PAPER

Error Analysis and Numerical Stabilization of the Fast H_∞ Filter

Tomonori KATSUMATA[†], **Kiyoshi NISHIYAMA**[†a)], ***Members***, ***and*** **Katsuaki SATOH**[††], ***Nonmember***

SUMMARY The fast H_∞ filter is developed by one of the authors, and its practical use in industries is expected. This paper derives a linear propagation model of numerical errors in the recursive variables of the fast H_∞ filter, and then theoretically analyzes the stability of the filter. Based on the analyzed results, a numerical stabilization method of the fast H_∞ filter is proposed with the error feedback control in the backward prediction. Also, the effectiveness of the stabilization method is verified using numerical examples.

key words: *fast* H_∞ *filter, error analysis, numerical stabilization, system identification*

1. Introduction

System identification is an experimental approach to the modeling of an unknown system, and is of great importance in the fields of signal processing, communication, and automatic control [1]. In many cases, the unknown system is modeled as an adaptive filter whose finite impulse response (FIR) are adjusted with N taps so that its output will perfectly match that of the unknown system except for observation noise. The most celebrated adaptive schemes for the FIR filter are the least mean square (LMS) and recursive least squares (RLS) algorithms [1]–[6]. The performance of such adaptive algorithms can be quantified by examining a number of characteristics, such as the a) accuracy of the obtained solution, b) convergence speed, c) tracking ability, d) computational complexity, and e) robustness to round off error accumulation [3]. Although the LMS algorithm is very simple, with a computational complexity of $O(N)$, and is relatively robust to numerical errors, it suffers from slow convergence, especially when applied to highly correlated excitation signals such as speech. In contrast, the RLS algorithm with $O(N^2)$ provides very fast convergence regardless of the characteristics of the input signals. Usually, the RLS algorithm and its fast versions use a forgetting factor ρ to improve the tracking performance for time-varying systems. However, the value of ρ has been typically determined empirically, without any evidence of optimality. In our previous work, this open problem is solved using the framework of H_∞ optimization [7], [8]. The resultant H_∞ filter, the hyper H_∞ filter, enables the forgetting factor ρ to be optimally determined by means of the so-called γ-iteration. Expectedly, the hyper H_∞ filter has possessed the robustness to disturbances (especially to near-end speech) and the extremely high convergence speed. The hyper H_∞ filter is applicable to the estimation problem of an infinite impulse response (IIR) filter as well as an FIR filter. However, the IIR filter often becomes numerically unstable when implemented with a finite precision. So, the FIR filter is widely used in practical applications, especially in communication and acoustic systems. Furthermore, the FIR filter is much favorable to deriving a fast algorithm of the adaptive filter which requires a shifting property for the input covariance matrix. Indeed, a fast algorithm of the hyper H_∞ filter for FIR systems, the fast H_∞ filter, has also been successfully developed, providing a computational complexity of $\boldsymbol{O}(N)$. However, the numerical behavior of the fast H_∞ filter has not been analyzed yet.

In this paper, we derive a linear propagation model of numerical errors in the recursive variables of the fast H_∞ filter, and then theoretically analyze the stability of the filter. Based on the analyzed results, we propose a numerical stabilization method of the fast H_∞ filter with the error feedback control in the backward prediction. Also, the effectiveness of the stabilization method is verified using numerical examples.

The remainder of this paper is organized as follows. Section 2 presents the hyper H_∞ filter. Section 3 shortly explains the fast H_∞ filter. In Sect. 4, an analysis of numerical errors appeared in the fast H_∞ filter is given. Based on the results of Sect. 4, we propose a numerical stabilization method of the fast H_∞ filter in Sect. 5. In Sect. 6, numerical examples are given to verify the effectiveness of the proposed method. Finally, we present our conclusions in Sect. 7.

2. The Hyper H_∞ Filter

The hyper H_∞ filter, which was originally proposed in [7], [8], can simultaneously optimize both the forgetting factor and the robustness to disturbances through the so-called γ-iteration. This filter, which differs from the ordinary H_∞ Filters [9], is effective for tracking time-varying systems with unknown dynamics of the state vector $\boldsymbol{x}_k$ [10], and furthermore a fast algorithm of the filter is successfully derived as seen in the next section.

Theorem 1: (The Hyper H_∞ Filter) For the following N-

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[†]The authors are with the Department of Electrical Engineering and Computer Science, Iwate University, Morioka-shi, 020-8551 Japan.
[††]The author is with Amenity Research Institute, Hachioji-shi, 192-0081 Japan.
a) E-mail: nisiyama@cis.iwate-u.ac.jp
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dimensional state-space model with system noise $\boldsymbol{w}_k$ and observation noise v_k:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{G}_k\boldsymbol{w}_k, \qquad \boldsymbol{w}_k, \boldsymbol{x}_k \in \mathcal{R}^N \tag{1}$$
$$y_k = \boldsymbol{H}_k\boldsymbol{x}_k + v_k, \qquad y_k, v_k \in \mathcal{R} \tag{2}$$
$$z_k = \boldsymbol{H}_k\boldsymbol{x}_k, \qquad z_k \in \mathcal{R},\ \boldsymbol{H}_k \in \mathcal{R}^{1\times N} \tag{3}$$

one possible level-γ_f hyper H_∞ filter to achieve

$$\sup_{\boldsymbol{x}_0,\{\boldsymbol{w}_i\},\{v_i\}} \cdot \frac{\sum_{i=0}^{k} \|e_{f,i}\|^2/\rho}{\|\boldsymbol{x}_0 - \check{\boldsymbol{x}}_0\|^2_{\boldsymbol{\Sigma}_0^{-1}} + \sum_{i=0}^{k} \|\boldsymbol{w}_i\|^2 + \sum_{i=0}^{k} \|v_i\|^2/\rho} < \gamma_f^2 \tag{4}$$

is represented by

$$\check{z}_{k|k} = \boldsymbol{H}_k\hat{\boldsymbol{x}}_{k|k} \tag{5}$$
$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{K}_{s,k}(y_k - \boldsymbol{H}_k\hat{\boldsymbol{x}}_{k-1|k-1}) \tag{6}$$
$$\boldsymbol{K}_{s,k} = \hat{\boldsymbol{\Sigma}}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}(\boldsymbol{H}_k\hat{\boldsymbol{\Sigma}}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}} + \rho)^{-1} \tag{7}$$
$$\hat{\boldsymbol{\Sigma}}_{k|k} = \hat{\boldsymbol{\Sigma}}_{k|k-1} - \hat{\boldsymbol{\Sigma}}_{k|k-1}\boldsymbol{C}_k^{\mathrm{T}}\boldsymbol{R}_{e,k}^{-1}\boldsymbol{C}_k\hat{\boldsymbol{\Sigma}}_{k|k-1}$$
$$\hat{\boldsymbol{\Sigma}}_{k+1|k} = \hat{\boldsymbol{\Sigma}}_{k|k}/\rho \tag{8}$$

where $\cdot^T$ denotes the transpose, $\cdot^{-1}$ the inverse, and

$$e_{f,i} = \check{z}_{i|i} - \boldsymbol{H}_i\boldsymbol{x}_i, \quad \hat{\boldsymbol{x}}_{0|0} = \check{\boldsymbol{x}}_0, \quad \hat{\boldsymbol{\Sigma}}_{1|0} = \boldsymbol{\Sigma}_0$$
$$\boldsymbol{R}_{e,k} = \boldsymbol{R} + \boldsymbol{C}_k\hat{\boldsymbol{\Sigma}}_{k|k-1}\boldsymbol{C}_k^{\mathrm{T}}$$
$$\boldsymbol{R} = \begin{bmatrix} \rho & 0 \\ 0 & -\rho\gamma_f^2 \end{bmatrix}, \quad \boldsymbol{C}_k = \begin{bmatrix} \boldsymbol{H}_k \\ \boldsymbol{H}_k \end{bmatrix} \tag{9}$$
$$0 < \rho = 1 - \chi(\gamma_f) \le 1, \quad \gamma_f > 1 \tag{10}$$

in which $\chi(\gamma_f)$ is a monotonically decreasing scalar function of γ_f such that $\chi(1) = 1$ and $\chi(\infty) = 0$, and the driving matrix $\boldsymbol{G}_k$ is generated by

$$\boldsymbol{G}_k\boldsymbol{G}_k^{\mathrm{T}} = \chi(\gamma_f)\hat{\boldsymbol{\Sigma}}_{k+1|k} = \frac{\chi(\gamma_f)}{\rho}\hat{\boldsymbol{\Sigma}}_{k|k}. \tag{11}$$

For the filter to exist, the parameter γ_f must satisfy

$$\hat{\boldsymbol{\Sigma}}_{i|i}^{-1} = \hat{\boldsymbol{\Sigma}}_{i|i-1}^{-1} + \frac{1 - \gamma_f^{-2}}{\rho}\boldsymbol{H}_i^{\mathrm{T}}\boldsymbol{H}_i > 0, \ i = 0, \ldots, k. \tag{12}$$

Proof : See [7], [8]. □

If γ_f is chosen to be as small as the existence condition allows, the H_∞ filter with the optimal $\boldsymbol{G}_k$ (or ρ) will be able to very quickly track the changed state vector $\boldsymbol{x}_k$. This is because the sum of the squared error $\|e_{f,i}\|^2$ must be subject to the inequality of (4), which prevents the maximum energy gain from growing to a value in excess of γ_f^2.

Note that the algorithm of the hyper H_∞ filter in the limit of $\gamma_f = \infty$ coincides with that of the Kalman filter with $\boldsymbol{G}_k = \boldsymbol{0}$.

3. The Fast H_∞ Filter

The computational complexity of the hyper H_∞ filter is $O(N^2)$, making it difficult to implement this filter in real-time applications. To overcome this difficulty, a fast H_∞ filter, whose complexity is $\boldsymbol{O}(N)$, has been derived for an FIR system with input u_k using a shifting property of the observation matrix such that $\boldsymbol{H}_{k+1} = [u_{k+1}, H_k(1), H_k(2), \ldots, H_k(N-1)]$ [7], [8].

Theorem 2: (The Fast H_∞ Filter) When the observation matrix $\boldsymbol{H}_k$ has the shifting property, the hyper H_∞ filter for the N-dimensional state-space model (1)–(3) with $\hat{\boldsymbol{\Sigma}}_{1|0} = \varepsilon_0\boldsymbol{I}$, $\varepsilon_0 > 0$ can be recursively performed at a computational complexity of $\boldsymbol{O}(N)$ per iteration as

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{K}_{s,k}(y_k - \boldsymbol{H}_k\hat{\boldsymbol{x}}_{k-1|k-1}) \tag{13}$$
$$\boldsymbol{K}_{s,k} = \frac{\boldsymbol{K}_k(:,1)}{1 + \gamma_f^{-2}\boldsymbol{H}_k\boldsymbol{K}_k(:,1)} \quad \in \mathcal{R}^{N\times 1} \tag{14}$$

in which the gain matrix $\boldsymbol{K}_k$ is recursively calculated as

$$\boldsymbol{K}_k = \boldsymbol{m}_k - \boldsymbol{D}_k\boldsymbol{\mu}_k \ \in \mathcal{R}^{N\times 2} \tag{15}$$
$$\boldsymbol{D}_k = \frac{\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k}{1 - \boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\eta}_k}$$
$$\begin{bmatrix} \boldsymbol{m}_k \\ \boldsymbol{\mu}_k \end{bmatrix} = \begin{bmatrix} S_k^{-1}\boldsymbol{e}_k^T \\ \boldsymbol{K}_{k-1} + \boldsymbol{A}_k S_k^{-1}\boldsymbol{e}_k^T \end{bmatrix}, \boldsymbol{m}_k \in \mathcal{R}^{N\times 2}$$
$$\boldsymbol{\eta}_k = \boldsymbol{c}_{k-N} + \boldsymbol{C}_k\boldsymbol{D}_{k-1}$$
$$S_k = \rho S_{k-1} + \boldsymbol{e}_k^T\boldsymbol{W}\tilde{\boldsymbol{e}}_k, \ \boldsymbol{e}_k = \boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_k$$
$$\boldsymbol{A}_k = \boldsymbol{A}_{k-1} - \boldsymbol{K}_{k-1}\boldsymbol{W}\tilde{\boldsymbol{e}}_k$$
$$\tilde{\boldsymbol{e}}_k = \boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1} \tag{16}$$

where $\boldsymbol{C}_k = \begin{bmatrix} \boldsymbol{H}_k \\ \boldsymbol{H}_k \end{bmatrix}$, $\boldsymbol{W} = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma_f^{-2} \end{bmatrix}$, and $\boldsymbol{c}_k \in \mathcal{R}^{2\times 1}$ is the first row of $\boldsymbol{C}_k = [\boldsymbol{c}_k, \ldots, \boldsymbol{c}_{k-N+1}]$, assuming that $\boldsymbol{c}_{k-i} = \boldsymbol{0}_{2\times 1}$ for $k - i < 0$. Also, ρ and γ_f are chosen as $0 < \rho = 1 - \chi(\gamma_f) \le 1$ and $\gamma_f > 1$. The recursions are initialized with $\boldsymbol{K}_0 = \boldsymbol{0}_{N\times 2}$, $\boldsymbol{A}_0 = \boldsymbol{0}$, $S_0 = \frac{1}{\varepsilon_0}$, $\boldsymbol{D}_0 = \boldsymbol{0}$, $\hat{\boldsymbol{x}}_{0|0} = \boldsymbol{0}$ where ε_0 is set to be a relatively large positive number, $\boldsymbol{0}$ denotes the zero vector, and $\boldsymbol{0}_{n\times m}$ the $n \times m$ zero matrix.

For the filter achieving (4) to exist, the value of γ_f must be chosen so as to satisfy the following scalar existence condition:

$$-\varrho\hat{\Xi}_i + \rho\gamma_f^2 > 0, \quad i = 0, \ldots, k \tag{17}$$

where ϱ and $\hat{\Xi}_i$ are defined by

$$\varrho = 1 - \gamma_f^2, \ \ \hat{\Xi}_i = \frac{\rho\boldsymbol{H}_i\boldsymbol{K}_{s,i}}{1 - \boldsymbol{H}_i\boldsymbol{K}_{s,i}} \tag{18}$$

respectively.

Proof : The key idea to derivation of the fast H_∞ filter is based on the following equations :

$$\boldsymbol{Q}_k\boldsymbol{K}_k = \boldsymbol{C}_k^T, \ \ \boldsymbol{Q}_k := \sum_{i=0}^{k}\rho^{k-i}\boldsymbol{C}_i^T\boldsymbol{W}\boldsymbol{C}_i \tag{19}$$
$$\check{\boldsymbol{Q}}_k\check{\boldsymbol{K}}_k = \check{\boldsymbol{C}}_k^T, \ \ \check{\boldsymbol{Q}}_k := \sum_{i=0}^{k}\rho^{k-i}\check{\boldsymbol{C}}_i^T\boldsymbol{W}\check{\boldsymbol{C}}_i \tag{20}$$

where

$$\check{\boldsymbol{C}}_k := [\boldsymbol{c}_k, \ldots, \boldsymbol{c}_{k-N}] = [\boldsymbol{C}_k \ \boldsymbol{c}_{k-N}]. \tag{21}$$

The details of the derivation are described in [7] and [8]. □

4. Error Analysis of the Fast H∞ Filter

When the gain matrix $\boldsymbol{K}_k$ is initialized with $\boldsymbol{K}_0 = \boldsymbol{0}_{N\times 2}$, the numerical stability of the fast H∞ filter is mainly governed by the following recursive variables of the algorithm:

$$\begin{aligned} \boldsymbol{A}_k &= \boldsymbol{A}_{k-1} - \boldsymbol{K}_{k-1}\boldsymbol{W}\tilde{\boldsymbol{e}}_k \\ S_k &= \rho S_{k-1} + \boldsymbol{e}_k^T\boldsymbol{W}\tilde{\boldsymbol{e}}_k \\ \boldsymbol{D}_k &= (\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)(1 - \boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\eta}_k)^{-1} \end{aligned} \tag{22}$$

where

$$\begin{aligned} &\boldsymbol{K}_k = \boldsymbol{m}_k - \boldsymbol{D}_k\boldsymbol{\mu}_k \\ &\begin{bmatrix} \boldsymbol{m}_k \\ \boldsymbol{\mu}_k \end{bmatrix} = \begin{bmatrix} S_k^{-1}\boldsymbol{e}_k^T \\ \boldsymbol{K}_{k-1} + \boldsymbol{A}_k S_k^{-1}\boldsymbol{e}_k^T \end{bmatrix} \\ &\tilde{\boldsymbol{e}}_k = \boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1}, \ \boldsymbol{e}_k = \boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_k \\ &\boldsymbol{\eta}_k = \boldsymbol{c}_{k-N} + \boldsymbol{C}_k\boldsymbol{D}_{k-1}. \end{aligned} \tag{23}$$

Then, the recursions of the forward linear prediction coefficient $\boldsymbol{A}_k$, the power of the forward prediction error S_k, and the backward linear prediction coefficient $\boldsymbol{D}_k$ can be represented by the following nonlinear system:

$$\Theta_k = \boldsymbol{f}_k(\Theta_{k-1}), \quad \Theta_k := \begin{bmatrix} \boldsymbol{A}_k \\ S_k \\ \boldsymbol{D}_k \end{bmatrix}.$$

Linearizing the nonlinear system, we obtain a propagation model of numerical errors $d\Theta_k$ of Θ_k as

$$\begin{bmatrix} d\boldsymbol{A}_k \\ dS_k \\ d\boldsymbol{D}_k \end{bmatrix} = \boldsymbol{F}_k \begin{bmatrix} d\boldsymbol{A}_{k-1} \\ dS_{k-1} \\ d\boldsymbol{D}_{k-1} \end{bmatrix} \tag{24}$$

where

$$\begin{aligned} &d\boldsymbol{A}_k := \hat{\boldsymbol{A}}_k - \boldsymbol{A}_k, \ dS_k := \hat{S}_k - S_k \\ &d\boldsymbol{D}_k := \hat{\boldsymbol{D}}_k - \boldsymbol{D}_k \end{aligned} \tag{25}$$

$$\boldsymbol{F}_k := \frac{\partial \boldsymbol{f}_k}{\partial \Theta_{k-1}^T} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1} & \boldsymbol{0} & \boldsymbol{0}_{N\times N} \\ \boldsymbol{F}_k^{21} & \rho & \boldsymbol{0}^T \\ \boldsymbol{F}_k^{31} & \boldsymbol{F}_k^{32} & \boldsymbol{F}_k^{33} \end{bmatrix}$$

$$\boldsymbol{F}_k^{33} = \frac{(\boldsymbol{I} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{C}_k)}{\beta_k} + \frac{(\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{C}_k}{\beta_k^2} \tag{26}$$

$$\beta_k = 1 - \boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\eta}_k. \tag{27}$$

Here, the recursive variables $\boldsymbol{A}_k$, S_k, and $\boldsymbol{D}_k$ are perturbed as $\hat{\boldsymbol{A}}_k = \boldsymbol{A}_k + d\boldsymbol{A}_k$, $\hat{S}_k = S_k + dS_k$, $\hat{\boldsymbol{D}}_k = \boldsymbol{D}_k + d\boldsymbol{D}_k$, respectively, each entry of $\boldsymbol{F}_k$ is derived as in Appendix A, and $\boldsymbol{I}$ denotes the identity matrix.

Now, noting that $\boldsymbol{F}_k$ is a block lower triangular matrix, we find

$$d\boldsymbol{A}_k = (\boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1})d\boldsymbol{A}_{k-1}. \tag{28}$$

Assuming that $\{\boldsymbol{c}_k\}$ is statistically stationary, i.e., $\boldsymbol{Q}_{k-1} \approx \boldsymbol{Q}_{k-2}$, the 1-1 block matrix of $\boldsymbol{F}_k$, $\boldsymbol{F}_k^{11}$, is approximated using $\boldsymbol{Q}_k = \rho\boldsymbol{Q}_{k-1} + \boldsymbol{C}_k^T\boldsymbol{W}\boldsymbol{C}_k$ as

$$\begin{aligned} &\boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1} \\ &= \boldsymbol{I} - \boldsymbol{Q}_{k-1}^{-1}\boldsymbol{C}_{k-1}^T\boldsymbol{W}\boldsymbol{C}_{k-1} \\ &= \boldsymbol{I} - \boldsymbol{Q}_{k-1}^{-1}(\boldsymbol{Q}_{k-1} - \rho\boldsymbol{Q}_{k-2}) \\ &\approx \rho\boldsymbol{I}. \end{aligned} \tag{29}$$

Namely, the eigenvalues of $\boldsymbol{F}_k^{11}$ are almost less than one when $\rho < 1$. This implies that the recursion of $d\boldsymbol{A}_k$ is stable.

Similarly, we obtain

$$dS_k = \rho dS_{k-1} + \boldsymbol{F}_k^{21}d\boldsymbol{A}_{k-1} \tag{30}$$

which is stable because of $\rho < 1$ and the stability of $d\boldsymbol{A}_k$. Therefore, the recursive variable S_k is also numerically stable.

However, the updating of $d\boldsymbol{D}_k$ is recursively performed by

$$d\boldsymbol{D}_k = \boldsymbol{F}_k^{33}d\boldsymbol{D}_{k-1} + \boldsymbol{F}_k^{32}dS_{k-1} + \boldsymbol{F}_k^{31}d\boldsymbol{A}_{k-1} \tag{31}$$

whose dynamics is given by

$$\begin{aligned} \boldsymbol{F}_k^{33} &= \frac{1}{\beta_k}\left\{(\boldsymbol{I} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{C}_k) + \frac{(\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{C}_k}{\beta_k}\right\} \\ &= \frac{1}{\beta_k}\left\{(\boldsymbol{I} + (\boldsymbol{D}_k\boldsymbol{\mu}_k - \boldsymbol{m}_k)\boldsymbol{W}\boldsymbol{C}_k)\right\} \\ &= \frac{\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{W}\boldsymbol{C}_k}{\beta_k} = \frac{\boldsymbol{I} - \boldsymbol{Q}_k^{-1}\boldsymbol{C}_k^T\boldsymbol{W}\boldsymbol{C}_k}{\beta_k} \\ &\approx \frac{\rho}{\beta_k}\boldsymbol{I}. \end{aligned} \tag{32}$$

The last equation means that if $\beta_k > \rho$ for any k, the backward prediction is stable. However, the stable condition may not be guaranteed under finite precision. This is the reason why the fast H∞ filter causes the numerical instability.

5. Numerical Stabilization of the Fast H∞ Filter

5.1 Preliminaries

Definition 1: The forward and backward prediction errors are newly defined by

$$\tilde{\boldsymbol{e}}_k := \boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1}, \ \tilde{\underline{\boldsymbol{e}}}_k := \boldsymbol{c}_{k-N} + \boldsymbol{C}_k\boldsymbol{D}_{k-1}. \tag{33}$$

Similarly, the forward and backward estimation errors and their powers are defined by

$$\boldsymbol{e}_k := \boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_k, \quad \underline{\boldsymbol{e}}_k := \boldsymbol{c}_{k-N} + \boldsymbol{C}_k\boldsymbol{D}_k \tag{34}$$

and

$$\overline{S}_k := \sum_{i=0}^{k}\rho^{k-i}\boldsymbol{\epsilon}_i^T\boldsymbol{W}\boldsymbol{\epsilon}_i, \quad \boldsymbol{\epsilon}_i := \boldsymbol{c}_i + \boldsymbol{C}_{i-1}\boldsymbol{A}_k$$

$$\underline{S}_k := \sum_{i=0}^{k} \rho^{k-i} \underline{\boldsymbol{\epsilon}}_i^T \boldsymbol{W} \underline{\boldsymbol{\epsilon}}_i, \ \ \underline{\boldsymbol{\epsilon}}_i := \boldsymbol{c}_{i-N} + \boldsymbol{C}_i \boldsymbol{D}_k \tag{35}$$

respectively.

We now consider a case that $\overline{S}_k$ and $\underline{S}_k$ are minimized by $\boldsymbol{A}_k$ and $\boldsymbol{D}_k$, respectively. Then, we have the following useful propositions.

Proposition 1: The backward estimation error $\underline{\boldsymbol{e}}_k$ is written with its own power $\underline{S}_k$ as

$$\underline{\boldsymbol{e}}_k = \underline{S}_k \boldsymbol{\mu}_k^T. \tag{36}$$

Proof : The inverse of $\check{\boldsymbol{Q}}_k$ partitioned with $\check{\boldsymbol{C}}_i = [\boldsymbol{C}_i \ \boldsymbol{c}_{i-N}]$ can be expressed with the inverses of its diagonal blocks as

$$\begin{aligned}
\check{\boldsymbol{Q}}_k^{-1} &= \begin{bmatrix} \boldsymbol{Q}_k & \underline{\boldsymbol{t}}_k \\ \underline{\boldsymbol{t}}_k^T & q_k^b \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{Q}_k^{-1}\underline{\boldsymbol{t}}_k \\ \boldsymbol{0}^T & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} \boldsymbol{Q}_k^{-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & (q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{Q}_k^{-1}\underline{\boldsymbol{t}}_k \\ \boldsymbol{0}^T & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} \boldsymbol{Q}_k^{-1} & \boldsymbol{0} \\ -(q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} & (q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{Q}_k^{-1} + \boldsymbol{Q}_k^{-1}\underline{\boldsymbol{t}}_k (q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} & -\boldsymbol{Q}_k^{-1}\underline{\boldsymbol{t}}_k (q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \\ -(q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} & (q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k)^{-1} \end{bmatrix}
\end{aligned} \tag{37}$$

using the matrix inversion for partitioned matrices lemma. Here, $\underline{\boldsymbol{t}}_k$ and q_k^b are defined, respectively, by

$$\underline{\boldsymbol{t}}_k := \sum_{i=0}^{k} \rho^{k-i} \boldsymbol{C}_i^T \boldsymbol{W} \boldsymbol{c}_{i-N}, \ q_k^b := \sum_{i=0}^{k} \rho^{k-i} \boldsymbol{c}_{i-N}^T \boldsymbol{W} \boldsymbol{c}_{i-N}. \tag{38}$$

On the other hand, minimizing $\underline{S}_k$ with respect to $\boldsymbol{D}_k$ provides

$$\begin{aligned}
\sum_{i=0}^{k} \rho^{k-i} \boldsymbol{C}_i^T \boldsymbol{W} \boldsymbol{C}_i \boldsymbol{D}_k &= -\sum_{i=0}^{k} \rho^{k-i} \boldsymbol{C}_i^T \boldsymbol{W} \boldsymbol{c}_{i-N} \\
\boldsymbol{Q}_k \boldsymbol{D}_k &= -\underline{\boldsymbol{t}}_k.
\end{aligned} \tag{39}$$

Then, since the power of the backward estimation error is expressed as

$$\begin{aligned}
\underline{S}_k &= q_k^b + 2\boldsymbol{D}_k^T \underline{\boldsymbol{t}}_k + \boldsymbol{D}_k^T \boldsymbol{Q}_k \boldsymbol{D}_k = q_k^b + \boldsymbol{D}_k^T \underline{\boldsymbol{t}}_k \\
&= q_k^b + \underline{\boldsymbol{t}}_k^T \boldsymbol{D}_k = q_k^b - \underline{\boldsymbol{t}}_k^T \boldsymbol{Q}_k^{-1} \underline{\boldsymbol{t}}_k
\end{aligned} \tag{40}$$

we can rewrite (37) as

$$\begin{aligned}
\check{\boldsymbol{Q}}_k^{-1} &= \begin{bmatrix} \boldsymbol{Q}_k^{-1} + \boldsymbol{D}_k \boldsymbol{D}_k^T \underline{S}_k^{-1} & \boldsymbol{D}_k \underline{S}_k^{-1} \\ \underline{S}_k^{-1} \boldsymbol{D}_k^T & \underline{S}_k^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{Q}_k^{-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & 0 \end{bmatrix} + \frac{1}{\underline{S}_k} \begin{bmatrix} \boldsymbol{D}_k \\ 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_k^T & 1 \end{bmatrix}.
\end{aligned} \tag{41}$$

Furthermore, multiplying both sides of (41) by $\check{\boldsymbol{C}}_k^T$ from the right-hand side provides

$$\begin{aligned}
\check{\boldsymbol{Q}}_k^{-1} \check{\boldsymbol{C}}_k^T &= \begin{bmatrix} \boldsymbol{Q}_k^{-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & 0 \end{bmatrix} \check{\boldsymbol{C}}_k^T \\
&\quad + \frac{1}{\underline{S}_k} \begin{bmatrix} \boldsymbol{D}_k \\ 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_k^T & 1 \end{bmatrix} \check{\boldsymbol{C}}_k^T \\
\check{\boldsymbol{K}}_k &= \begin{bmatrix} \boldsymbol{K}_k \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{D}_k \\ 1 \end{bmatrix} \frac{\underline{\boldsymbol{e}}_k^T}{\underline{S}_k}.
\end{aligned} \tag{42}$$

From the $N+1$th raw of $\check{\boldsymbol{K}}_k = [\boldsymbol{m}_k^T \ \boldsymbol{\mu}_k^T]^T$, we find

$$\underline{\boldsymbol{e}}_k^T = \underline{S}_k \check{\boldsymbol{K}}_k(N+1,:) = \underline{S}_k \boldsymbol{\mu}_k \tag{43}$$

whose transpose completes the proof of Proposition 1. □

Proposition 2: The powers of the forward and backward estimation errors are expressed by

$$\overline{S}_k = \frac{\det\{\check{\boldsymbol{Q}}_k\}}{\det\{\boldsymbol{Q}_{k-1}\}}, \quad \underline{S}_k = \frac{\det\{\check{\boldsymbol{Q}}_k\}}{\det\{\boldsymbol{Q}_k\}} \tag{44}$$

where $\det\{\cdot\}$ denotes the determinant of matrix.

Proof : Taking the determinant of both sides of the following identity equation for $\check{\boldsymbol{Q}}_k = \begin{bmatrix} q_k^f & \boldsymbol{t}_k^T \\ \boldsymbol{t}_k & \boldsymbol{Q}_{k-1} \end{bmatrix}$ partitioned with $\check{\boldsymbol{C}}_i = [\boldsymbol{c}_i \ \boldsymbol{C}_{i-1}]$:

$$\begin{aligned}
&\begin{bmatrix} q_k^f & \boldsymbol{t}_k^T \\ \boldsymbol{t}_k & \boldsymbol{Q}_{k-1} \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{0}^T \\ -\boldsymbol{Q}_{k-1}^{-1} \boldsymbol{t}_k & \boldsymbol{Q}_{k-1}^{-1} \end{bmatrix} \\
&\quad = \begin{bmatrix} q_k^f - \boldsymbol{t}_k^T \boldsymbol{Q}_{k-1}^{-1} \boldsymbol{t}_k & \boldsymbol{t}_k^T \boldsymbol{Q}_{k-1}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}
\end{aligned} \tag{45}$$

we have

$$\det\{\check{\boldsymbol{Q}}_k\} \det\{\boldsymbol{Q}_{k-1}^{-1}\} = q_k^f - \boldsymbol{t}_k^T \boldsymbol{Q}_{k-1}^{-1} \boldsymbol{t}_k \tag{46}$$

in which $\boldsymbol{t}_k$ and q_k^f are defined, respectively, by

$$\boldsymbol{t}_k := \sum_{i=0}^{k} \rho^{k-i} \boldsymbol{C}_{i-1}^T \boldsymbol{W} \boldsymbol{c}_i, \ q_k^f := \sum_{i=0}^{k} \rho^{k-i} \boldsymbol{c}_i^T \boldsymbol{W} \boldsymbol{c}_i. \tag{47}$$

Here the properties of determinant, $\det\{\boldsymbol{AB}\} = \det\{\boldsymbol{A}\}\det\{\boldsymbol{B}\}$,

$\det\left\{\begin{bmatrix} A & 0 \\ * & B \end{bmatrix}\right\} = \det\{A\}\det\{B\}$, are used for the calculation of (46).

On the other hand, when $\overline{S}_k$ is minimized by A_k, we have

$$Q_{k-1}A_k = -t_k \quad (A_k = -Q_{k-1}^{-1}t_k). \tag{48}$$

Substituting (48) to the right-hand side of (46) provides

$$\det\{\breve{Q}_k\}\det\{Q_{k-1}^{-1}\} = q_k^f + t_k^T A_k. \tag{49}$$

Furthermore, using

$$\begin{aligned}\overline{S}_k &= \sum_{i=0}^{k}\rho^{k-i}(c_i + C_{i-1}A_k)^T W(c_i + C_{i-1}A_k) \\ &= \sum_{i=0}^{k}\rho^{k-i}c_i^T W c_i + 2A_k^T\sum_{i=0}^{k}\rho^{k-i}C_{i-1}^T W c_i \\ &\quad + A_k^T\left(\sum_{i=0}^{k}\rho^{k-i}C_{i-1}^T W C_{i-1}\right)A_k \\ &= \sum_{i=0}^{k}\rho^{k-i}c_i^T W c_i + 2A_k^T t_k + A_k^T Q_{k-1}A_k \\ &= q_k^f + t_k^T A_k \end{aligned} \tag{50}$$

we can rewrite (49) as

$$\det\{\breve{Q}_k\}\det\{Q_{k-1}^{-1}\} = \overline{S}_k. \tag{51}$$

Then, recalling $\det\{A^{-1}\} = 1/\det\{A\}$, we have

$$\overline{S}_k = \frac{\det\{\breve{Q}_k\}}{\det\{Q_{k-1}\}}. \tag{52}$$

Similarly, the power of backward estimation error is given by

$$\underline{S}_k = \frac{\det\{\breve{Q}_k\}}{\det\{Q_k\}}. \tag{53}$$

□

Proposition 3: The backward prediction error $\tilde{\underline{e}}_k$ is expressed with the backward estimation error $\underline{e}_k$ as

$$\tilde{\underline{e}}_k = \frac{1-\gamma_f^{-2}}{\rho}\zeta_k\underline{e}_k \tag{54}$$

where

$$\zeta_k = H_k P_{k-1} H_k^T + \rho/(1-\gamma_f^{-2}). \tag{55}$$

Proof : The backward linear prediction coefficient D_k minimizing $\underline{S}_k$ satisfies

$$Q_k D_k = -\underline{t}_k. \tag{56}$$

Hence, substituting the following equations:

$$Q_k = \rho Q_{k-1} + C_k^T W C_k$$

$$\underline{t}_k = \rho\underline{t}_{k-1} + C_k^T W c_{k-N} \tag{57}$$

to (56) and using

$$\tilde{K}_k := \rho^{-1}Q_{k-1}^{-1}C_k^T \tag{58}$$

we have

$$\begin{aligned}&(\rho Q_{k-1} + C_k^T W C_k)D_k = -\rho\underline{t}_{k-1} - C_k^T W c_{k-N} \\ &\rho Q_{k-1}D_k + C_k^T W(c_{k-N} + C_k D_k) = -\rho\underline{t}_{k-1} \\ &\rho Q_{k-1}(D_k + \rho^{-1}Q_{k-1}^{-1}C_k^T W\underline{e}_k) = -\rho\underline{t}_{k-1} \\ &Q_{k-1}(D_k + \tilde{K}_k W\underline{e}_k) = -\underline{t}_{k-1}.\end{aligned} \tag{59}$$

Then, recalling $Q_{k-1}D_{k-1} = -\underline{t}_{k-1}$, we find

$$D_k = D_{k-1} - \tilde{K}_k W\underline{e}_k. \tag{60}$$

Similarly, arranging (56) at time $k-1$ as

$$\begin{aligned}&(\rho^{-1}Q_k - \rho^{-1}C_k^T W C_k)D_{k-1} \\ &\quad = -(\rho^{-1}\underline{t}_k - \rho^{-1}C_k^T W c_{k-N}) \\ &\quad \rho^{-1}Q_k D_{k-1} - \rho^{-1}C_k^T W(c_{k-N} + C_k D_{k-1}) \\ &\quad = -\rho^{-1}\underline{t}_k \\ &\quad Q_k D_{k-1} - C_k^T W\tilde{\underline{e}}_k = -\underline{t}_k \\ &\quad Q_k(D_{k-1} - Q_k^{-1}C_k^T W\tilde{\underline{e}}_k) = -\underline{t}_k\end{aligned} \tag{61}$$

and comparing the last equation with (56), we obtain

$$D_k = D_{k-1} - K_k W\tilde{\underline{e}}_k, \quad K_k = Q_k^{-1}C_k^T. \tag{62}$$

On the other hand, the recursion of $P_k^{-1} := Q_k$:

$$\begin{aligned}P_k^{-1} &= \rho P_{k-1}^{-1} + C_k^T W C_k \\ &= \rho P_{k-1}^{-1} + (1-\gamma_f^{-2})H_k^T H_k\end{aligned}$$

allows the Riccati equation in the hyper H_∞ filter to reduce to

$$\begin{aligned}P_k &= (\rho P_{k-1}^{-1} + H_k^T(1-\gamma_f^{-2})H_k)^{-1} \\ &= \frac{1}{\rho}P_{k-1} - \frac{1}{\rho}P_{k-1}H_k^T \\ &\quad \times\left(H_k\frac{1}{\rho}P_{k-1}H_k^T + \frac{1}{1-\gamma_f^{-2}}\right)^{-1}H_k\frac{1}{\rho}P_{k-1} \\ &= \Bigg(P_{k-1} - P_{k-1}H_k^T \\ &\quad \times\left(H_k P_{k-1}H_k^T + \frac{\rho}{1-\gamma_f^{-2}}\right)^{-1}H_k P_{k-1}\Bigg)\Big/\rho.\end{aligned} \tag{63}$$

Multiplying both sides of (63) by H_k^T, we have the following relationship between $K_k(:,1)$ and $\tilde{K}_k(:,1)$:

$$\begin{aligned}K_k(:,1) &= \tilde{K}_k(:,1)\left(1 - \frac{H_k P_{k-1}H_k^T}{H_k P_{k-1}H_k^T + \rho/(1-\gamma_f^{-2})}\right) \\ &= \tilde{K}_k(:,1)\frac{\rho/(1-\gamma_f^{-2})}{\zeta_k}\end{aligned} \tag{64}$$

where

$$\zeta_k = \boldsymbol{H}_k \boldsymbol{P}_{k-1} \boldsymbol{H}_k^T + \rho/(1-\gamma_f^{-2}). \quad (65)$$

Besides, comparing (60) with (62) in terms of $\boldsymbol{D}_k - \boldsymbol{D}_{k-1}$, we find

$$\tilde{\boldsymbol{K}}_k \boldsymbol{W} \underline{\boldsymbol{e}}_k = \boldsymbol{K}_k \boldsymbol{W} \underline{\tilde{\boldsymbol{e}}}_k. \quad (66)$$

Then, substituting (64) into (66) provides

$$\tilde{\boldsymbol{K}}_k \boldsymbol{W} \underline{\boldsymbol{e}}_k = \frac{\rho/(1-\gamma_f^{-2})}{\zeta_k} \tilde{\boldsymbol{K}}_k \boldsymbol{W} \underline{\tilde{\boldsymbol{e}}}_k. \quad (67)$$

From this, through some calculations, we can obtain

$$\underline{\tilde{\boldsymbol{e}}}_k = \frac{(1-\gamma_f^{-2})\zeta_k}{\rho} \underline{\boldsymbol{e}}_k. \quad (68)$$

Note that $\tilde{\boldsymbol{K}}_k(:,1) = \tilde{\boldsymbol{K}}_k(:,2)$, $\boldsymbol{K}_k(:,1) = \boldsymbol{K}_k(:,2)$, $\underline{e}_k(1,1) = \underline{e}_k(2,1)$, and $\underline{\tilde{e}}_k(1,1) = \underline{\tilde{e}}_k(2,1)$. □

Proposition 4: The backward prediction error $\underline{\tilde{\boldsymbol{e}}}_k$ is expressed with the power of the forward estimation error $\overline{S}_k$ as

$$\underline{\tilde{\boldsymbol{e}}}_k = \rho^{-N} \overline{S}_k \boldsymbol{\mu}_k^T. \quad (69)$$

Proof : From Proposition 1, the backward estimation error $\underline{\boldsymbol{e}}_k$ can be written as

$$\underline{\boldsymbol{e}}_k = \underline{S}_k \boldsymbol{\mu}_k^T. \quad (70)$$

Starting with this, we derive an alternative expression of the backward prediction error $\underline{\tilde{\boldsymbol{e}}}_k$.

First, we arrange the Riccati equation of (63) as

$$\begin{aligned}
\boldsymbol{P}_k &= \left(\boldsymbol{P}_{k-1} - \boldsymbol{P}_{k-1}\boldsymbol{H}_k^T\left(\boldsymbol{H}_k\boldsymbol{P}_{k-1}\boldsymbol{H}_k^T \right.\right.\\
&\qquad \left.\left. + \frac{\rho}{1-\gamma_f^{-2}}\right)^{-1} \boldsymbol{H}_k \boldsymbol{P}_{k-1}\right)/\rho \\
&= \left(\boldsymbol{P}_{k-1} - \frac{\boldsymbol{P}_{k-1}\boldsymbol{H}_k^T\boldsymbol{H}_k\boldsymbol{P}_{k-1}}{\zeta_k}\right)/\rho \quad (71)
\end{aligned}$$

where

$$\zeta_k = \boldsymbol{H}_k\boldsymbol{P}_{k-1}\boldsymbol{H}_k^T + \frac{\rho}{1-\gamma_f^{-2}}. \quad (72)$$

Multiplying both sides of (71) by $\boldsymbol{P}_{k-1}^{-1}$ from the right-hand side provides

$$\boldsymbol{P}_k\boldsymbol{P}_{k-1}^{-1} = \left(\boldsymbol{I} - \frac{\boldsymbol{P}_{k-1}\boldsymbol{H}_k^T}{\zeta_k}\boldsymbol{H}_k\right)/\rho. \quad (73)$$

Next, taking the determinant of both sides of (73), from $\det\{\boldsymbol{AB}\} = \det\{\boldsymbol{A}\}\det\{\boldsymbol{B}\}$, $\det\{\boldsymbol{I}+\boldsymbol{AB}\} = \det\{\boldsymbol{I}+\boldsymbol{BA}\}$, and (72), we have

$$\begin{aligned}
&\det\{\boldsymbol{P}_k\}\det\{\boldsymbol{P}_{k-1}^{-1}\} \\
&= \det\left\{\boldsymbol{I} - \frac{\boldsymbol{P}_{k-1}\boldsymbol{H}_k^T\boldsymbol{H}_k}{\zeta_k}\right\}\det\{\rho^{-1}\boldsymbol{I}\} \\
&= \det\left\{1 - \frac{\boldsymbol{H}_k\boldsymbol{P}_{k-1}\boldsymbol{H}_k^T}{\zeta_k}\right\}\rho^{-N} \\
&= \det\left\{1 - \left(\zeta_k - \frac{\rho}{1-\gamma_f^{-2}}\right)/\zeta_k\right\}\rho^{-N} \\
&= \rho^{-N}\det\left\{\frac{\rho}{\zeta_k(1-\gamma_f^{-2})}\right\} = \frac{\rho^{-N+1}}{\zeta_k(1-\gamma_f^{-2})}. \quad (74)
\end{aligned}$$

The last equation of (74) leads to the following relationship:

$$\frac{(1-\gamma_f^{-2})}{\rho^{-N+1}}\zeta_k = \frac{\det\{\boldsymbol{P}_k^{-1}\}}{\det\{\boldsymbol{P}_{k-1}^{-1}\}} = \frac{\det\{\boldsymbol{Q}_k\}}{\det\{\boldsymbol{Q}_{k-1}\}}. \quad (75)$$

On the other hand, recalling Proposition 2, namely

$$\overline{S}_k = \frac{\det\{\check{\boldsymbol{Q}}_k\}}{\det\{\boldsymbol{Q}_{k-1}\}}, \quad \underline{S}_k = \frac{\det\{\check{\boldsymbol{Q}}_k\}}{\det\{\boldsymbol{Q}_k\}} \quad (76)$$

we can express (75) as

$$\frac{(1-\gamma_f^{-2})}{\rho^{-N+1}}\zeta_k = \frac{\overline{S}_k}{\underline{S}_k}. \quad (77)$$

Then, arranging (77) for $\underline{S}_k$, we have

$$\underline{S}_k = \frac{\rho^{-N+1}}{(1-\gamma_f^{-2})}\frac{\overline{S}_k}{\zeta_k}. \quad (78)$$

Last, using (78) with Propositions 1 and 3, we obtain an alternative expression of the backward prediction error $\underline{\tilde{\boldsymbol{e}}}_k$ as

$$\begin{aligned}
\underline{\tilde{\boldsymbol{e}}}_k &= \frac{1-\gamma_f^{-2}}{\rho}\zeta_k\underline{\boldsymbol{e}}_k = \frac{1-\gamma_f^{-2}}{\rho}\zeta_k\underline{S}_k\boldsymbol{\mu}_k^T \\
&= \frac{1-\gamma_f^{-2}}{\rho}\zeta_k\frac{\rho^{-N+1}}{(1-\gamma_f^{-2})}\frac{\overline{S}_k}{\zeta_k}\boldsymbol{\mu}_k^T = \rho^{-N}\overline{S}_k\boldsymbol{\mu}_k^T.
\end{aligned}$$

□

Proposition 5: The minimum forward estimation error power $\overline{S}_k$ is equal to S_k in (16).

Proof : Recalling (50), we obtain

$$\overline{S}_k = \sum_{i=0}^{k}\rho^{k-i}\boldsymbol{c}_i^T\boldsymbol{W}\boldsymbol{c}_i + \boldsymbol{A}_k^T\boldsymbol{t}_k. \quad (79)$$

Then, using (16) and (48), we can arrange (79) as

$$\begin{aligned}
\overline{S}_k &= \sum_{i=0}^{k}\rho^{k-i}\boldsymbol{c}_i^T\boldsymbol{W}\boldsymbol{c}_i + (\boldsymbol{A}_{k-1} - \boldsymbol{K}_{k-1}\boldsymbol{W}\tilde{\boldsymbol{e}}_k)^T\boldsymbol{t}_k \\
&= \sum_{i=0}^{k-1}\rho^{k-i}\boldsymbol{c}_i^T\boldsymbol{W}\boldsymbol{c}_i + \boldsymbol{c}_k^T\boldsymbol{W}\boldsymbol{c}_k + \rho\boldsymbol{A}_{k-1}^T\boldsymbol{t}_{k-1} \\
&\quad + \boldsymbol{A}_{k-1}^T\boldsymbol{C}_{k-1}^T\boldsymbol{W}\boldsymbol{c}_k - \tilde{\boldsymbol{e}}_k^T\boldsymbol{W}\boldsymbol{K}_{k-1}^T\boldsymbol{t}_k
\end{aligned}$$

$$
\begin{aligned}
&= \rho\overline{S}_{k-1} + (\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1})^T \boldsymbol{W}\boldsymbol{c}_k \\
&\quad -\tilde{\boldsymbol{e}}_k^T \boldsymbol{W}\boldsymbol{K}_{k-1}^T \boldsymbol{t}_k \\
&= \rho\overline{S}_{k-1} + \tilde{\boldsymbol{e}}_k^T \boldsymbol{W}(\boldsymbol{c}_k - \boldsymbol{K}_{k-1}^T \boldsymbol{t}_k) \\
&= \rho\overline{S}_{k-1} + \tilde{\boldsymbol{e}}_k^T \boldsymbol{W}(\boldsymbol{c}_k - \boldsymbol{C}_{k-1}\boldsymbol{Q}_{k-1}^{-1} \boldsymbol{t}_k) \\
&= \rho\overline{S}_{k-1} + \tilde{\boldsymbol{e}}_k^T \boldsymbol{W}(\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_k) \\
&= \rho\overline{S}_{k-1} + \tilde{\boldsymbol{e}}_k^T \boldsymbol{W}\boldsymbol{e}_k. \qquad (80)
\end{aligned}
$$

Comparing the last equation with S_k in (16), we find that $\overline{S}_k$ agrees with S_k. □

From Propositions 4 and 5, we see that the backward prediction error $\boldsymbol{\eta}_k$ is theoretically equal to $\rho^{-N}S_k\boldsymbol{\mu}_k^T$ expect for numerical errors.

5.2 A Numerical Stabilization Method of the Fast H$_\infty$ Filter

The analyzed results of the numerical error propagation in Sect. 4 have suggested that the recursion of the backward linear prediction coefficient $\boldsymbol{D}_k$ may be often unstable. Especially, as β_k in (32) is close to zero due to the error accumulation, the recursive variable $\boldsymbol{D}_k$ becomes unstable because of dramatically increasing the eigenvalues of $\boldsymbol{F}_k^{33}$, then risking the filter stability.

To improve the instability, we correct $\boldsymbol{D}_{k-1}$ so as to minimize the numerical error $\boldsymbol{\eta}_k - \rho^{-N}S_k\boldsymbol{\mu}_k^T$, but to be as close to the originally computed value as possible. This requirement is expressed with the minimization of $J(\check{\boldsymbol{D}}_{k-1})$ defined as

$$
\begin{aligned}
&J(\check{\boldsymbol{D}}_{k-1}) \\
&:= \sum_{i=0}^{k} \rho^{k-i}[(\boldsymbol{c}_{i-N}+\boldsymbol{C}_i\check{\boldsymbol{D}}_{k-1}) - (\boldsymbol{c}_{i-N}+\boldsymbol{C}_i\boldsymbol{D}_{k-1})]^T \\
&\qquad \times \boldsymbol{W}[(\boldsymbol{c}_{i-N}+\boldsymbol{C}_i\check{\boldsymbol{D}}_{k-1}) - (\boldsymbol{c}_{i-N}+\boldsymbol{C}_i\boldsymbol{D}_{k-1})] \\
&\qquad + \kappa(\boldsymbol{c}_{k-N} + \boldsymbol{C}_k\check{\boldsymbol{D}}_{k-1} - \rho^{-N}S_k\boldsymbol{\mu}_k^T)^T \\
&\qquad \times \boldsymbol{W}(\boldsymbol{c}_{k-N} + \boldsymbol{C}_k\check{\boldsymbol{D}}_{k-1} - \rho^{-N}S_k\boldsymbol{\mu}_k^T) \\
&= (\check{\boldsymbol{D}}_{k-1} - \boldsymbol{D}_{k-1})^T \boldsymbol{Q}_k (\check{\boldsymbol{D}}_{k-1} - \boldsymbol{D}_{k-1}) \\
&\qquad + \kappa\check{\boldsymbol{\xi}}_k^T \boldsymbol{W}\check{\boldsymbol{\xi}}_k, \quad \kappa \geq 0 \qquad (81)
\end{aligned}
$$

where

$$
\check{\boldsymbol{\xi}}_k := \check{\boldsymbol{\eta}}_k - \rho^{-N}S_k\boldsymbol{\mu}_k^T, \quad \check{\boldsymbol{\eta}}_k := \boldsymbol{c}_{k-N} + \boldsymbol{C}_k\check{\boldsymbol{D}}_{k-1}. \qquad (82)
$$

Setting the derivative of $J(\check{\boldsymbol{D}}_{k-1})$ to $\boldsymbol{0}$ gives

$$
\begin{aligned}
\frac{\partial J(\check{\boldsymbol{D}}_{k-1})}{\partial \check{\boldsymbol{D}}_{k-1}} &= 2\boldsymbol{Q}_k(\check{\boldsymbol{D}}_{k-1} - \boldsymbol{D}_{k-1}) + 2\kappa\boldsymbol{C}_k^T\boldsymbol{W}\check{\boldsymbol{\xi}}_k \\
&= \boldsymbol{0} \qquad (83)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\check{\boldsymbol{D}}_{k-1} &= \boldsymbol{D}_{k-1} - \kappa\boldsymbol{Q}_k^{-1}\boldsymbol{C}_k^T\boldsymbol{W}\check{\boldsymbol{\xi}}_k \\
&= \boldsymbol{D}_{k-1} - \kappa\boldsymbol{K}_k\boldsymbol{W}\check{\boldsymbol{\xi}}_k. \qquad (84)
\end{aligned}
$$

Then, replacing $\boldsymbol{D}_{k-1}$ in (62) with the corrected $\check{\boldsymbol{D}}_{k-1}$ and using (84), we can also correct $\boldsymbol{D}_k$ as

$$
\begin{aligned}
\boldsymbol{D}_k &= \check{\boldsymbol{D}}_{k-1} - \boldsymbol{K}_k\boldsymbol{W}\check{\boldsymbol{\eta}}_k \\
&= (\boldsymbol{D}_{k-1} - \kappa\boldsymbol{K}_k\boldsymbol{W}\check{\boldsymbol{\xi}}_k) - \boldsymbol{K}_k\boldsymbol{W}\check{\boldsymbol{\eta}}_k \\
&= \boldsymbol{D}_{k-1} - \boldsymbol{K}_k\boldsymbol{W}(\check{\boldsymbol{\eta}}_k + \kappa\check{\boldsymbol{\xi}}_k) \\
&= \boldsymbol{D}_{k-1} - \boldsymbol{K}_k\boldsymbol{W}\{\check{\boldsymbol{\eta}}_k + \kappa(\check{\boldsymbol{\eta}}_k - \rho^{-N}S_k\boldsymbol{\mu}_k^T)\}. \qquad (85)
\end{aligned}
$$

Note that (84), nonexplicit form of $\check{\boldsymbol{D}}_{k-1}$, works well to concentrate the corrections in the second term of the right-hand side of the last equation in (85), and $\underline{\tilde{\boldsymbol{e}}}_k$ when $\boldsymbol{D}_{k-1}$ is replaced with $\check{\boldsymbol{D}}_{k-1}$ is equal to $\check{\boldsymbol{\eta}}_k$.

For simplicity, assuming that $\check{\boldsymbol{\eta}}_k \approx \boldsymbol{\eta}_k$, i.e., the required correction is small, we obtain

$$
\boldsymbol{D}_k = \boldsymbol{D}_{k-1} - \boldsymbol{K}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k \qquad (86)
$$

from (85) where

$$
\tilde{\boldsymbol{\eta}}_k = \boldsymbol{\eta}_k + \kappa(\boldsymbol{\eta}_k - \rho^{-N}S_k\boldsymbol{\mu}_k^T). \qquad (87)
$$

Furthermore, substituting $\boldsymbol{K}_k = \boldsymbol{m}_k - \boldsymbol{D}_k\boldsymbol{\mu}_k$ to (86), we obtain a new feasible update equation of $\boldsymbol{D}_k$ as

$$
\begin{aligned}
\boldsymbol{D}_k &= \boldsymbol{D}_{k-1} - (\boldsymbol{m}_k - \boldsymbol{D}_k\boldsymbol{\mu}_k)\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k \\
\boldsymbol{D}_k &= \boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k + \boldsymbol{D}_k\boldsymbol{\mu}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k \\
\boldsymbol{D}_k(1 - \boldsymbol{\mu}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k) &= \boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k \\
\boldsymbol{D}_k &= \frac{\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k}{1 - \boldsymbol{\mu}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k}. \qquad (88)
\end{aligned}
$$

Namely, the numerical stabilization of the recursive variable $\boldsymbol{D}_k$ is accomplished by merely replacing $\boldsymbol{\eta}_k$ in (22) with $\tilde{\boldsymbol{\eta}}_k$. This $\tilde{\boldsymbol{\eta}}_k$ includes the backward prediction errors computed in two different ways, $\boldsymbol{\eta}_k$ and $\rho^{-N}S_k\boldsymbol{\mu}_k^T$, as

$$
\tilde{\boldsymbol{\eta}}_k = \boldsymbol{\eta}_k + \kappa(\boldsymbol{\eta}_k - \rho^{-N}S_k\boldsymbol{\mu}_k^T) \qquad (89)
$$

which will yield a feedback mechanism to influence the error propagation dynamics.

Then, the quantity β_k playing the important role in the backward linear prediction is replaced with $\tilde{\beta}_k$ as

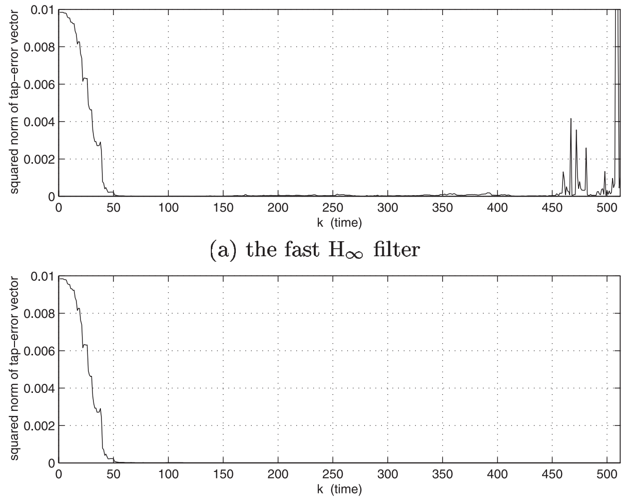
$$
\begin{aligned}
\tilde{\beta}_k &= 1 - \boldsymbol{\mu}_k\boldsymbol{W}\tilde{\boldsymbol{\eta}}_k \\
&= \beta_k - \kappa\boldsymbol{\mu}_k\boldsymbol{W}(\boldsymbol{\eta}_k - \rho^{-N}S_k\boldsymbol{\mu}_k^T). \qquad (90)
\end{aligned}
$$

A statistical analysis suggests that $\tilde{\beta}_k$ is more apart from zero than β_k (Appendix B). Consequently, the numerical error, which causes the instability, contributes to preventing $\tilde{\beta}_k$ from being close to zero, rescuing the recursion of $\boldsymbol{D}_k$ from the worst scenario.

Hence, it is expected that the feedback control of (89) for the backward prediction error $\boldsymbol{\eta}_k$ improves the numerical stability of the fast H$_\infty$ filter since the error feedback control prevents $\tilde{\beta}_k$ from approaching to zero. The modified filter is referred to as the stabilized fast H$_\infty$ filter (SFHF) hereafter.

6. Numerical Examples

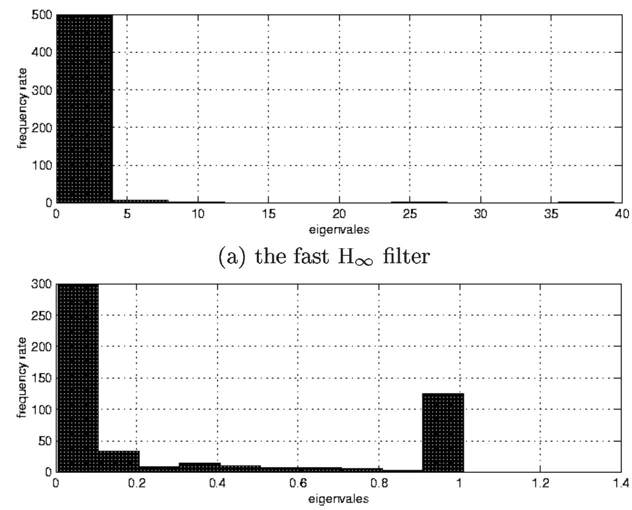
The performance of the proposed stabilization method for the fast H$_\infty$ filter is evaluated using identification of a finite



(a) the fast H_∞ filter

(b) the stabilized fast H_∞ filter ($\kappa = 1.0$)

Fig. 1 Convergence property of each filter; $\gamma_f = 2.2$, $N = 64$, $L = 512$.



(a) the fast H_∞ filter

(b) the stabilized fast H_∞ filter ($\kappa = 1.0$)

Fig. 2 Distribution of the maximum eigenvalues $\{\{\max_i\{|\lambda_i\{\boldsymbol{F}_k^{33}\}|\}\}_{k=1}^L$ of the transition matrix $\boldsymbol{F}_k^{33}$; $\gamma_f = 2.2$, $N = 64$, $L = 512$.

Table 1 Comparison of the maximum eigenvalues of the averaged error transition matrix for each filter ; $\gamma_f = 2.2$, $N = 64$, $L = 512$.

	the FHF	the SFHF ($\kappa = 1.0$)
$\max_i\{|\lambda_i\{\bar{\boldsymbol{F}}^{33}\}|\}$	1.1620	0.3218

impulse response (FIR) system such as widely used in echo cancellers. Also, the forgetting factor is chosen as $\rho = 1 - \gamma_f^{-2}$ for simplicity, and all calculations are executed using MATLAB 5.3.

As an unknown system to be identified, we consider an echo path whose impulse response $\{h_i\}$ consists of { 0.0, 0.008, −0.012, 0.064, 0.013, −0.052 −0.007, 0.039, 0.011, 0.0, −0.002, −0.009 −0.016, −0.013, −0.001, 0.004, 0.015, 0.013 0.007, 0.0, −0.001, −0.002, −0.001, 0.0 } for $i < 24$ and zero otherwise ($24 \le i < N$). The observed echo is given by

$$y_k = \sum_{i=0}^{N-1} h_i u_{k-i} + v_k, \quad k = 1, 2, \ldots, L \tag{91}$$

where v_k is a stationary, white Gaussian noise with zero mean and standard deviation $\sigma_v = 1.0 \times 10^{-4}$, N denotes the length of the adjustable impulse response (tap number), and L stands for the length of observation data. Note that the state-space model of the observed echo y_k becomes time-varying due to $\boldsymbol{H}_k = [\ u_k, \ u_{k-1}, \ \ldots, \ u_{k-(N-1)}\ ]$ with a shifting property such that $\boldsymbol{H}_{k+1} = [u_{k+1}, H_k(1), H_k(2), \ldots, H_k(N-1)]$, provided that $\boldsymbol{x}_k = [\ h_0, \ h_1, \ \ldots, \ h_{N-1}\ ]^T$. The received signal (tap inputs) $\{u_k\}$ is generated by the following autoregressive (AR) model:

$$u_k = \alpha_1 u_{k-1} + \alpha_2 u_{k-2} + w'_k \tag{92}$$

where $\alpha_1 = 0.7, \alpha_2 = 0.1$, and w'_k is a stationary, white Gaussian noise with zero mean and standard deviation $\sigma_{w'} = 0.04$.

Figure 1 demonstrates the differences in stability between the fast H_∞ filter and the stabilized fast H_∞ filter ($\kappa = 1.0$) using the squared norm of tap error vector $\|\hat{\boldsymbol{x}}_{k|k} - \boldsymbol{x}_k\|^2$ as a measure, where $N = 64$, $\gamma_f = 2.2$, $\chi(\gamma_f) = \gamma_f^{-2}$, and $\varepsilon_0 = 1.0$.

Figure 2 shows the distribution of the maximum eigenvalues of $\boldsymbol{F}_k^{33}$ for a period of 1 to L, i.e., the histogram of $\{\max_i\{|\lambda_i\{\boldsymbol{F}_k^{33}\}|\}\}_{k=1}^L$, for each filter, where λ_i denotes the eigenvalue, and $\boldsymbol{F}_k^{33}$ the error transition matrix of the backward linear prediction coefficient. Furthermore, the maximum eigenvalue of the averaged transition matrix $\bar{\boldsymbol{F}}^{33} = 1/L \sum_{k=1}^L \boldsymbol{F}_k^{33}$, $\max_i\{|\lambda_i\{\bar{\boldsymbol{F}}^{33}\}|\}$, is listed in Table 1. From Fig. 2 and Table 1, we see that the error feedback control drastically decreases the eigenvalues of the error transition matrix $\boldsymbol{F}_k^{33}$, and succeeds to numerically stabilize the fast H_∞ filter.

Fortunately, the modification of the backward prediction error $\boldsymbol{\eta}_k$ has not given a significant effect on the overall performance of the fast H_∞ filter although the optimality of $\boldsymbol{D}_k$ is different from the original one, and an additional computational burden for the modification was also negligible.

7. Conclusions

In this paper, we have derived a linear propagation model of numerical errors in the recursive variables of the fast H_∞ filter, and then analyzed the stability of the filter using the first-order error propagation model. Additionally, based on the analyzed results, we have proposed a numerical stabilization method of the fast H_∞ filter with the error feedback control in the backward prediction. The effectiveness of the stabilization method was verified using numerical examples. Also, in our recent experiments, the stabilized fast H_∞ filter has succeeded in continuously running without restart on a DSP for days.

Further analysis of the behavior of $\tilde{\beta}_k$ will be one of our future works.

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Appendix A: Derivation of the Error Transition Matrix

Each block entry of the error transition matrix $\boldsymbol{F}_k$ is derived as follows:

$$\begin{aligned}\boldsymbol{F}_k^{11} &= \frac{\partial \boldsymbol{A}_k}{\partial \boldsymbol{A}_{k-1}^T} = \frac{\partial(\boldsymbol{A}_{k-1} - \boldsymbol{K}_{k-1}\boldsymbol{W}\tilde{\boldsymbol{e}}_k)}{\partial \boldsymbol{A}_{k-1}^T} \\ &= \frac{\partial\{\boldsymbol{A}_{k-1} - \boldsymbol{K}_{k-1}\boldsymbol{W}(\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1})\}}{\partial \boldsymbol{A}_{k-1}^T} \\ &= \frac{\partial \boldsymbol{A}_{k-1}}{\partial \boldsymbol{A}_{k-1}^T} - \frac{\partial(\boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{c}_k)}{\partial \boldsymbol{A}_{k-1}^T} \\ &\quad - \frac{\partial(\boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1})}{\partial \boldsymbol{A}_{k-1}^T} \\ &= \boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1} \qquad \text{(A· 1)}\end{aligned}$$

$$\begin{aligned}\boldsymbol{F}_k^{22} &= \frac{\partial S_k}{\partial S_{k-1}} = \frac{\partial(\rho S_{k-1})}{\partial S_{k-1}} + \frac{\partial(\boldsymbol{e}_k^T\boldsymbol{W}\tilde{\boldsymbol{e}}_k)}{\partial S_{k-1}} \\ &= \rho\frac{\partial S_{k-1}}{\partial S_{k-1}} = \rho \qquad \text{(A· 2)}\end{aligned}$$

$$\begin{aligned}\boldsymbol{F}_k^{33} &= \frac{\partial \boldsymbol{D}_k}{\partial \boldsymbol{D}_{k-1}^T} \\ &= \frac{\partial}{\partial \boldsymbol{D}_{k-1}^T}\left(\frac{\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k}{\beta_k}\right) \\ &= \frac{\partial(\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)}{\partial \boldsymbol{D}_{k-1}^T}\beta_k^{-1} \\ &\quad + (\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)\frac{\partial \beta_k^{-1}}{\partial \boldsymbol{D}_{k-1}^T} \\ &= (\boldsymbol{I} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{C}_k)\beta_k^{-1} \\ &\quad + (\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)\frac{\partial \beta_k^{-1}}{\partial \beta_k}\frac{\partial \beta_k}{\partial \boldsymbol{D}_{k-1}^T} \\ &= \frac{(\boldsymbol{I} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{C}_k)}{\beta_k} \\ &\quad + (\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)(-\beta_k^{-2})(-\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{C}_k) \\ &= \frac{(\boldsymbol{I} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{C}_k)}{\beta_k} \\ &\quad + \frac{(\boldsymbol{D}_{k-1} - \boldsymbol{m}_k\boldsymbol{W}\boldsymbol{\eta}_k)\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{C}_k}{\beta_k^2} \qquad \text{(A· 3)}\end{aligned}$$

where

$$\beta_k = 1 - \boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\eta}_k,\ \boldsymbol{\eta}_k = \boldsymbol{c}_{k-N} + \boldsymbol{C}_k\boldsymbol{D}_{k-1}. \qquad \text{(A· 4)}$$

Also, noting

$$\begin{aligned}S_k &= \rho S_{k-1} + [\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1} \\ &\quad - \boldsymbol{C}_{k-1}\boldsymbol{K}_{k-1}\boldsymbol{W}(\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1})]^T \\ &\quad \times\boldsymbol{W}(\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1}) \\ &= \rho S_{k-1} + [\boldsymbol{c}_k - \boldsymbol{C}_{k-1}\boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{c}_k \\ &\quad + \boldsymbol{C}_{k-1}(\boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1})\boldsymbol{A}_{k-1})]^T \\ &\quad \times\boldsymbol{W}(\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1}) \qquad \text{(A· 5)}\end{aligned}$$

we can obtain the explicit form of $\boldsymbol{F}_k^{21}$ as

$$\begin{aligned}\boldsymbol{F}_k^{21} &= \frac{\partial S_k}{\partial \boldsymbol{A}_{k-1}^T} \\ &= (\boldsymbol{c}_k + \boldsymbol{C}_{k-1}\boldsymbol{A}_{k-1})^T \\ &\quad \times\boldsymbol{W}\boldsymbol{C}_{k-1}(\boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1}) \\ &\quad + [\boldsymbol{c}_k - \boldsymbol{C}_{k-1}\boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{c}_k \\ &\quad + \boldsymbol{C}_{k-1}(\boldsymbol{I} - \boldsymbol{K}_{k-1}\boldsymbol{W}\boldsymbol{C}_{k-1})\boldsymbol{A}_{k-1})]^T \\ &\quad \times\boldsymbol{W}\boldsymbol{C}_{k-1}. \qquad \text{(A· 6)}\end{aligned}$$

Similarly, we can calculate $\boldsymbol{F}_k^{31} = \frac{\partial \boldsymbol{D}_k}{\partial \boldsymbol{A}_{k-1}^T}$ and $\boldsymbol{F}_k^{32} = \frac{\partial \boldsymbol{D}_k}{\partial S_{k-1}}$.

Appendix B: Behavior Analysis of $\tilde{\beta}_k$

When the backward and forward estimations are sufficiently performed, $\boldsymbol{\eta}_k$ and $\boldsymbol{e}_k$ almost become roundoff errors. So, suppose that $\boldsymbol{\eta}_k$ and $\boldsymbol{e}_k$ are zero-mean random vectors being mutually independent. Indeed, these properties were also observed experimentally in our examples. Then, using (23), $S_k > 0$, and $1 - \gamma_f^{-2} > 0$, we have

$$\begin{aligned}\mathrm{E}\{\tilde{\beta}_k\} &= \mathrm{E}\{\beta_k\} - \kappa\mathrm{E}\{\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\eta}_k\} + \kappa\rho^{-N}\mathrm{E}\{S_k\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\mu}_k^T\} \\ &= \mathrm{E}\{\beta_k\} + \kappa\rho^{-N}\mathrm{E}\{S_k\boldsymbol{\mu}_k\boldsymbol{W}\boldsymbol{\mu}_k^T\}\end{aligned}$$

$$= \mathrm{E}\{\beta_k\} + \kappa\rho^{-N}(1-\gamma_f^{-2})\mathrm{E}\{S_k\mu_k(1,1)^2\}$$
$$\geq \mathrm{E}\{\beta_k\}$$

where $\mathrm{E}\{\cdot\}$ denotes expectation. Note that $\boldsymbol{\mu}_k \boldsymbol{W} \boldsymbol{\mu}_k^T = (1 - \gamma_f^{-2})\mu_k(1,1)^2$ holds due to $\mu_k(1,1) = \mu_k(1,2)$, which stems from $\boldsymbol{C}_k = [\boldsymbol{H}_k^T\ \boldsymbol{H}_k^T]^T$.

This implies that $\tilde{\beta}_k$ is more apart from zero than β_k due to $\mathrm{E}\{\beta_k\} > 0$. Indeed, in our examples, $\tilde{\beta}_k$ takes the value more than 1, whereas β_k almost takes the value in the range of 0 and 1.

Tomonori Katsumata was born in Chiba, Japan in 1973. He received the B.E. and M.E. degrees from Yamanashi University in 1997 and 2003, respectively. He is currently pursuing the doctoral degree at Iwate University. His research interests are in system identification and echo canceller.

Kiyoshi Nishiyama was born in Tokyo, Japan in 1957. He received the M.E. degree in electrical engineering from Chiba University in 1985 and the degree of Dr.Eng. from Tokyo Institute of Technology in 1991. He joined the Department of Computer and Information Sciences, Iwate University in 1998. He is currently a Professor. His research interests are in estimation, optimization, and machine learning.

Katsuaki Satoh was born in Tokyo, Japan in 1947. He graduated from Yamanashi University. From 1970 to 1991 he worked for the Acoustic Research Lab. of Matsushita Electric Co., Ltd. Since 1991, he has been the President of Amenity Research Institute Co., Ltd.