SPECTRAL ANALYSIS OF RANDOM SPARSE MATRICES

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ABSTRACT. We study $n \times n$ random symmetric matrices whose entries above
the diagonal are iid random variables each of which takes 1 with probability
p and 0 with probability $1 - p$, for a given density parameter $p = \alpha/n$ for
sufficiently large α. For a given such matrix A, we consider a matrix A' that
is obtained by removing some rows and corresponding columns with too many
value 1 entries. Then for this A', we show that the largest eigenvalue is asymp-
totically close to $\alpha + 1$ and its eigenvector is almost parallel to all one vector
$(1, \dots, 1)$.

1. INTRODUCTION

We study the largest eigenvalues and their eigenvectors of random symmetric
and sparse $0, 1$-matrices. *Spectral analysis* (of matrices) is in general to ana-
lyze eigenvalues and eigenvectors (of given matrices). Random matrices and
their spectral analysis play key roles in many problems in information sciences.
In particular, analyzing random symmetric $0, 1$-matrices has been shown quite
important for the design and analysis of combinatorial algorithms such as graphs,
etc; see, for example, a survey [1] of Alon for relation to graph algorithms, and a
paper [3] for relation to SAT algorithms. Because of its importance, several de-
tail spectral investigations have been made for random symmetric $0, 1$-matrices.
Yet our understanding is still limited; many important questions have been left
open, and there seem to exist some interesting but unknown spectral properties.
In fact, using powerful approximation methods developed in statistical physics,
many interesting and useful spectral properties have been shown *approximately
under certain heuristic assumptions*. We have been trying to give rigorous justi-
fications to such approximate analysis, and this paper reports one set of results
on the largest eigenvalue and its eigenvector of a random symmetric and sparse
$0, 1$-matrix.

For explaining previous related work, we first introduce some notions and nota-
tions for discussing the results precisely. Let $\mathbf{N}$ denote the set of positive integers.

Throughout this paper, we will use $n \in \mathbf{N}$ as a size parameter, a parameter determining the size of matrices and vectors. Let $[n]$ denote the set $\{1, ..., n\}$. Our analysis is stated asymptotically; that is, we consider the situation where n is sufficiently large. In particular, by "w.h.p." we mean that a given event holds with probability approaching to 1 when $n \to \infty$.

Our target object is an $n \times n$ random symmetric matrix $A = (a_{ij})$ whose diagonal entries are zeros and whose entries above the diagonal are iid random variables, each taking value 1 with probability p and value 0 with probability $1 - p$. A parameter p is called a *density parameter*. In this paper, we consider "sparse" matrices, where p is small and A has *on average* not so many entries with value 1. Precisely, we consider the case where the density parameter is defined by $p = \alpha/n$ for some sufficiently large $\alpha > 0$.

The largest eigenvalue and its eigenvector are the most important subjects of the spectral analysis. In this paper, we use λ_1 and $\boldsymbol{\chi}_1$ to denote the largest eigenvalue of A and the normalized eigenvector of this eigenvalue. Roughly speaking, we may expect that λ_1 is close to pn and that $\boldsymbol{\chi}_1$ is almost parallel to vector $\overline{\mathbf{1}} = (1, ..., 1)/\sqrt{n}$, or more quantitatively $|\langle \boldsymbol{\chi}_1, \overline{\mathbf{1}} \rangle|$ is close to 1. (Throughout this paper, we will use a to denote the quantity $|\langle \boldsymbol{\chi}_1, \overline{\mathbf{1}} \rangle|$.) To see these fundamental spectral properties, let us consider a matrix $\widehat{A}$ whose each entry takes the expectation of the corresponding entry of A; that is, $\widehat{A}$'s each diagonal entry is 0, and its each nondiagonal entry is p. For this $\widehat{A}$, it is easy to see that $\widehat{A}\mathbf{1} = p(n-1)\mathbf{1}$. That is, $p(n-1) \approx pn$ is one of the eigenvalues of $\widehat{A}$ and its eigenvector is parallel to $\mathbf{1}$; furthermore, it is not so hard to show that $p(n-1)$ is the largest eigenvalue. Thus, it seems that $\lambda_1 \approx pn$ and $a \approx 1$ *on average*. In fact, results supporting this intuition have been shown in the literature, and they have been used as key facts for the spectral analysis of algorithms. The detail analysis, however, has not been done completely, in particular for the sparse case that we will study in this paper. Note that these properties may not hold in the sparse case. One reason is, as pointed out in the literature (e.g., [2]), the fact that the possibility to have some rows (and columns by symmetry) with unusually many value 1 entries is not so small; such rows and columns cause certain irregularity, which affects λ_1 and $\boldsymbol{\chi}_1$. But even in this case, we can still expect similar properties if we remove such rows and columns.

Recently, by using powerful approximation methods developed in statistical physics, various spectral properties of random sparse matrices have been demonstrated [7]. Although their analysis uses certain approximations and heuristic hypotheses, the results that match very well to computer experiments seem quite informative and useful. For the above explained fundamental properties, they showed that if $\alpha > 1$, then we have

$$\lambda_1 \approx \alpha + 1, \quad \text{and} \quad a \approx \sqrt{1 - \frac{1}{\alpha}}$$

for random symmetric $0, 1$-matrices created with density parameter $p = \alpha/n$ (and then modified by removing rows and columns with too many 1 entries). In this paper, we prove that these relations asymptotically hold w.h.p. for sufficiently large α.

Note that $\alpha = pn$. Thus, $\lambda_1 \approx \alpha + 1 = pn + 1$ is different from our intuition $\lambda_1 \approx pn$ by $+1$. One can derive a similar estimate with the $+1$ term from the earlier result of [6] for the dense case where $p > \Omega(1)$. More recently, Krivelevich and Sudakov [8] analyzed the largest eigenvalue for the sparse case. While their result include some other case, the relation that we can show from their result is $\lambda_1 = (1 + o(1))pn$, which is still not sharp enough to show this $+1$ term. Our result is the first one that gives a sharper estimate deriving the $+1$ term.

2. Preliminaries and Key Lemma

As mentioned in Introduction, we consider an $n \times n$ random symmetric matrix A whose diagonal entries are zeros and whose entries above the diagonal are iid random variables, each taking value 1 with probability p and value 0 with probability $1 - p$. We will use symbols $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, ... to denote unit vectors. On the other hand, for general vectors, e.g., $\overline{\boldsymbol{v}}$, by $\overline{\boldsymbol{v}}$, we denote a unit vector $\boldsymbol{v}/\|\boldsymbol{v}\|$ that is obtained from $\boldsymbol{v}$ by normalization. Let $\mathbf{1}$ and $\overline{\mathbf{1}}$ denote all one vector $(1, 1, ..., 1)$ and its normalization respectively. For each i, $1 \leq i \leq n$, let $\boldsymbol{\chi}_i$ denote the iTh largest eigenvector of A.

We rely on the analysis of random $0, 1$-matrices given by Alon and Kahale [2], which is based on the approach given in [5]. More specifically, we base the following lemma given by Coja-Oghlan et al. in [3, Lemma 45]. (Below, we use, e.g., $\#S$ to denote the number of elements in the set S.)

Lemma 1. *Let A be a random matrix as explained above with density parameter $p = \alpha/n$. For some sufficiently large constant d_0, if $\alpha \geq d_0$, then for any constant $c_1 > 1$, there exists some $c_2 > 0$ such that the following three statements hold w.h.p. for the random matrix A.*

(1) Let $V'' = \{i \in \{1, ..., n\} | \sum_{j=1}^{n} a_{ij} > c_1 pn\}$, and $V' = \{1, ..., n\} - V''$. Let $n' = \#V'$ and $A' = (a_{ij})_{i,j \in V'}$ be the induced $n' \times n'$ matrix. Then we have $n' \geq (1 - \exp(-pn/c_2))n$.

(2) For any unit vector $\boldsymbol{\xi}$ such that $\boldsymbol{\xi} \perp \mathbf{1}$, we have $\|A'\boldsymbol{\xi}\| \leq c_2\sqrt{pn'}$.

(3) $\|A'\overline{\mathbf{1}} - pn'\overline{\mathbf{1}}\| \leq c_2\sqrt{pn'}$.

The lemma states first that the number of rows (resp., columns) with unusually many 1 entries is not so large w.h.p. (for sufficiently large α). It then states that after removing such rows and columns the remaining matrix, i.e., the matrix A' of the lemma, behaves w.h.p. more or less similar to the "average" matrix. This A' will be the target of our analysis of this paper. (In the next section, we will use A to denote this A'.)

We prepare some more technical lemmas. Here we introduce our order notations that will be used throughout this paper. For any *positive* function $f(n)$, we use $O(f(n))$ and $o(f(n))$ to denote *some positive* functions $f_1(n)$ and $f_2(n)$ that are defined in each context to satisfy $\exists c_1 > 0, \exists n_1 \in \mathbf{N}, \forall n \geq n_1 \, [\, f_1(n) \leq c_1 f(n) \,]$ and $\forall c > 0, \exists n_2 \in \mathbf{N}, \forall n \geq n_2 \, [\, f_2(n) \leq cf(n) \,]$ respectively. Note that we consider only positive functions, which is different from the standard order notations. This is for simplifying expressions and derivations. For example, for our notations, it does make sense to write expressions such as $f(n) < o(1)$ or use derivations such as $(\sqrt{pn} + o(1))^2 < pn + o(1)$ (because $pn < O(1)$).

Our first technical lemma is a main lemma of this paper. This is a stronger version of the statement (3) of Lemma 1.

Lemma 2. *W.h.p. we have*

$$\big| \, \|A\mathbf{1} - pn\mathbf{1}\|^2 - (p - p^2)n^2 \, \big| \;\; \leq \;\; pn^{1.5+\delta}.$$

Hence, by choosing $\delta < 0.5$, we immediately have

$$\big| \, \|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\| - \sqrt{pn} \, \big| \;\; < \;\; o(1).$$

As shown below, for proving these bounds, we do not need to modify a given random matrix by removing rows and columns with too many 1's as explained above. In other words, the modification was essential for proving (2) of Lemma 1. On the other hand, our real target matrix is A' that is obtained by removing rows of V' (and corresponding columns). Then we need bounds similar to the above for such A', and for this, the following corollary is given. Below let $\mathbf{1}' = (1, \ldots, 1)/\sqrt{n'}$. (Its proof, which idea is essentially given in [3, Proof of Lemma 39], is stated in Appendix.)

Corollary 3. *W.h.p. we have*

$$\big| \, \|A'\overline{\mathbf{1}} - pn'\overline{\mathbf{1}'}\| - \sqrt{pn'} \, \big| \;\; < \;\; O((pn')^{-2}) + o(1).$$

Now consider the proof of Lemma 2. Analysis of this type becomes much easier if we may assume that a given matrix is "completely random", i.e., all nondiagonal entries are iid random variables. On the other hand, our target random matrix A is symmetric, and there are correlations between a_{ij} and a_{ji} for all pairs of i and j, $1 \leq i < j \leq n$. In fact, in the previous work [3, 4] some interesting techniques have been introduced to avoid such correlations. Here we take more direct approach and prove the lemma by considering these correlations. But in order to show the outline, we prove the lemma here ignoring the correlations; a tedious but rather straightforward argument for taking care the correlations is stated in Appendix.

Proof of Lemma 2 (Outline). For each $i \in [n]$, let $X_i = \sum_j a_{ij}$. Note that these are not iid random variables. But let us assume here that they were independent. (There is only one point in the whole proof where this independence

is used, which will be reminded explicitly below.) Next we let $Y_i = (X_i - \mu)^2$, where $\mu = \mathrm{E}[X_i] = p(n-1)$, and let $Y = \sum_i Y_i$. Then we have

$$
\begin{aligned}
\mathrm{E}[Y_i] &= \mathrm{E}[X_i^2] - \mu^2 \\
&= p(n-1) + p^2(n-1)(n-2) - p^2(n-1)^2 \\
&= p(n-1) - p^2(n-1) = (p - p^2)(n-1).
\end{aligned}
$$

Hence, we have $\mathrm{E}[Y] = (p - p^2)n(n-1)$.

On the other hand, following the argument of [3, Proof of Lemma 39], we have $\mathrm{V}[Y_i] \leq \mathrm{E}[Y_i^2] < O((pn)^2)$, which is bounded by $O(1)$ because pn is a constant for our choice of p. Now if all $Y_1, ..., Y_n$ were independent, then we would have

$$
\tag{1} \mathrm{V}[Y] = \sum_i \mathrm{V}[Y_i] \leq n\mathrm{E}[Y_i^2] < O(n).
$$

Then by using Chebyshev bound, we can show that Y is close to $\mathrm{E}[Y]$. More precisely, we have

$$
\begin{aligned}
\Pr[\,|Y - \mathrm{E}[Y]| > pn^{1.5+\delta}\,] &< \frac{\mathrm{V}[Y]}{(pn^{1.5+\delta})^2} \\
&< \frac{O(n)}{(\Omega(1)n^{0.5+\delta})^2} < o(1).
\end{aligned}
$$

Note, on the other hand, that what we want to estimate is $\sum_i (X_i - pn)^2 = \sum_i (X_i - \mu - p)^2$. Let $Y_i' = (X_i - \mu - p)^2$. Then noting

$$
\begin{aligned}
Y_i' &= (X_i - \mu - p)^2 = (X_i - \mu)^2 - 2p(X_i - \mu) + p^2 \\
&= Y_i - 2p(X_i - \mu) + p^2,
\end{aligned}
$$

we have

$$
\mathrm{E}[Y_i'] = \mathrm{E}[Y_i] + p^2 = (p - p^2)n + (2p^2 - p).
$$

Also we have

$$
\begin{aligned}
\mathrm{V}[Y_i'] &= \mathrm{E}\left[(Y_i' - \mathrm{E}[Y_i'])^2\right] \\
&= \mathrm{E}\left[\left((Y_i - 2p(X_i - \mu) + p^2) - (\mathrm{E}[Y_i] + p^2)\right)^2\right] \\
&= \mathrm{E}\left[\left((Y_i - \mathrm{E}[Y_i]) - 2p(X_i - \mu)\right)^2\right] \\
&= \mathrm{V}[Y_i] + 4p^2\mathrm{V}[X_i] \\
&< O(1) + O(p^2) < O(1).
\end{aligned}
$$

Hence, for $Y' = \sum_i Y_i'$, we can again show that Y' is close to $\mathrm{E}[Y']$ w.h.p., and from this we have

$$
\begin{aligned}
o(1) &> \Pr\left[\,|Y' - \mathrm{E}(Y')| > 2pn^{1.5+\delta}\,\right] \\
&= \Pr\left[\,|Y' - ((p - p^2)n^2 + (2p^2 - p)n)| > 2pn^{1.5+\delta}\,\right] \\
&\geq \Pr\left[\,|Y' - (p - p^2)n^2| > pn^{1.5+\delta}\,\right],
\end{aligned}
$$

where the last bound holds for any sufficiently large n.

Finally consider the real situation where $Y_1, ..., Y_n$ are not independent. Even in this case, as shown in Appendix, we still have $V[Y] < O(n)$ and similarly $V[Y'] < O(n)$. Therefore, essentially the same analysis as above is sufficient for the lemma. $\square$

Let us state some more small technical lemmas. In the following, let A' and n' be the random matrix and its size given in Lemma 1, and let $\mathbf{1}'$ denote the vector $(1, \dots, 1)$ with n' components. We assume that the part (1) of Lemma 1 holds.

Lemma 4. *For any unit vector $\boldsymbol{\xi}$ that is perpendicular to $\mathbf{1}'$, and for any unit vector $\boldsymbol{\eta}$, w.h.p. we have*

$$|\langle A'\boldsymbol{\xi}, \boldsymbol{\eta}\rangle| \ \leq \ c_2\sqrt{pn'}.$$

Proof. Let $A'\boldsymbol{\xi} = \sum_i a_i \boldsymbol{\chi}'_i$, by using eigenvectors $\boldsymbol{\chi}'_1, \boldsymbol{\chi}'_2, ..., \boldsymbol{\chi}'_{n'}$ of A', Then from Lemma 1 (2), w.h.p. we have

$$\sum_i a_i^2 \ = \ \|A'\boldsymbol{\xi}\|^2 \ \leq \ c_2 pn'.$$

On the other hand, letting $\boldsymbol{\eta} = \sum_i b_i \boldsymbol{\chi}'_i$, we have $\sum_i b_i^2 = 1$. Then the lemma is shown by

$$\langle A'\boldsymbol{\xi}, \boldsymbol{\eta}\rangle^2 \ = \ \left\langle \sum_i a_i \boldsymbol{\chi}'_i, \sum_j b_j \boldsymbol{\chi}'_j \right\rangle^2 \ = \ \left(\sum_i a_i b_i\right)^2$$

$$\leq \ \left(\sum_i a_i^2\right)\left(\sum_j b_j^2\right) \ \leq \ (c_2)^2 pn'. \qquad \square$$

By using this lemma, we can bound eigenvalues as follows.

Corollary 5. *For some constant c_3 (e.g., letting $c_3 = 3$), w.h.p. we have*

$$|\lambda'_i| \ \leq \ c_3 c_2 \sqrt{pn'}$$

for any i, $2 \leq i \leq n'$.
Remark. *The coefficient $c_3 = 3$ is a bit loose. In fact, we have $|\lambda'_i| \leq c_2\sqrt{pn'}$ for all i, $2 \leq i \leq n' - 1$.*

Proof. We make use of the following Courant-Fischer characterization (recall that $\boldsymbol{\xi}$ denote a unit vector):

$$\lambda'_i \ = \ \max_{S:\dim(S)=i} \min_{\boldsymbol{\xi}\in S} \langle A'\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \ = \ \min_{S:\dim(S)=n'-i+1} \max_{\boldsymbol{\xi}\in S} \langle A'\boldsymbol{\xi}, \boldsymbol{\xi}\rangle.$$

First for bounding λ'_2 and λ'_{n-1}, we use $S_0 = \{\boldsymbol{v} | \boldsymbol{v} \perp \mathbf{1}'\}$. Then w.h.p. we have

$$\lambda'_2 \ \leq \ \max_{\boldsymbol{\xi}\in S_0} \langle A\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \ \leq \ c_2\sqrt{pn'}, \text{ and}$$

$$\lambda'_{n'-1} \ \geq \ \min_{\boldsymbol{\xi}\in S_0} \langle A\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \ \geq \ -c_2\sqrt{pn'}.$$

Finally for bounding $\lambda'_{n'}$, we let $\boldsymbol{\xi}_*$ to denote the one that defines $\lambda'_n = \langle A\boldsymbol{\xi}_*, \boldsymbol{\xi}_* \rangle$. Let $\boldsymbol{\xi}_* = a\overline{\mathbf{1}'} + b\boldsymbol{\eta}$, where $a = \langle \boldsymbol{\xi}_*, \overline{\mathbf{1}'} \rangle$ (and thus $|a|, |b| \leq 1$ and $\boldsymbol{\eta} \perp \overline{\mathbf{1}'}$). Then by using (i) $\langle A'\overline{\mathbf{1}'}, \overline{\mathbf{1}'} \rangle \geq 1$, (ii) the symmetry of A', and (iii) Lemma 4, we have

$$
\begin{aligned}
\lambda'_{n'} &= \langle A'\boldsymbol{\xi}_*, \boldsymbol{\xi}_* \rangle = \langle A'(a\overline{\mathbf{1}'} + b\boldsymbol{\eta}), a\overline{\mathbf{1}'} + b\boldsymbol{\eta} \rangle \\
&= a^2 \langle A'\overline{\mathbf{1}'}, \overline{\mathbf{1}'} \rangle + ab(\langle A'\boldsymbol{\eta}, \overline{\mathbf{1}'} \rangle + \langle A'\overline{\mathbf{1}'}, \boldsymbol{\eta} \rangle) + b^2 \langle A'\boldsymbol{\eta}, \boldsymbol{\eta} \rangle \\
&= a^2 \langle A'\overline{\mathbf{1}'}, \overline{\mathbf{1}'} \rangle + 2ab(\langle A'\boldsymbol{\eta}, \overline{\mathbf{1}'} \rangle + b^2 \langle A'\boldsymbol{\eta}, \boldsymbol{\eta} \rangle \\
&\geq -(2ab + b^2)\sqrt{pn'},
\end{aligned}
$$

which is our desired bound. $\qquad\square$

On the other hand, since $\langle A'\mathbf{1}', \mathbf{1}' \rangle$ is $\sum_{ij} a_{ij}$, we have the following from the Chernoff bound. This shows that $\lambda'_1 > pn' - o(1)$.

Lemma 6. *For any δ, $0 < \delta \leq 1/2$, w.h.p. we have*

$$
pn' - pn'^{1-\delta} \leq \langle A'\overline{\mathbf{1}'}, \overline{\mathbf{1}'} \rangle \leq pn' + pn'^{1-\delta}.
$$

3. Main Analysis

As explained above, we will analyze matrix A' that is defined in Lemma 1 by removing rows of V'' (and the corresponding columns) from matrix A generated randomly with the density parameter p. We may assume that Lemma 1 (1) holds. For simplifying our notation, throughout this section we will use A and n (instead of A' and n') to denote a random matrix of this type and its size respectively. Also throughout this section, we assume that α is sufficiently large, or, more specifically, $\alpha > (c_2)^2$, where c_2 is the constant of Lemma 1.

For stating our result succinctly we introduce some more order notations. By $o_\alpha(1)$ and $o_{\alpha^+}(1)$ we denote *some positive* functions $\varepsilon_1(\alpha)$ and $\varepsilon_2(\alpha)$ that are defined in each context to satisfy $\varepsilon_1(\alpha) \leq d_1/\alpha^{b_1}$ and $\varepsilon_2(\alpha) \leq d_2/\alpha^{1.5}$ respectively with some constants $b_1, d_1, d_2 > 0$ independent from n and α. That is, while $o_\alpha(1)$ goes to 0 when α goes ∞, $o_{\alpha^+}(1)$ is a function that goes to 0 faster than $1/\alpha$. For any positive function $f(\alpha)$, if it satisfies that $\exists c > 0$, $\alpha_0 > 0$, $\forall \alpha > \alpha_0$, then we have

$$
\begin{aligned}
\frac{1}{f(\alpha) - o_\alpha(1)} &\leq \frac{1}{f(\alpha)} + o_\alpha(1), \text{ and} \\
\frac{1}{f(\alpha) + o_\alpha(1)} &\geq \frac{1}{f(\alpha)} - o_\alpha(1).
\end{aligned}
$$

These derivations are useful in the following analysis. With this notation, the bound of Corollary 3 can be restated as

$$
(2) \qquad \qquad | \, \|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\| - \sqrt{pn} \, | \ < \ o_{\alpha^+}(1) + o(1).
$$

What is essential for our analysis is the relation between $\boldsymbol{\chi}_1$ and $\overline{\mathbf{1}}$. In the following we fix $a_1, ..., a_n$, b, and $\boldsymbol{\chi}$ to those satisfying

$$\overline{\mathbf{1}} \;=\; a_1\boldsymbol{\chi}_1 + \cdots + a_n\boldsymbol{\chi}_n \;=\; a_1\boldsymbol{\chi}_1 + b\boldsymbol{\chi}.$$

Here recall that the overlap a is defined by $a = \langle \boldsymbol{\chi}_1, \overline{\mathbf{1}} \rangle$. Hence, we have $a_1 = a$ and

$$\boldsymbol{\chi}_1 \;=\; a\overline{\mathbf{1}} + b\boldsymbol{\eta}$$

for some $\boldsymbol{\eta}$ such that $\boldsymbol{\eta} \perp \overline{\mathbf{1}}$. Note that $a^2 + b^2 = 1$ and $b^2 = \sum_{i\geq 2} a_i^2$. Thus, for bounding a (which is one of our goals), we will estimate $\sum_{i\geq 2} a_i^2 \ (= b^2)$ and $\sqrt{1-b^2} \ (= a)$.

Here the key tool is the bound (2) applied to the following relation:

$$
\begin{aligned}
\|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\|^2 \;&=\; \left\| A\left(\sum_i a_i\boldsymbol{\chi}_i\right) - pn\left(\sum_i a_i\boldsymbol{\chi}_i\right) \right\|^2 \\
&=\; \left\| \sum_i (a_i\lambda_i - a_i \cdot pn)\boldsymbol{\chi}_i \right\|^2 \\
(3) \qquad &=\; \sum_i a_i^2(\lambda_i - pn)^2.
\end{aligned}
$$

Theorem 1. *W.h.p. we have*

$$(4) \qquad\qquad a \;\geq\; \sqrt{1 - \frac{1}{\alpha} - o_{\alpha^+}(1) - o(1)}.$$

Proof. By using (2) and (3), we have

$$
\begin{aligned}
pn + o_{\alpha^+}(1) &+ o(1) \\
&>\; \sum_i a_i^2(\lambda_i - pn)^2 \;\geq\; \sum_{i\geq 2} a_i^2(\lambda_i - pn)^2 \\
&\geq\; \sum_{i\geq 2} a_i^2(c_2\sqrt{pn} - pn)^2 \;=\; \sum_{i\geq 2} a_i^2(\sqrt{pn} - c_2)^2 pn.
\end{aligned}
$$

Hence,

$$(5) \qquad b^2 \;=\; \sum_{i\geq 2} a_i^2 \;<\; \frac{1 + o_{\alpha^+}(1) + o(1)}{(\sqrt{pn} - c_2)^2} \;<\; \frac{1}{(\sqrt{\alpha} - c_2)^2} + o_{\alpha^+}(1) + o(1),$$

and then our desired bound (4) follows. $\square$

Next we give the following upper bound for λ_1.

Theorem 2. *W.h.p. we have*

$$(6) \qquad\qquad \lambda_1 \;<\; \alpha + 1 + o_\alpha(1) + o(1).$$

Proof. We first show that

$$(7) \qquad \lambda_1 \; < \; \alpha + \frac{\sqrt{\alpha}}{a(\sqrt{\alpha} - c_2)} + o_{\alpha^+}(1) + o(1)$$

holds w.h.p. Let $\boldsymbol{w} = A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}$. We estimate $\langle A\overline{\mathbf{1}}, \boldsymbol{\chi}_1 \rangle$ in the following two ways:

$$
\begin{aligned}
\langle A\overline{\mathbf{1}}, \boldsymbol{\chi}_1 \rangle &= \langle A(a\boldsymbol{\chi}_1 + b\boldsymbol{\chi}), \boldsymbol{\chi}_1 \rangle \\
&= \langle a\lambda_1 \boldsymbol{\chi}_1 + bA\boldsymbol{\chi}, \boldsymbol{\chi}_1 \rangle \;=\; a\lambda_1, \text{ and} \\
\langle A\overline{\mathbf{1}}, \boldsymbol{\chi}_1 \rangle &= \langle A\overline{\mathbf{1}}, a\overline{\mathbf{1}} + b\boldsymbol{\eta} \rangle \\
&= a\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}} \rangle + b\langle pn\overline{\mathbf{1}} + \boldsymbol{w}, \boldsymbol{\eta} \rangle \\
&= a\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}} \rangle + b\langle \boldsymbol{w}, \boldsymbol{\eta} \rangle \;\leq\; a\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}} \rangle + b\|\boldsymbol{w}\| \\
&< a\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}} \rangle + b\sqrt{pn} + o_{\alpha^+}(1) + o(1) \qquad \text{(by (2) and } |b| \leq 1) \\
&< a\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}} \rangle + \frac{\sqrt{pn}}{\sqrt{pn} - c_2} + o_{\alpha^+}(1) + o(1). \qquad \text{(by (5))}
\end{aligned}
$$

The bound (7) follows from these two bounds.

Next we derive (6). First note that from (4) we have

$$\frac{1}{a^2} \; < \; \left(1 + \frac{1}{\alpha} + o_{\alpha^+}(1) + o(1)\right)^2.$$

Using this bound, the desired bound is shown by

$$
\begin{aligned}
\lambda_1 \; &< \; \alpha + \frac{1}{a} \cdot \frac{\sqrt{\alpha}}{\sqrt{\alpha} - c_2} + o_{\alpha^+}(1) + o(1) \\
&< \; \alpha + \frac{1}{a}\left(1 + o_\alpha(1)\right) + o_{\alpha^+}(1) + o(1) \\
&< \; \alpha + \left(1 + \frac{1}{\alpha} + o_{\alpha^+}(1) + o(1)\right)\left(1 + o_\alpha(1)\right) \\
&< \; \alpha + 1 + o_\alpha(1) + o(1). \qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Theorem 3. *W.h.p. we have*

$$(8) \qquad a \; < \; \sqrt{1 - \frac{1}{\alpha} + o_{\alpha^+}(1) + o(1)}.$$

Thus, within our approximation, this upper bound matches to the lower bound (4).

Proof. Here again using (2) and (3), we have

$$
\begin{aligned}
pn - o_{\alpha^+}(1) - o(1) \;&<\; \sum_i a_i^2 (\lambda_i - pn)^2 \\
&=\; a^2(\lambda_1 - pn)^2 + \sum_{i \geq 2} a_i^2(\lambda_i - pn)^2 \\
&<\; \frac{pn}{(\sqrt{pn} - c_2)^2} + o_{\alpha^+}(1) + o(1) + \sum_{i \geq 2} a_i^2 (pn + c_3 c_2 \sqrt{pn})^2 .
\end{aligned}
$$

Here for deriving the last inequality, we used bounds for λ_1: a lower bound $pn - o(1)$ is from Lemma 6, and an upper bound is (7).

Hence we have

$$
\begin{aligned}
b^2 \;=\; \sum_{i \geq 2} a_i^2 \;&>\; \frac{1 - (\sqrt{pn} - c_2)^{-2}}{(c_2 c_3 + \sqrt{pn})^2} - o_{\alpha^+}(1) - o(1) \\
&=\; \frac{1 - (\sqrt{\alpha} - c_2)^{-2}}{(c_2 c_3 + \sqrt{\alpha})^2} - o_{\alpha^+}(1) - o(1) .
\end{aligned}
$$

Hence,

$$
a \;<\; \sqrt{1 - \frac{1 - (\sqrt{\alpha} - c_2)^{-2}}{(c_2 c_3 + \sqrt{\alpha})^2}} + o_{\alpha^+}(1) + o(1) .
$$

Then the bound (8) of the lemma follows. $\qquad\square$

Finally we show the following lower bound for λ_1.

Theorem 4. *W.h.p. we have*

$$
(9) \qquad\qquad \lambda_1 \;>\; \alpha + 1 - o_\alpha(1) - o(1) .
$$

Proof. We start by estimating $\|A\overline{\mathbf{1}}\|^2$. Let $\boldsymbol{w}' = A\overline{\mathbf{1}} - \langle A\overline{\mathbf{1}}, \overline{\mathbf{1}}\rangle\overline{\mathbf{1}}$. Then since $\boldsymbol{w}' \perp \overline{\mathbf{1}}$, we have

$$
\begin{aligned}
\|A\overline{\mathbf{1}}\|^2 \;&=\; \langle A\overline{\mathbf{1}}, \overline{\mathbf{1}}\rangle^2 + \|\boldsymbol{w}'\|^2 \;\geq\; (pn - pn^{1-\delta})^2 + \|\boldsymbol{w}'\|^2 \\
(10) \qquad &>\; (pn)^2 + \|\boldsymbol{w}'\|^2 - o(1) .
\end{aligned}
$$

On the other hand, we let $\boldsymbol{w} = A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}$, and then we have

$$
\boldsymbol{w}' + \Big(\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}}\rangle - pn \Big)\overline{\mathbf{1}} \;=\; \boldsymbol{w} .
$$

Hence, by the triangular inequality, we have

$$
\begin{aligned}
\|\boldsymbol{w}'\| \;&\geq\; \|\boldsymbol{w}\| - \left\| \Big(\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}}\rangle\overline{\mathbf{1}} - pn \Big)\overline{\mathbf{1}} \right\| \\
&=\; \|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\| - \left\| \Big(\langle A\overline{\mathbf{1}}, \overline{\mathbf{1}}\rangle\overline{\mathbf{1}} - pn \Big)\overline{\mathbf{1}} \right\| \\
&>\; (\sqrt{pn} - o_{\alpha^+}(1) - o(1)) - \|(pn^{1-\delta})\overline{\mathbf{1}}\| ,
\end{aligned}
$$

where (2) and Lemma 6 are used to get this bound for some $\delta > 0$. Thus, we have

$$\|\boldsymbol{w}'\| > \sqrt{pn} - o_{\alpha^+}(1) - o(1).$$

Then by (10), we have

(11) $$\|A\bar{\mathbf{1}}\|^2 > (pn)^2 + pn - o_{\alpha^+}(1) - o(1).$$

On the other hand, we have

$$\|A\bar{\mathbf{1}}\|^2 = a^2\lambda_1^2 + \sum_{i\geq 2} a_i^2\lambda_i^2 \leq a^2\lambda_1^2 + b^2(c_3c_2)^2pn.$$

Hence, by using (11), we have

$$\begin{aligned}
\lambda_1^2 &> \frac{1}{a^2}\cdot(pn)^2 + pn - b^2(c_3c_2)^2pn - o_{\alpha^+}(1) - o(1) \\
&= \frac{\alpha^2}{a^2} + \alpha - b^2(c_3c_2)^2\alpha - o_{\alpha^+}(1) - o(1).
\end{aligned}$$

Now we analyze below the first and the third terms of this bound. For the third term, by using the upper bound (5), we can show some constant $d_1 > 0$ exists such that

$$\begin{aligned}
b^2(c_3c_2)^2\alpha &> \left(\frac{1}{\alpha} + o_{\alpha^+}(1) + o(1)\right)(c_2c_2)^2\alpha \\
&> d_1 + o_\alpha(1) + o(1)
\end{aligned}$$

holds. On the other hand, by using (8), we have some d_2 with which the first term is bounded by

$$\begin{aligned}
\frac{\alpha^2}{a^2} &\geq \frac{1}{1 - 1/\alpha + o_{\alpha^+}(1) + o(1)}\cdot\alpha^2 \\
&= \left(1 + \frac{1 - \alpha\cdot o_{\alpha^+}(1)}{\alpha - 1 + \alpha\cdot o_{\alpha^+}(1)}\right)\cdot\alpha^2 \\
&\geq \alpha^2 + \frac{\alpha^2 - \alpha^3\cdot o_{\alpha^+}(1)}{\alpha + \alpha\cdot o_{\alpha^+}(1)} \\
&> \alpha^2 + \alpha\left(\frac{1}{1 + o_{\alpha^+}(1)}\right) - d_2\alpha^{-0.5}.
\end{aligned}$$

From these bounds, for any $\epsilon > 0$, we have

$$\begin{aligned}
\lambda_1^2 \geq{}& \alpha^2 + 2\alpha\left(\frac{1 - \epsilon}{1 + o_{\alpha^+}(1)}\right) + \left(\frac{1 - \epsilon}{1 + o_{\alpha^+}(1)}\right)^2 \\
&+ \frac{2\alpha\epsilon}{1 + o_{\alpha^+}(1)} - \left(\frac{1 - \epsilon}{1 + o_{\alpha^+}(1)}\right)^2 - d_1 - d_2\alpha^{-0.5} - o_{\alpha^+}(1) - o(1),
\end{aligned}$$

where the sum of the last six terms is positive by taking $\epsilon = d_3\alpha^{-1.5} + o_{\alpha^+}(1) + o(1)$ for some sufficiently large constant $d_3 > 0$.

Therefore, with this d_3, we have

$$\lambda_1 \; > \; \alpha + \left(\frac{1 - d_3\alpha^{-1.5} - o_{\alpha^+}(1) - o(1)}{1 + o_{\alpha^+}(1)} \right) \; > \; \alpha + 1 - o_{\alpha^+}(1) - o(1). \quad \square$$

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Appendix

A. Analysis of $V[Y]$. Here we give a detail analysis of $V[Y]$ that is omitted in the proof of Lemma 2. We use the same notations, but let us recall important ones.

$$X_i \; = \; \sum_j a_{ij}, \quad \mu \; = \; E[X_i] \; = \; p(n-1) \; < \; O(1),$$

$$Y_i \; = \; (X_i - \mu)^2, \text{ and } Y \; = \; \sum_i Y_i.$$

For simplifying our notation, we introduce here the following notations:

$$(12) \qquad \begin{aligned} E_{X2} &= E[X_1^2] = p(n-1) + p^2(n-1)(n-2), \\ E_{Y1} &= E[Y_1] = E[(X_1 - \mu)^2] = E_{X2} - \mu^2, \text{ and} \\ E_{Y2} &= E[Y_1^2] < O((pn)^2) < O(1). \end{aligned}$$

By symmetry we have $E_{X2} = E[X_i^2]$, $E_{Y1} = E[Y_i]$, and $E_{Y2} = E[Y_i^2]$ for all $i \in [n]$.

Our goal is to estimate $V[Y]$. Let $\nu = E[Y]$; note that $\nu = nE_{Y1}$. First by definition we have

$$V[Y] = E[(Y - \nu)^2] = E[Y^2 - 2Y\nu + \nu^2]$$

$$= E\left[\left(\sum_i Y_i\right)^2\right] - 2E[Y]\nu + \nu^2 = E\left[\left(\sum_i Y_i\right)^2\right] - \nu^2$$

$$= E\left[\sum_i Y_i^2\right] + E\left[\sum_{i \neq i'} Y_i \cdot Y_{i'}\right] - \nu^2$$

$$= nE_{Y2} + n(n-1)E[Y_1 \cdot Y_2] - n^2(E_{Y1})^2$$

(13)
$$\leq nE_{Y2} + n^2\left(E[Y_1 \cdot Y_2] - (E_{Y1})^2\right),$$

where by symmetry the choice of Y_1 and Y_2 is not essential.

If we could assume that Y_1 and Y_2 are independent, then we would have

$$(13) = nE_{Y2} + n^2\left(E[Y_1] \cdot E[Y_2] - (E_{Y1})^2\right) = nE_{Y2}$$

Thus, for our estimation, we consider the difference between $E[Y_1 \cdot Y_2]$ and $(E_{Y1})^2$.

Again by definition, we have

$$E[Y_1 \cdot Y_2]$$
$$= E\left[(X_1 - \mu)^2(X_2 - \mu)^2\right]$$
$$= E\left[(X_1^2 - 2\mu X_1 + \mu^2)(X_2^2 - 2\mu X_2 + \mu^2)\right]$$
$$= E\left[X_1^2 X_2^2 - 2\mu X_1^2 X_2 + \mu^2 X_1^2\right.$$
$$\left. -2\mu X_1 X_2^2 + 4\mu^2 X_1 X_2 - 2\mu^3 X_1 + \mu^2 X_2^2 - 2\mu^3 X_2 + \mu^4\right]$$

(14)
$$= E[X_1^2 X_2^2] - 4\mu E[X_1^2 X_2] + 4\mu^2 E[X_1 X_2] + 2\mu^2 E_{X2} - 4\mu^4 + \mu^4.$$

We estimate expectations $E[X_1^2 X_2^2]$, $E[X_1^2 X_1]$, and $E[X_1 X_2]$.

First consider $E[X_1 X_2]$. Since $X_1 = \sum_j a_{1j}$ and $X_2 = \sum_j a_{2j}$, the correlation between X_1 and X_2 is caused only by $a_{12} = a_{21}$; all the other variables are independent of each other. In the following, let us use the following notations:

$$S_{a1} = \sum_{j \neq 2} a_{1j}, \text{ and } S_{a2} = \sum_{j \neq 1} a_{2j}.$$

Also let a' denote $a_{12} = a_{21}$. Hence, we have $X_1 = S_{a1} + a'$ and $X_2 = S_{a2} + a'$, and it holds that

$$E[X_1 X_2] = E\left[(S_{a1} + a')(S_{a2} + a')\right]$$
$$= E[S_{a1} S_{a2}] + 2E[a' S_{a1}] + E[a'^2]$$
$$= (p(n-2))^2 + 2p(p(n-2)) + p$$

(15)
$$= p^2 n(n-2) + p = \mu^2 - p^2 + p.$$

Next consider $\mathrm{E}[X_1^2 X_2]$. Again using our notations, we can evaluate it as

$$
\begin{aligned}
\mathrm{E}[X_1^2 X_2] &= \mathrm{E}\left[X_1^2 S_{\mathrm{a}2} + (S_{\mathrm{a}1} + a')^2 a'\right] \\
&= E_{\mathrm{X}2} \cdot \mathrm{E}[S_{\mathrm{a}2}] + \mathrm{E}\left[a'(S_{\mathrm{a}1})^2\right] + \mathrm{E}\left[2a'^2 S_{\mathrm{a}1}\right] + \mathrm{E}[a'^3] \\
&= E_{\mathrm{X}2} \cdot (\mu - p) + p\left(p^2(n-2)(n-3) + 3p(n-2) + 1\right)
\end{aligned}
$$

Here we compare the last term of (16) and

$$
E_{\mathrm{X}2} = p(n-1) + p^2(n-1)(n-2). \quad \text{(by (12))}
$$

Then we have

$$
\begin{aligned}
(16) &= E_{\mathrm{X}2} \cdot (\mu - p) + p\left(E_{\mathrm{X}2} + 2pn + 1 - 2p^2(n-2) - 5p)\right) \\
&= E_{\mathrm{X}2} \cdot \mu + 2p^2 n + p - 2p^3(n-2) - 5p^2.
\end{aligned}
$$

The last one is $\mathrm{E}[X_1^2 X_2^2]$. This can be stated as

$$
\begin{aligned}
\mathrm{E}[X_1^2 X_2^2] &= \mathrm{E}\left[X_1^2 (S_{\mathrm{a}2} + a')^2\right] \\
&= \mathrm{E}\left[X_1^2 \left((S_{\mathrm{a}2})^2 + 2a' S_{\mathrm{a}2} + a'^2\right)\right] \\
&= E_{\mathrm{X}2} \cdot \mathrm{E}\left[(S_{\mathrm{a}2})^2\right] + \mathrm{E}\left[\left((S_{\mathrm{a}1})^2 + 2a' S_{\mathrm{a}1} + a'^2\right)\left(2a' S_{\mathrm{a}2} + a'^2\right)\right] \\
&= E_{\mathrm{X}2} \cdot \mathrm{E}\left[(S_{\mathrm{a}2})^2\right] \\
&\quad + \mathrm{E}\left[2a'(S_{\mathrm{a}1})^2 S_{\mathrm{a}2}\right] + \mathrm{E}\left[a'^2 (S_{\mathrm{a}1})^2\right] \\
&\quad + \mathrm{E}\left[4a'^2 (S_{\mathrm{a}1})(S_{\mathrm{a}2})\right] + \mathrm{E}\left[2a'^3 S_{\mathrm{a}1}\right] \\
&\quad + \mathrm{E}\left[2a'^3 S_{\mathrm{a}2}\right] + \mathrm{E}[a'^4] \\
&= E_{\mathrm{X}2} \cdot E'_{\mathrm{X}2} + 2p\mu' E'_{\mathrm{X}2} + p E'_{\mathrm{X}2} + 4p\mu'^2 + 4p\mu' + p,
\end{aligned}
$$

where we let E'_{X2} and μ' denote $\mathrm{E}[(S_{\mathrm{a}1})^2]$ $(= \mathrm{E}[(S_{\mathrm{a}2})^2])$ and $\mathrm{E}[S_{\mathrm{a}1}]$ $(= \mathrm{E}[S_{\mathrm{a}2}])$ respectively.

Here note that

$$
E_{\mathrm{X}2} = \mathrm{E}[X_1^2] = \mathrm{E}\left[(S_{\mathrm{a}1}^2 + 2a' S_{\mathrm{a}1} + a'^2\right] = E'_{X2} + 2p\mu' + p.
$$

Hence $E_{\mathrm{X}2} \cdot E_{\mathrm{X}2}$ is expressed as

$$
\begin{aligned}
E_{\mathrm{X}2} \cdot E_{\mathrm{X}2} &= E_{\mathrm{X}2} \cdot (E'_{X2} + 2p\mu' + p) \\
&= E_{\mathrm{X}2} \cdot E'_{X2} + (E'_{X2} + 2p\mu' + p)(2p\mu' + p) \\
&= E_{\mathrm{X}2} \cdot E'_{X2} + 2p\mu' E'_{X2} + p E'_{X2} + 4p^2 \mu'^2 + 2p^2 \mu' + p^2.
\end{aligned}
$$

Comparing this with (16), we can easily see that

$$
(16) \qquad \mathrm{E}[X_1^2 X_2^2] = (E_{\mathrm{X}2})^2 + p(4\mu'^2 + 4\mu' + 1) - p^2(4\mu'^2 + 2\mu' + 1)
$$

Now by (15), (16), and (16), we can restate (14) as

$$
\begin{aligned}
\mathrm{E}[Y_1 \cdot Y_2] \quad &(= (14)) \\
&\leq (E_{\mathrm{X}2})^2 - 2\mu^2 E_{\mathrm{X}2} + \mu^4 + p(4\mu'^2 + 4\mu' + 1) + 4\mu(2p^3(n-2) + 5p^2) + 4\mu^2 p.
\end{aligned}
$$

On the other hand, since we have

$$\left(E_{Y1}\right)^2 \;=\; \left(E_{X2} - \mu^2\right)^2 \;=\; \left(E_{X2}\right)^2 - 2\mu^2 E_{X2} + \mu^4,$$

and $pn = \alpha < O(1)$, it holds that

$$E[Y_1 \cdot Y_2] \;<\; E[Y_1] \cdot E[Y_2] + p \cdot O(1),$$

which implies

(17) $$V[Y] \;<\; nE_{Y2} + n^2 p \cdot O(1) \;<\; O(n).$$

This is our desired bound.

B. A proof of Corollary 3. Recall that A is an n-by-n symmetric matrix while A' is n'-by-n' (and symmetric). Also recall that we use $\mathbf{1}$ and $\mathbf{1'}$ to denote all one vectors with n components and n' components respectively, and that $\overline{\mathbf{1}}$ and $\overline{\mathbf{1'}}$ are the corresponding normalized verctors.

We note two facts used here: $\left|\left\|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\right\| - \sqrt{pn}\right| = o(1)$, and $n' \geq (1-\exp(-pn/c_2))n$. As a first step to showing Corollary 3, we derive the following bound:

$$\left|\left\|A'\overline{\mathbf{1'}} - pn\overline{\mathbf{1'}}\right\| - \sqrt{pn}\right|$$
$$\leq \;\left|\left\|A'\overline{\mathbf{1'}} - pn\overline{\mathbf{1'}}\right\| - \left\|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\right\|\right| + \left|\left\|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\right\| - \sqrt{pn}\right|$$
$$\leq \;\left|\left\|A'\overline{\mathbf{1'}} - pn\overline{\mathbf{1'}}\right\| - \left\|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\right\|\right| + o(1).$$

For the first term of the last formula above, we invoke a claim from [3], precisely the formula (28) therein. That is,

$$\forall S \subset V : |S| \leq (pn)^{-10}n \left[\sum_{v \in V}\sum_{u \in S} a_{vu} \leq (pn)^{-5}n\right].$$

Let $x_v = \sum_{u \in V'} a_{vu}$ and $y_v = \sum_{u \in V''} a_{vu}$. Then, $\sum_{v \in V'} y_v \leq (pn)^{-5}n$. Thus,

$$\|A'\mathbf{1'} - pn\mathbf{1'}\|^2 - \|A\mathbf{1} - pn\mathbf{1}\|^2 \;=\; \sum_{v \in V'}(x_v - pn)^2 - \sum_{v \in V}(x_v + y_v - pn)^2$$
$$\leq \;\sum_{v \in V'}\left((x_v - pn)^2 - (x_v + y_v - pn)^2\right)$$
$$\leq \;\sum_{v \in V'} 2pny_v \;\leq\; 2pn(pn)^{-5}n \;=\; 2n(pn)^{-4}.$$

Thus, noting that A and A' are n-by-n and n'-by-n' respectively, we have

$$\left\|A'\overline{\mathbf{1'}} - pn\overline{\mathbf{1'}}\right\|^2 \frac{n'}{n} - \left\|A\overline{\mathbf{1}} - pn\overline{\mathbf{1}}\right\|^2 \leq 2(pn)^{-4}.$$

This implies

$$(1 - \exp(-pn/c_2)) \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\|^2 - \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2 \leq 2(pn)^{-4}$$

$$\Longleftrightarrow \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\|^2 - \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2 / (1 - \exp(-pn/c_2))$$
$$\leq 2(pn)^{-4} / (1 - \exp(-pn/c_2))$$

$$\Longleftrightarrow \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\|^2 - \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2 (1 + 2\exp(-pn/c_2))$$
$$\leq 2(pn)^{-4}(1 + 2\exp(-pn/c_2))$$

$$\Longleftrightarrow \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\|^2 - \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2$$
$$\leq 2(pn)^{-4}(1 + 2\exp(-pn/c_2)) + 2\exp(-pn/c_2) \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2$$

$$\Longleftrightarrow \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\|^2 - \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2$$
$$\leq 2(pn)^{-4}(1 + 2\exp(-pn/c_2)) + 2\exp(-pn/c_2)(pn + o(1))$$

$$\Longleftrightarrow \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\|^2 - \left\| A\overline{\mathbf{1}} - pn\overline{\mathbf{1}} \right\|^2 \leq 3(pn)^{-4}.$$

Thus,

$$\left| \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\| - \sqrt{pn} \right| \leq O((pn)^{-2}) + o(1).$$

Next, using this, we derive

$$\left| \left\| A'\overline{\mathbf{1}'} - pn'\overline{\mathbf{1}'} \right\| - \sqrt{pn} \right| = \left| \left\| (A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'}) + (pn\overline{\mathbf{1}'} - pn'\overline{\mathbf{1}'}) \right\| - \sqrt{pn} \right|$$
$$\leq \left| \left\| A'\overline{\mathbf{1}'} - pn\overline{\mathbf{1}'} \right\| - \sqrt{pn} \right| + \left\| pn\overline{\mathbf{1}'} - pn'\overline{\mathbf{1}'} \right\|$$
$$\leq O((pn)^{-2}) + (pn - pn') + o(1)$$
$$= O((pn)^{-2}) + o(1).$$

Finally, the target bound is derivied by

$$\left| \left\| A'\overline{\mathbf{1}'} - pn'\overline{\mathbf{1}'} \right\| - \sqrt{pn'} \right| \leq \left| \left\| A'\overline{\mathbf{1}'} - pn'\overline{\mathbf{1}'} \right\| - \sqrt{pn} \right| + \left| \sqrt{pn} - \sqrt{pn'} \right|$$
$$\leq O((pn)^{-2}) + \left| \sqrt{pn} - \sqrt{pn'} \right| + o(1)$$
$$\leq O((pn)^{-2}) + o(1),$$

where the last iequality comes from

$$
\begin{aligned}
\sqrt{pn} - \sqrt{pn'} \le \alpha^{-2} \iff & \sqrt{\alpha} - \sqrt{\alpha(1 - \exp(-\alpha/c_2))} \le \alpha^{-2} \\
\iff & 1 - (1 - \exp(-\alpha/c_2))^{1/2} \le \alpha^{-2-1/2} \\
\iff & 1 - \alpha^{-5/2} \le (1 - \exp(-\alpha/c_2))^{1/2} \\
\iff & (1 - \alpha^{-5/2})^2 \le 1 - \exp(-\alpha/c_2) \\
\iff & 1 - 2\alpha^{-5/2} + \alpha^{-5} \le 1 - \exp(-\alpha/c_2) \\
\iff & \exp(-\alpha/c_2) \le 2\alpha^{-5/2} - \alpha^{-5}
\end{aligned}
$$

since we let $\alpha = pn$.

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