \gamma^* \\ &\overset{(1)}{=} d^*. \end{aligned}$$

3.2 Analysis of the Distance Defining Set

Before we construct the upper bound defining set $\mathcal{T}$, let us further characterize the distance defining set $\mathcal{T}^*$. Let $\mathcal{T}_j^* = \mathcal{T}^* \cap \mathcal{N}_j$, $j \in [s^*]$, such that $\mathcal{T}^* = \bigsqcup_{j=1}^{s^*} \mathcal{T}_j^*$, where $\mathcal{N}_j$ denotes the symbol index set corresponding to the m_j local groups encoded by the $[r_j + \delta_j - 1, r_j, \delta]_q$ MDS code in Construction 1. Also define integers p_j^* and q_j^* such that

$$|\mathcal{T}_j^*| = p_j^*(r_j + \delta_j - 1) + q_j^*,$$

and $0 \le q_j^* \le r_j + \delta_j - 2$.

Consider a set $\mathcal{T}_j' \subset \mathcal{N}_j$ with $|\mathcal{T}_j'| = |\mathcal{T}_j^*|$, that corresponds to the entire symbols from some p_j^* local groups and some q_j^* symbols from another local group. By Remark 3, we clearly have $\mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j^*) \ge \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j')$. We further claim that

$$\mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j^*) = \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j'), \tag{4}$$

i.e., $\mathcal{T}_j^*$ is a worst case set in terms of evaluation point rank. Suppose that $\mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j^*) > \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j')$. Then, we can construct a set $\hat{\mathcal{T}} = (\mathcal{T}^* \setminus \mathcal{T}_j^*) \sqcup \mathcal{T}_j'$ with $|\hat{\mathcal{T}}| = |\mathcal{T}^*|$ such that

$$\begin{aligned} \mathrm{rank}_{\mathrm{E}}(\hat{\mathcal{T}}) &\overset{(a)}{=} \sum_{j' \in [s^*] \setminus \{j\}} \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_{j'}^*) + \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j') \\ &< \sum_{j'=1}^{s^*} \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_{j'}^*) \\ &\overset{(a)}{=} \mathrm{rank}_{\mathrm{E}}(\mathcal{T}^*) \\ &\overset{(b)}{=} \mathrm{rank}_{\mathrm{G}}(\mathcal{T}^*) \\ &= k - 1, \end{aligned}$$

which leads to

$$\mathrm{rank}_{\mathrm{G}}(\hat{\mathcal{T}}) \overset{(b)}{=} \mathrm{rank}_{\mathrm{E}}(\hat{\mathcal{T}}) < k - 1,$$

where (a) and (b) are due to Remark 2 and Lemma 1, respectively. The fact that $\hat{\mathcal{T}}$ can be enlarged while still satisfying $\mathrm{rank}_{\mathrm{G}}(\hat{\mathcal{T}}) \le k - 1$ is contradictory to the precondition on $\mathcal{T}^*$ to be of largest cardinality such that $\mathrm{rank}_{\mathrm{G}}(\mathcal{T}^*) = k - 1$, and the claim of (4) is proved.

We also claim that

$$q_j^* < r_j. \tag{5}$$

Suppose otherwise that $r_j \le q_j^* \le r_j + \delta_j - 2$, and consider again the sets $\mathcal{T}_j'$ and $\hat{\mathcal{T}}$ above, where it is clear that $\mathrm{rank}_{\mathrm{G}}(\hat{\mathcal{T}}) = \mathrm{rank}_{\mathrm{E}}(\hat{\mathcal{T}}) = k - 1$. Note that, due to Lemma 2, Remark 2, and Lemma 1, $\hat{\mathcal{T}}$ can be enlarged by adding one more symbol from the local group corresponding to the q_j^* symbols, while still satisfying $\mathrm{rank}_{\mathrm{G}}(\hat{\mathcal{T}}) = k - 1$, which again is a contradiction.

Now, we get

$$\begin{aligned} \mathrm{rank}_{\mathrm{G}}(\mathcal{T}^*) &\overset{(a)}{=} \mathrm{rank}_{\mathrm{E}}(\mathcal{T}^*) \\ &\overset{(b)}{=} \sum_{j=1}^{s^*} \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j^*) \\ &\overset{(4)}{=} \sum_{j=1}^{s^*} \mathrm{rank}_{\mathrm{E}}(\mathcal{T}_j') \\ &\overset{(c)}{=} \sum_{j=1}^{s^*} (p_j^* r_j + q_j^*), \end{aligned} \tag{6}$$

where (a) and (b) come from Lemma 1 and Remark 2, respectively, and (c) is due to Remark 2, Lemma 2, and (5). We also have

$$\gamma^* = |\mathcal{T}^*| - \mathrm{rank}_{\mathrm{G}}(\mathcal{T}^*) = \sum_{j=1}^{s^*} p_j^*(\delta_j - 1). \tag{7}$$

3.3 Construction of the Upper Bound Defining Set

Finally, let us construct the upper bound defining set $\mathcal{T}$ by first writing $\mathcal{T} = \bigsqcup_{j=1}^{s^*} \mathcal{T}_j$, where $\mathcal{T}_j = \mathcal{T} \cap \mathcal{N}_j$, $j \in [s^*]$, and using Algorithm 1. It is easy to see that it is always possible to make the set $\mathcal{U}$ in Step 7 of the algorithm, since $|\mathcal{Q}_l| \le l(r_j + \delta_j - 1)$ and therefore $|\mathcal{N}_j \setminus \mathcal{Q}_l| \ge \delta_j - 1$.

Two properties of Algorithm 1 are derived, which are required in showing that the set $\mathcal{T}$ results in the required upper bound. We only discuss the case where the condition in Step 3 is satisfied, since it is trivial that the properties hold in the other case. First note that, since $\mathcal{Q}_l = \mathcal{Q}_{l-1} \cup \mathcal{S}_i$, we have

Algorithm 1 Used in the Proof of Theorem 1

1: Let $Q_0 = \emptyset$, $l = 0$
2: **repeat**
3: **if** $\exists i \in \mathcal{N}_j \setminus Q_l$ such that $\mathrm{rank}_\mathrm{G}(Q_l \sqcup \{i\}) > \mathrm{rank}_\mathrm{G}(Q_l)$ **then**
4: $l = l + 1$
5: $Q_l = Q_{l-1} \cup \mathcal{S}_i$
6: **else**
7: Choose an arbitrary set $\mathcal{U} \subset \mathcal{N}_j \setminus Q_l$ such that $|\mathcal{U}| = \delta_j - 1$
8: $l = l + 1$
9: $Q_l = Q_{l-1} \sqcup \mathcal{U}$
10: **end if**
11: **until** $l = p_j^*$
12: $\mathcal{T}_j = Q_l$

Algorithm 2 Used in deriving (9)

Let $\hat{Q} = Q_{l-1}$, $\mathcal{K} = \emptyset$, $\mathcal{R} = \emptyset$
while $\exists i' \in Q_l \setminus \hat{Q}$ such that $\mathrm{rank}_\mathrm{G}(\hat{Q} \sqcup \{i'\}) > \mathrm{rank}_\mathrm{G}(\hat{Q})$ **do**
$\hat{Q} = \hat{Q} \sqcup \{i'\}$
$\mathcal{K} = \mathcal{K} \sqcup \{i'\}$
end while
$\mathcal{R} = Q_l \setminus \hat{Q}$

$$\begin{aligned} \mathrm{rank}_\mathrm{G}(Q_l) - \mathrm{rank}_\mathrm{G}(Q_{l-1}) &\le \mathrm{rank}_\mathrm{G}(\mathcal{S}_i) \\ &\overset{(a)}{\le} r_j, \end{aligned} \tag{8}$$

$l \in [p_j^*]$, where (a) is due to Remark 1. Also, we claim that

$$|Q_l| - |Q_{l-1}| \ge \mathrm{rank}_\mathrm{G}(Q_l) - \mathrm{rank}_\mathrm{G}(Q_{l-1}) + \delta_j - 1. \tag{9}$$

To see why, consider Algorithm 2, where the incremental symbols in Step 5 of Algorithm 1 are categorized into either rank-contributing or redundant symbols by the sets $\mathcal{K}$ and $\mathcal{R}$, respectively. It is clear that the erasure of symbols corresponding to the set $\mathcal{E} = \mathcal{R} \sqcup \{i'\} \subset \mathcal{S}_i \subset Q_l$ with some $i' \in \mathcal{K}$ are not correctable from the remaining symbols of Q_l due to the incremental rank by $i' \in \mathcal{E}$. The same argument holds for $\mathcal{S}_i$ as $\mathcal{S}_i \subset Q_l$. Since $d(\mathcal{C}|_{\mathcal{S}_i}) \ge \delta_j$, it must be true that $|\mathcal{E}| = |Q_l \setminus Q_{l-1}| - |\mathcal{K}| + 1 \ge \delta_j$, resulting in (9).

We now have

$$\begin{aligned} \mathrm{rank}_\mathrm{G}(\mathcal{T}_j) &= \sum_{l=1}^{p_j^*} (\mathrm{rank}_\mathrm{G}(Q_l) - \mathrm{rank}_\mathrm{G}(Q_{l-1})) \\ &\overset{(8)}{\le} p_j^* r_j, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \gamma_j &= |\mathcal{T}_j| - \mathrm{rank}_\mathrm{G}(\mathcal{T}_j) \\ &= \sum_{l=1}^{p_j^*} (|Q_l| - |Q_{l-1}|) - \sum_{l=1}^{p_j^*} (\mathrm{rank}_\mathrm{G}(Q_l) - \mathrm{rank}_\mathrm{G}(Q_{l-1})) \\ &\overset{(9)}{\ge} p_j^*(\delta_j - 1). \end{aligned} \tag{11}$$

We complete the proof by noting that (2) and (3) hold as

$$\begin{aligned} \mathrm{rank}_\mathrm{G}(\mathcal{T}) &\le \sum_{j=1}^{s^*} \mathrm{rank}_\mathrm{G}(\mathcal{T}_j) \\ &\overset{(10)}{\le} \sum_{j=1}^{s^*} p_j^* r_j \le \sum_{j=1}^{s^*} (p_j^* r_j + q_j^*) \\ &\overset{(6)}{=} \mathrm{rank}_\mathrm{G}(\mathcal{T}^*) \\ &= k - 1, \end{aligned}$$

and

$$\begin{aligned} \gamma = |\mathcal{T}| - \mathrm{rank}_\mathrm{G}(\mathcal{T}) &\ge \sum_{j=1}^{s^*} (|\mathcal{T}_j| - \mathrm{rank}_\mathrm{G}(\mathcal{T}_j)) \\ &\overset{(11)}{\ge} \sum_{j=1}^{s^*} p_j^*(\delta_j - 1) \\ &\overset{(7)}{=} \gamma^*. \end{aligned}$$

Acknowledgments

This work is in part supported by SNU Eng-Med Collaboration Grant, Basic Science Research Program (NRF-2017R1A2B2007102) through NRF funded by MSIP, Technology Innovation Program (10051928) funded by MOTIE, Bio-Mimetic Robot Research Center funded by DAPA (UD130070ID), INMAC, and BK21-plus.

References

[1] M. Sathiamoorthy, M. Asteris, D. Papailiopoulos, A.G. Dimakis, R. Vadali, S. Chen, and D. Borthakur, "Xoring elephants: Novel erasure codes for big data," Proc. 39th international conference on Very Large Data Bases, vol.6, no.5, pp.325–336, 2013.

[2] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, "On the locality of codeword symbols," IEEE Trans. Inf. Theory, vol.58, no.11, pp.6925–6934, Nov. 2012.

[3] N. Prakash, G.M. Kamath, V. Lalitha, and P.V. Kumar, "Optimal linear codes with a local-error-correction property," Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on, pp.2776–2780, July 2012.

[4] G.M. Kamath, N. Prakash, V. Lalitha, and P.V. Kumar, "Codes with local regeneration and erasure correction," IEEE Trans. Inf. Theory, vol.60, no.8, pp.4637–4660, Aug. 2014.

[5] A. Zeh and E. Yaakobi, "Bounds and constructions of codes with multiple localities," 2016 IEEE International Symposium on Information Theory (ISIT), pp.640–644, July 2016.

[6] B. Chen, S.T. Xia, and J. Hao, "Locally repairable codes with multiple (r_i, δ_i)-localities," 2017 IEEE International Symposium on Information Theory (ISIT), pp.2038–2042, June 2017.

[7] S. Kadhe and A. Sprintson, "Codes with unequal locality," 2016 IEEE International Symposium on Information Theory (ISIT), pp.435–439, July 2016.

[8] G. Kim and J. Lee, "Locally repairable codes with unequal local erasure correction," IEEE Trans. Inf. Theory, vol.64, no.11, pp.7137–7152, Nov. 2018.

[9] N. Silberstein, A.S. Rawat, O.O. Koyluoglu, and S. Vishwanath, "Optimal locally repairable codes via rank-metric codes," Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on, pp.1819–1823, July 2013.

[10] A.S. Rawat, O.O. Koyluoglu, N. Silberstein, and S. Vishwanath, "Optimal locally repairable and secure codes for distributed storage systems," IEEE Trans. Inf. Theory, vol.60, no.1, pp.212–236, Jan. 2014.