LETTER

Computational Complexity of Generalized Forty Thieves

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SUMMARY Forty Thieves is a solitaire game with two 52-card decks. The object is to move all cards from ten tableau piles of four cards to eight foundations. Each foundation is built up by suit from ace to king of the same suit, and each tableau pile is built down by suit. You may move the top card from any tableau pile to a tableau or foundation pile, and from the stock to a foundation pile. We prove that the generalized version of Forty Thieves is NP-complete.

key words: *computational complexity, NP-completeness, puzzle*

1. Introduction

Forty Thieves is a solitaire game with two 52-card decks. The object is to move all cards from ten tableau piles of four cards to eight foundations (see Fig. 1). Each foundation is built up by suit from ace to king of the same suit, and each tableau pile is built down by suit. You may move the top card from any tableau pile to a tableau or foundation pile, and from the stock to a foundation pile. (You can play Forty Thieves online at many sites on the Internet.)

A card is *exposed* if no cards cover it. Exposed cards in tableau piles may be moved to a foundation (resp. tableau pile) if they are one rank higher (resp. lower) than the top card of the foundation (resp. tableau pile). (Here, empty foundations are regarded as cards of rank zero.) If no cards may be moved, then the top card of the stock may be moved to a foundation or to the waste. The cards in the waste position cannot be reused. Once no more cards can be moved, the game ends. The aim of the game is to move all cards in the tableau piles to eight foundations.

Figure 1 is an initial layout of Forty Thieves, where cards of the first and second decks are denoted as ♠A, ♠2, ♠3, ⋯ and ♠A′, ♠2′, ♠3′, ⋯, respectively. (1) In the figure, ♠A is immediately moved to a foundation. Then, one of the two exposed cards ♠2 and ♠2′ can be moved to the foundation, since they are one rank higher than ♠A. (2) One of the two cards ♢4 and ♢4′ can be moved to the fifth tableau pile, since they are one rank lower than the top card ♢5. (3) If ♠2 was moved to the foundation at step (1) and if ♢4 was moved to the fifth tableau pile at step (2), then ♡2′ and ♡A are exposed. Consequently, ♡A, ♡2′, and ♡3 are moved to a foundation in that order. At this point, there are no cards in tableau piles which can be moved to foundations. (4) Fortunately, if the top card ♡6 of the stock is discarded, and ♢A′ is moved to a foundation, then (5) four cards ♢2′, ♢3, ♢4, and ♢5 can be moved to the foundation.

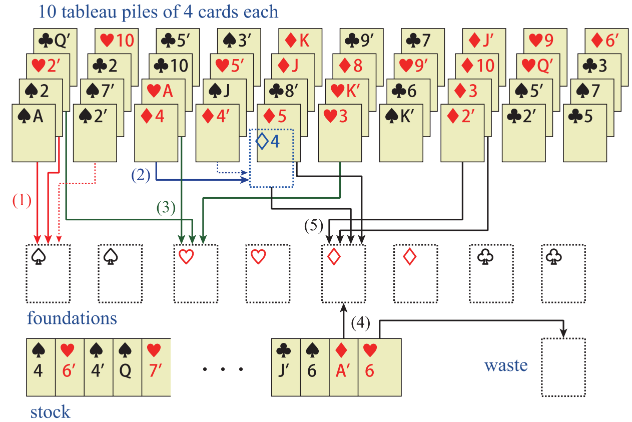

Fig. 1 Initial layout of Forty Thieves.

In this paper, we consider the generalized version of Forty Thieves, which uses two *generalized* $4k$*-card decks*. A $4k$-card deck includes k ranks of each of the four suits, spades (♠), hearts (♡), diamonds (♢), and clubs (♣). The instance of the *Generalized Forty Thieves* is the initial layout of $4 \times l$ cards and a stock of s cards, where l is an integer and $s = 8k - 4l$. The *Generalized Forty Thieves Problem* is to decide whether the player can move all of the $4 \times l$ cards to the foundations. We will show that the problem is NP-complete, even if the number of stock cards is zero. It is not difficult to show that the problem belongs to NP, since each card can be moved at most twice.

There has been a huge amount of literature on the computational complexities of games and puzzles. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [4]. Recently, Block Sum [3], Pyramid [7], String Puzzle [9], Tantrix Match [10], Yosenabe [6], and Zen Puzzle Garden [5] were shown to be NP-complete, and Chat Noir [8] is PSPACE-complete.

2. Reduction from 3SAT to Generalized Forty Thieves

The definition of 3SAT is mostly from [2]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a

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set of literals over U, such as $\{x_1, x_2, \overline{x_3}\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. An instance of 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over U such that $|c_j| = 2$ or $|c_j| = 3$ for each c_j. The 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. (An example of C is given in the caption of Fig. 3.) It is known that 3SAT is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in C [1].

We present a polynomial-time transformation from an arbitrary instance C of 3SAT to tableau piles of cards such that C is satisfiable if and only if all cards can be moved to the foundations. Let n and m be the numbers of variables and clauses of C, respectively. Without loss of generality, we can assume n and m are divisible by four and two, respectively. We use two $4k$-card decks, where $k = 15m - 2n + 4$.

Each variable $x_i \in U$ is transformed into six non-white cards in Fig. 2. (See also Fig. 3 when $n = m = 4$. Those cards for x_1 are ♠1, ♠2, ♠2′, ♠9, ♠12, ♠15.) Figure 2 consists of cards ♠1, ♠3, …, ♠$2i-1$, …, ♠$2n-1$ (labeled with $x_1, x_2, \ldots, x_i, \ldots, x_n$); ♠$2i$, ♠$2i'$ (labeled with “$x_i = 1$”,“$x_i = 0$”); $\square 2n+3j_1-2$, $\square 2n+3j_2-2$, $\square 2n+3j_3-2$ (labeled with $c_{j_1}, c_{j_2}, c_{j_3}$) where $\square \in \{♠, ♡, ♢\}$; and three dummy cards. This figure implies that x_i appears in c_{j_1} positively and in c_{j_2} and c_{j_3} negatively. If this is the first (resp. second, third) appearance of a card of rank $2n + 3j - 2$, the suit is ♠ (resp. ♡, ♢). (For example, ♠9, ♡9, and ♢9 labeled with c_1 appears in the green cards of Fig. 3, since c_1 contains x_1, x_2, and x_3.)

Suppose ♠$2i-1$ is placed on the top of a foundation (see Fig. 2). If ♠$2i$ is moved to the foundation, then card $2n + 3j_1 - 2$ (labeled with c_{j_1}) will be exposed. This situation implies the assignment $x_i = 1$. On the other hand, if ♠$2i'$ is moved to the foundation, then card $2n + 3j_2 - 2$ followed by $2n + 3j_3 - 2$ (labeled with c_{j_2} and c_{j_3}) will be exposed. This implies $x_i = 0$.

Figure 3 consists of $4 \times n/4$ yellow cards, n sets of 4×2 cards for n variables containing yellow, green, and white cards, and $4 \times n$ grey cards. Grey cards are dummy, which can be moved to foundations at the beginning of the game.

Let p be the number of size-two clauses. Suppose $|c_1| = |c_2| = \cdots = |c_{m-p}| = 3$ and $|c_{m-p+1}| = \cdots = |c_{m-1}| = |c_m| = 2$. Then, the $(4 \times 2n)$-card area of Fig. 3 contains no cards denoted by ♢$2n+3j-2$, where $m-p+1 \le j \le m$. In Fig. 7, the area of $2 \times 3(m-n)/2$ green cards is filled with those p ♢-cards. (Note that $p = 3(m-n)$ because the number of literals is $3n = 3(m-p) + 2p$.)

Figure 4 is a clause gadget for c_j. (a) If ♠$2n + 3j - 2$ (labeled with c_j) is moved to a foundation (see also ♠9 with c_1 when $n = 4$ in Fig. 3), then ♠$2n+3j-1$ and ♠$2n+3j$ can be moved to the foundation (see also ♠10 and ♠11 in Fig. 6), and red card ♣$5j-4$ is exposed (see ♣1 in Fig. 6).

(b) If ♡$2n + 3j - 2$ or ♢$2n + 3j - 2$ (labeled with c_j) is moved to the foundation (see also ♡9 or ♢9 in Fig. 3), then $\{$♡$2n + 3j - 1$, ♡$2n + 3j\}$ or $\{$♢$2n + 3j - 1$, ♢$2n + 3j\}$ can be moved to the foundation ($\{$♡10, ♡11$\}$ or $\{$♢10, ♢11$\}$ in Fig. 6), respectively. In this case, blue card ♣$5m + 2j$ or ♣$5m + 2j'$ is exposed (♣22 or ♣22′ in Fig. 6).

(a) If red card ♣$5j-4$ of Fig. 5 is moved to a foundation (see ♣1 in Fig. 6), then red cards ♣$5j-3$, ♣$5j-2$, ♣$5j-1$, ♣$5j$ can be moved to the foundation (♣2, ♣3, ♣4, ♣5 in Fig. 6). This situation implies that clause c_j is satisfied.

(b) If blue card ♣$5m + 2j$ or ♣$5m + 2j'$ of Fig. 4 is exposed, then blue card ♣$5m + 2j - 1$ can be moved on it

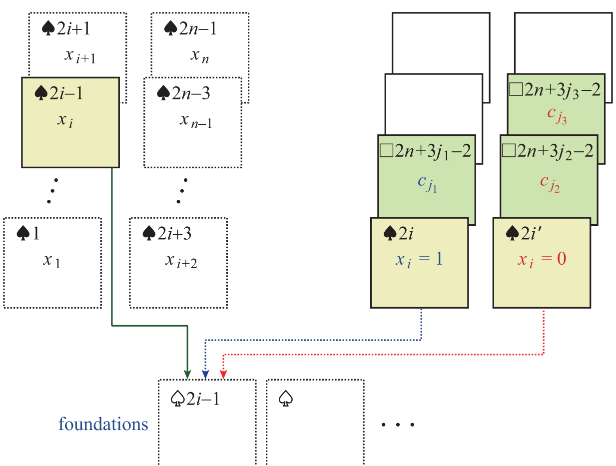

Fig. 2 Variable gadget for x_i.

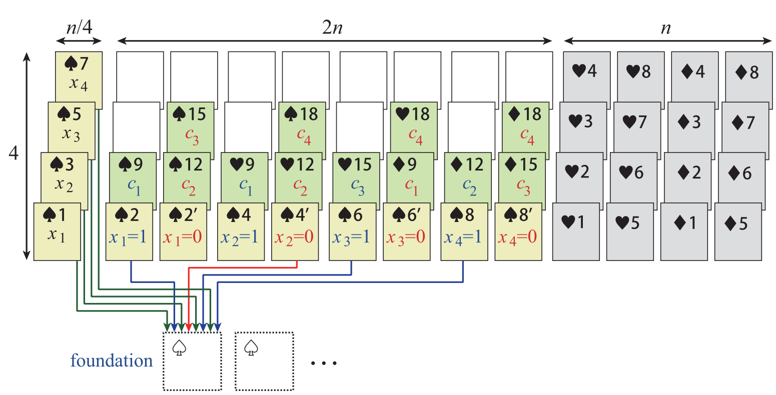

Fig. 3 Variable gadget when $n = m = 4$ for $U = \{x_1, x_2, x_3, x_4\}$ and $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_3, \overline{x_4}\}$, and $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$.

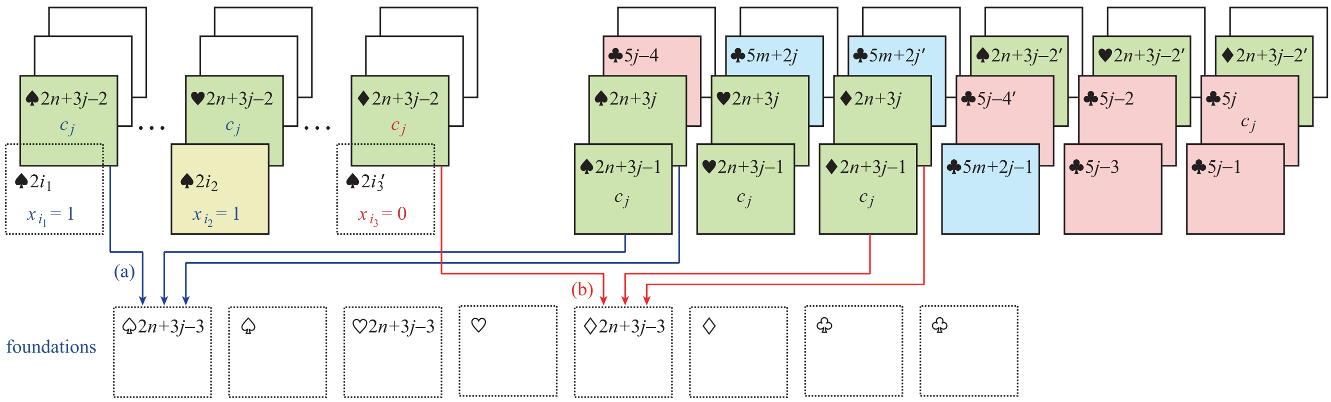

Fig. 4 Clause gadget for c_j.

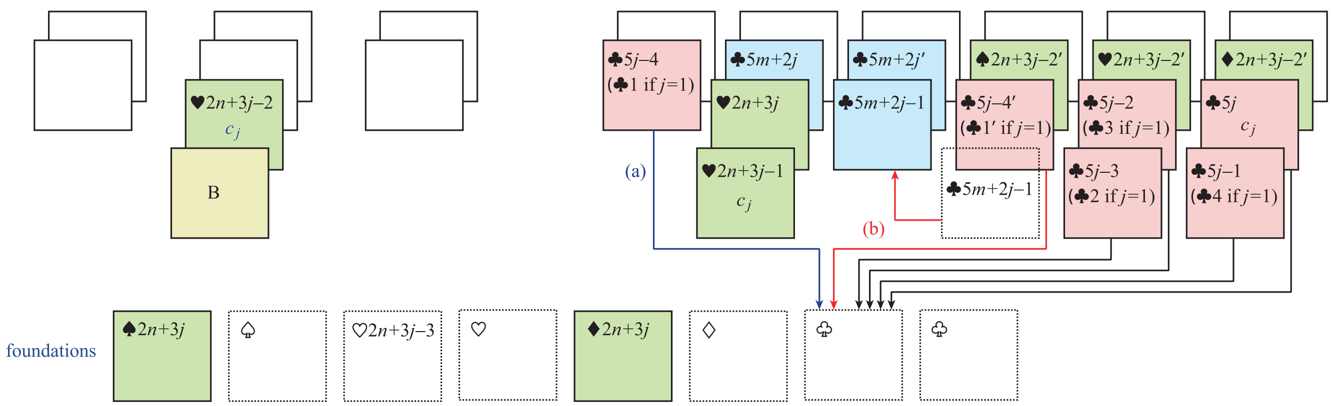

Fig. 5 Card ♣$5j$ is exposed if one of {♠$2n+3j-2$, ♡$2n+3j-2$, ◇$2n+3j-2$} in Fig. 4 is exposed.

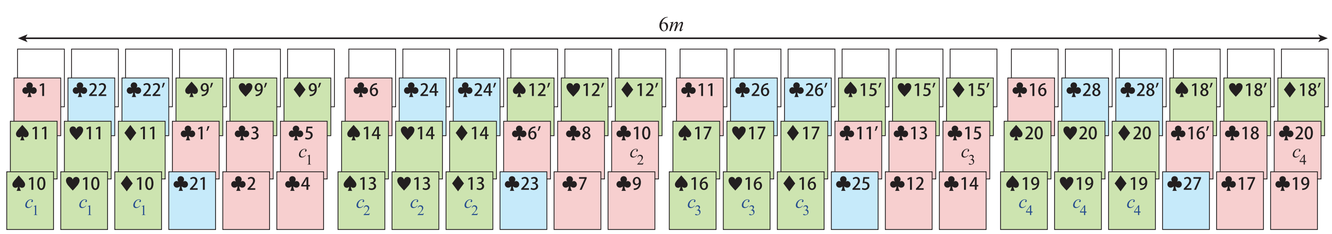

Fig. 6 Clause gadget for $C = \{c_1, c_2, \ldots, c_m\}$ when $n = m = 4$.

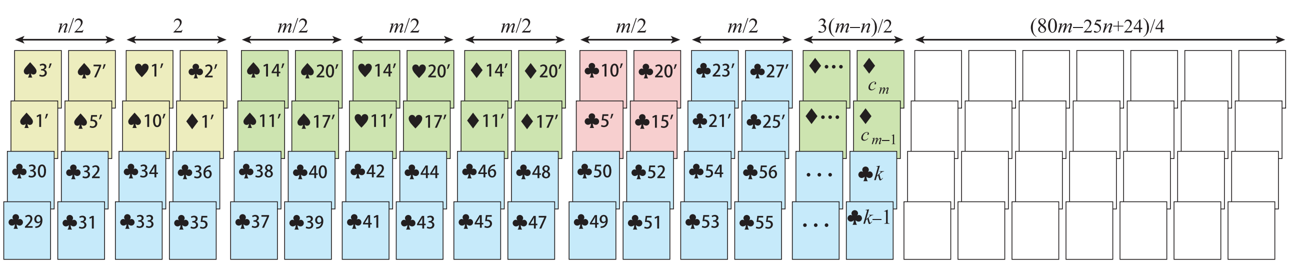

Fig. 7 Cards in the bottom two layers are released after the top two layers are removed.

(see Fig. 5), and red card ♣$5j-4'$ is exposed (♣$1'$ in Fig. 6). Then, red cards ♣$5j-4'$, ♣$5j-3$, ♣$5j-2$, ♣$5j-1$, ♣$5j$ can be moved to the foundation. This also implies c_j is satisfied. Red card ♣$5j$ is called the *target card for clause* c_j.

Since $k = 15m-2n+4$ ($= 56$ when $n = m = 4$), our two $4k$-card decks contain $120m - 16n + 32$ cards in total. The numbers of non-white cards in Figs. 3, 6, and 7 are $10n$, $18m$ and $16m - 4n + 8$, respectively. The number of the unused cards is $(120m-16n+32)-(10n+18m+(16m-4n+8)) = 86m - 22n + 24$. In Figs. 3, 6, and 7, there are $3n + 6m + (80m-25n+24)$ white cards, which are filled up with those $86m - 22n + 24$ unused cards. The $86m - 22n + 24$ cards are arranged so that all the remaining cards can trivially be moved to the foundations after the target card ♣$5m$ (= ♣20 when $m = 4$) is removed.

3. NP-Completeness of Generalized Forty Thieves

In this section, we will show that the instance C of 3SAT is satisfiable if and only if all cards in the tableau piles can be moved to the foundations.

Assume that the instance C of 3SAT is satisfiable. When ♠$2i-1$ is placed on the top of a foundation for every $i \in \{1, 2, \ldots, n\}$ (see ♠1, ♠3, ♠5, ♠7 in Fig. 3), either ♠$2i$ or ♠$2i'$ can be moved to the foundation.

Suppose x_i appears in c_{j_1} positively and in c_{j_2} and c_{j_3} negatively. If ♠$2i$ is moved to the foundation, then a card with label c_{j_1} is exposed. If ♠$2i'$ is moved, then a card c_{j_2} (followed by a card c_{j_3}) is exposed. We call such a pair of cards c_{j_2} and c_{j_3} *sequentially exposed cards*. Since C is satisfiable, we can choose ♠$2i$ or ♠$2i'$ so that at least one of the c_j-cards of rank $2n + 3j - 2$ is exposed or sequentially exposed, for every $j \in \{1, 2, \ldots, m\}$. (In Fig. 3, cards ♠2, ♠4′, ♠6, ♠8 are moved to the foundation, and ♠9, ♡12, ♠18, ♡15, ◇12 are exposed or sequentially exposed.)

Suppose ♠$2n$ or ♠$2n'$ is placed on the top of the foundation (see ♠8 or ♠8′ in Fig. 3), and grey cards $\{$♡$1,$ ♡$2, \ldots,$ ♡$2n\}$ and $\{$◇$1,$ ◇$2, \ldots,$ ◇$2n\}$ are piled up on foundations. In the following explanation, we use the 3SAT-instance C given in the caption of Fig. 3 for simplicity.

Since c_1 is satisfied by $x_1 = 1$, card ♠9 in Fig. 3 is exposed and is moved to the foundation. Then, in Fig. 6, cards ♠10, ♠11; ♣1, ♣2, ♣3, ♡9′, ♡10, ♡11; ♣4, ♣5, ◇9′, ◇10, ◇11 can be moved to foundations. (In this case, ♠9′ is not removed in the procedure of this paragraph.)

Since c_2 is satisfied by $x_2 = 0$ and $x_4 = 1$, cards ♡12 and ◇12 in Fig. 3 are exposed and are moved to the foundations. Then, ♡13, ♡14 and ◇13, ◇14 can be moved to the foundations, and thus blue cards ♣24 and ♣24′ are exposed. Now, ♣23 can be moved on ♣24 or ♣24′, and therefore ♣6′ is exposed and moved to the foundation. Next, cards ♠12′, ♠13, ♠14; ♣7, ♣8; ♣9, ♣10 can be moved to foundations. (Cards ♡12′ and ◇12′ are not removed in this paragraph.)

By continuing this observation, one can see that all the target cards ♣5, ♣10, …, ♣$5m$ can be moved to the foundation under the assumption that all the clauses $c_1, c_2, \ldots, c_m$ are satisfied.

Once the target card ♣$5m$ for c_m is moved to the foundation, then all of the remaining cards in Figs. 3, 6, and 7 are moved to the foundations, since white cards were arranged so that all the remaining cards can trivially be moved to the foundations after the target card is removed. Hence, if the instance C of 3SAT is satisfiable, then all cards in the tableau piles can be moved to the foundations.

Assume the player can move all cards in the tableau piles to the foundations. Consider the target card ♣20 for c_4. This card can be moved to a foundation only if ♣16 or ♣16′ is moved to the foundation. Consider a configuration when exactly one of {♣16, ♣16′} is moved to the foundation, and all of {♣17, ♣17′, ♣18, ♣18′, ⋯, ♣k, ♣k'} are in tableau piles.

Suppose ♣16 is moved to the foundation (and ♣16′ is in a tableau pile). Card ♣16 can be moved to the foundation only if ♠20 and ♠19 are already removed. Card ♠19 can be removed only if ♠18 in Fig. 3 is moved to the foundation, since (i) ♠19 and ♠20 belong to a single pile, (ii) ♠20′ is under {♣39, ♣40} in Fig. 7, and (iii) ♠18′ is under ♣16′.

Suppose ♣16′ is moved to the foundation (and ♣16 is in a tableau pile). Card ♣16′ is moved to the foundation only if blue card ♣27 is moved on either ♣28 or ♣28′, since none of the cards {♣17, ♣17′, ♣18, ♣18′, ⋯, ♣k, ♣k'} has been moved to foundations. From the same reason as the previous paragraph, card ♣28 or ♣28′ is exposed only if ♡18 or ◇18 in Fig. 3 is moved to the foundation, respectively.

One of {♣16, ♣16′} is moved to the foundation only if ♣15 (labeled with c_3) is moved to the foundation, since ♣15′ is under {♣51, ♣52} in Fig. 7. By continuing this observation from c_m to c_1, one can see that cards ♣$5m$, …, ♣10, ♣5 can be moved to the foundation only if yellow cards removed from the set {♠2, ♠2′, ♠4, ♠4′, …, ♠$2n$, ♠$2n'$} in Fig. 3 indicate the truth assignment satisfying all clauses of C. (From Fig. 3, one can see that $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfy all the clauses.)

At the beginning of the game, exactly one of {♠$2i$, ♠$2i'$} is moved to the foundation for every $i \in \{1, 2, \ldots, n\}$, since {♠1′, ♠3′, …, ♠$2n-1'$} are under {♣$7m+1$, ♣$7m+2$, ⋯} (see {♣29, ♣30, ♣31, ♣32} in Fig. 7). Four cards ♠10′, ♡1′, ◇1′, ♣2′ under {♣33, ♣34, ♣35, ♣36} in Fig. 7 interrupt four foundations during the procedures for variable and clause gadgets of Figs. 3 and 6. Hence, if the player can move all cards in the tableau piles to the foundations, then C is satisfiable.

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