

Computational Complexity of Generalized Golf Solitaire*

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SUMMARY Golf is a solitaire game, where the object is to move all cards from a 5×8 rectangular layout of cards to the foundation. A top card in each column may be moved to the foundation if it is either one rank higher or lower than the top card of the foundation. If no cards may be moved, then the top card of the stock may be moved to the foundation. We prove that the generalized version of Golf Solitaire is NP-complete.

key words: computational complexity, NP-completeness, puzzle

1. Introduction

Golf is a solitaire game, where the object is to move all cards from a 5×8 rectangular layout of cards to the foundation (see Fig. 1). In the rectangular layout, the eight cards of the first layer partially cover the second layer, which partially cover the third layer, and so on. Initially, one card is placed on the table as a foundation, and the remaining 11 cards are piled up, which are the stock. (You can play Golf Solitaire at the web site [1].)

A card is *exposed* if no cards cover it. Exposed cards in the rectangular layout may be moved to the foundation if they are either one rank higher or one rank lower than the top card of the foundation, regardless of suit. If no cards may be moved, then the top card of the stock may be moved to the foundation. Once the stock is exhausted and no more cards can be moved, the game ends. The aim of the game is to move all cards of the rectangular layout to the foundation.

In Fig. 1, card $\spadesuit 10$ is initially on the table as a foundation. Since card $\clubsuit J$ in the first layer is one rank higher than $\spadesuit 10$, it can be moved to the foundation. Then, $\heartsuit Q$, $\heartsuit K$, and $\diamondsuit Q$ can be moved to the foundation. At this point, there are no cards which can be moved to the foundation. Fortunately, if the top card $\heartsuit A$ of the stock is moved to the foundation, then $\clubsuit 2$, $\heartsuit 3$, and $\diamondsuit 2$ can be moved to the foundation.

In this paper, we consider the generalized version of Golf Solitaire. The *generalized $4k$ -card deck* includes k ranks of each of the four suits, spades (\spadesuit), hearts (\heartsuit), diamonds (\diamondsuit), and clubs (\clubsuit). The instance of the *Generalized Golf Problem* is the initial layout of $p \times q$ cards, an initial foundation card, and a stock of s cards, where $s = 4k - pq - 1$. The problem is to decide whether the player can move all of

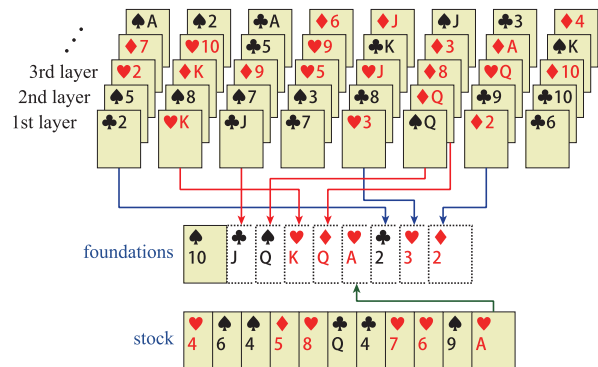


Fig. 1 Initial layout of Golf Solitaire.

the $p \times q$ cards to the foundation. We will show that the Generalized Golf Problem is NP-complete. It is not difficult to show that the problem belongs to NP, since the player can move at most $4k - 1$ cards to the foundation.

There has been a huge amount of literature on the computational complexities of games. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [7]. Recently, Block Sum [6], Kaboodle [2], Kurodoko [14], Magnet Puzzle [15], Pandemic [16], Pyramid [10], Shisen-Sho [13], String Puzzle [12], Yosenabe [9], and Zen Puzzle Garden [8] were shown to be NP-complete. Furthermore, it is known that Chat Noir [11] and Rolling Block Maze [3] are PSPACE-complete.

2. Reduction from 3SAT to Generalized Golf Solitaire

The definition of 3SAT is mostly from [5]. Let $U = \{x_1, x_2, \dots, x_n\}$ be a set of Boolean variables. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U , then x and \bar{x} are literals over U . The value of \bar{x} is 1 (true) if and only if x is 0 (false). A clause over U is a set of literals over U . It represents the disjunction of those literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of 3SAT is a collection $C = \{c_1, c_2, \dots, c_m\}$ of clauses over U such that $|c_j| \leq 3$ for each $c_j \in C$. The 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C . This problem is known to be NP-complete.

For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $c_1 = \{x_1, x_2, \bar{x}_3\}$, $c_2 = \{\bar{x}_1, \bar{x}_2, x_4\}$, $c_3 = \{\bar{x}_1, x_3, \bar{x}_4\}$, $c_4 = \{\bar{x}_2, \bar{x}_3, \bar{x}_4\}$ provide an instance of 3SAT. For this in-

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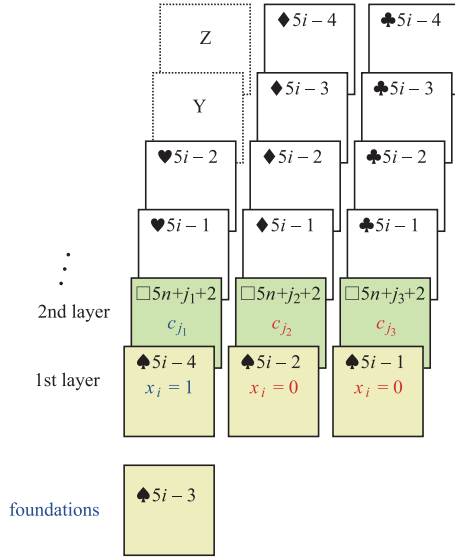


Fig. 2 Variable gadget for x_i .

stance, the answer is “yes,” since there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfying all clauses. It is known that 3SAT is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in C [4].

2.1 Transformation from an Instance of 3SAT to an Initial Layout of Cards

We present a polynomial-time transformation from an arbitrary instance C of 3SAT to a rectangular layout of cards, a foundation card, and a stock of cards such that C is satisfiable if and only if all cards in the rectangular layout and stock can be moved to the foundation. Note that the instances of 3SAT considered in this paper have the restriction explained just before Sect. 2.1.

Let n and m be the numbers of variables and clauses of C , respectively. Without loss of generality, we can assume that n is even and m is divisible by four. Each variable $x_i \in U$ is transformed into 6×3 cards shown in Fig. 2. (See also Fig. 4 when $n = m = 4$. The 6×3 cards for x_1 are the leftmost three columns.)

The first layer of Fig. 2 consists of cards $\spadesuit 5i-4$, $\spadesuit 5i-2$, and $\spadesuit 5i-1$ (labeled with “ $x_i = 1$ ”, “ $x_i = 0$ ”, and “ $x_i = 0$ ”, respectively). If variable x_i appears in clause c_{j_1} positively and in c_{j_2} and c_{j_3} negatively, then the three cards of the second layer have ranks $5n + j_1 + 2$, $5n + j_2 + 2$, and $5n + j_3 + 2$, respectively. If this is the first (resp. second, third) appearance of a card of rank $5n + j + 2$, the suit is \spadesuit (resp. \heartsuit , \diamondsuit). (For example, $\clubsuit 23$, $\heartsuit 23$, and $\diamondsuit 23$ labeled with c_1 appears in that order in the second layer of Fig. 4, since clause c_1 contains variables x_1 , x_2 , and x_3 in numerical order.)

Since cards $\heartsuit 5i-1$, $\heartsuit 5i-2$, $\diamondsuit 5i-1$, $\diamondsuit 5i-2$, $\diamondsuit 5i-3$, $\diamondsuit 5i-4$, $\clubsuit 5i-1$, $\clubsuit 5i-2$, $\clubsuit 5i-3$, and $\clubsuit 5i-4$ are useless for the simulation of variable x_i , they are arranged in a 4×3 layout of the variable gadget (see white cards in Fig. 2). Here, the suit and the rank of cards in two empty boxes Y and Z are

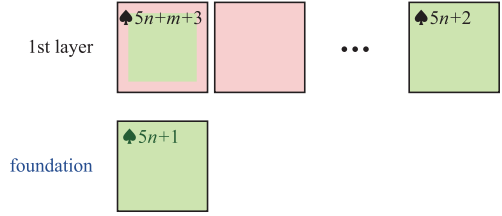


Fig. 3 Conjunction gadget for clauses c_1, c_2, \dots, c_m .

defined later. (Later, one can see 4×3 white cards of Fig. 2 can be removed trivially at the end of the game.)

Suppose $\spadesuit 5i-3$ is on the top of the foundation (see Fig. 2). If $\spadesuit 5i-4$ is moved to the foundation, then card $5n + j_1 + 2$ (labeled with c_{j_1}) will be exposed. This situation implies the assignment $x_i = 1$. On the other hand, if $\spadesuit 5i-2$ and $\spadesuit 5i-1$ are moved to the foundation, then cards $5n + j_2 + 2$ and $5n + j_3 + 2$ (labeled with c_{j_2} and c_{j_3}) will be exposed. This implies $x_i = 0$. (In Fig. 5, card $\spadesuit 2$ in the foundation and $\spadesuit 7$, $\spadesuit 12$, $\spadesuit 17$ in the stock are used for this purpose.)

We arrange n sets of the 6×3 cards in a line for the n variables (see the $6 \times 3n$ cards in Fig. 4). In the second layer, there exist exactly three cards of label c_j for every $j \in \{1, 2, \dots, m\}$. If at least one of the three cards is exposed for every c_j (see Fig. 6), then all clauses of C are satisfied.

Figure 3 is a conjunction gadget for the m clauses. Suppose that $\spadesuit 5n+1$ is placed on the top of the foundation (see also $\spadesuit 21$ in the foundation of Fig. 6). Then, card $\spadesuit 5n+2$ at the rightmost position in the first layer can be moved to the foundation (see $\spadesuit 22$ in Fig. 6). If a card of rank $5n + j + 2$ (labeled with c_j) has been exposed for every $j \in \{1, 2, \dots, m\}$, then cards of ranks $5n+3, 5n+4, \dots, 5n+m+2$ (labeled with c_1, c_2, \dots, c_m) can be moved to the foundation in that order (see $\spadesuit 23, \heartsuit 24, \heartsuit 25, \spadesuit 26$ in Fig. 6). Now, card $\spadesuit 5n+m+3$ can be moved to the foundation (see $\spadesuit 27$ in Fig. 6). (Later, one can see that $\spadesuit 5n+m+3$ can be moved if and only if all clauses are satisfied.) $\spadesuit 5n+m+3$ is called a *target card*.

Figure 4 is the initial layout of Golf Solitaire transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_3, \overline{x_4}\}$, and $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. In this figure, red cards of ranks 27 through 30 (ranks $5n+m+3$ through $6n+m+2$ in general) are dummy so that the set of cards forms a rectangular layout. Once, the target card $\spadesuit 27$ is moved to the top of the foundation, all red cards can easily be moved to the foundation.

In Fig. 4, blue cards $\heartsuit 1, \heartsuit 6, \heartsuit 11, \heartsuit 16$ in the stock and $\heartsuit 2, \heartsuit 7, \heartsuit 12, \heartsuit 17$ in the rectangular layout are used for removing all the remaining yellow cards in the first layer. For example, if card $\heartsuit 1$ is moved from the stock to the foundation, then card $\heartsuit 2$ can be moved to the foundation, and thus either $\{\spadesuit 1\}$ or $\{\spadesuit 3, \spadesuit 4\}$ can be moved to the foundation. (In case of Fig. 6, $\{\spadesuit 3, \spadesuit 4\}$ will be moved.)

Similarly, in Fig. 4, cards $\heartsuit 21, \heartsuit 21$ in the stock and $\heartsuit 22, \heartsuit 22$ in the sixth layer are used for removing all the remaining green cards in the second layer. Without loss of generality, we can assume that the instance $C = \{c_1, c_2, \dots, c_m\}$ of 3SAT is such that $3 \geq |c_1| \geq |c_2| \geq \dots \geq$

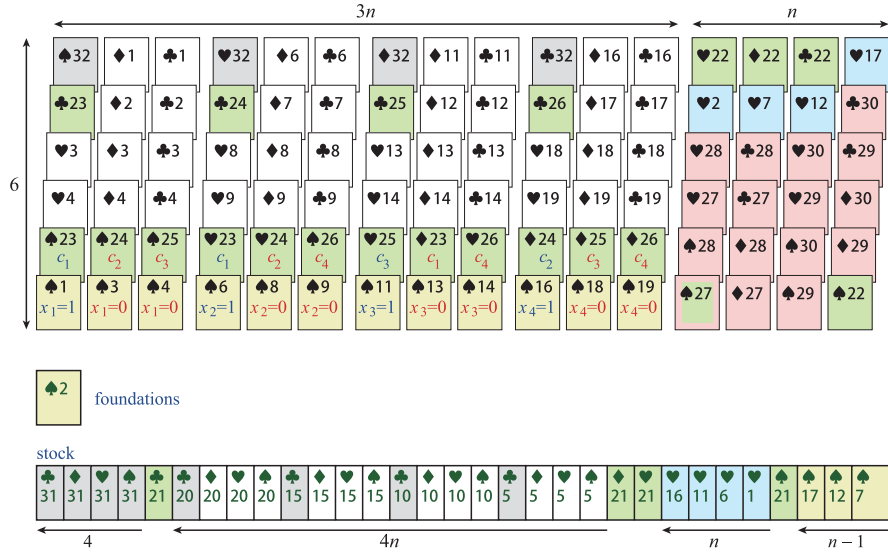


Fig. 4 The initial layout of Golf Solitaire transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, \bar{x}_3\}$, $c_2 = \{\bar{x}_1, \bar{x}_2, x_4\}$, $c_3 = \{\bar{x}_1, x_3, \bar{x}_4\}$, and $c_4 = \{x_2, \bar{x}_3, \bar{x}_4\}$. The numbers of variables and clauses are $n = 4$ and $m = 4$, respectively. The total number of cards is $4k = 30n + 8 = 4 \cdot 32$.

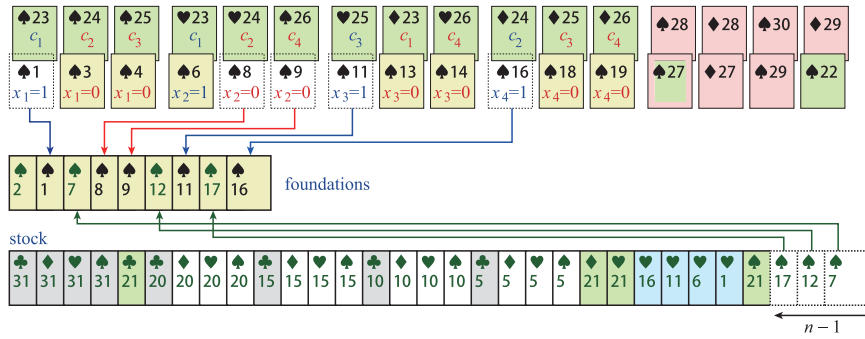


Fig. 5 Assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$.

$|c_m| \geq 2$. (We need this assumption for removing all green cards in the second layer sequentially.)

White cards $\spadesuit 5, \heartsuit 5, \diamondsuit 5$; $\spadesuit 10, \heartsuit 10, \diamondsuit 10$; $\spadesuit 15, \heartsuit 15, \diamondsuit 15$; $\spadesuit 20, \heartsuit 20, \diamondsuit 20$ in the stock are used for removing all white cards in the rectangular layout. Grey cards $\spadesuit 5, \spadesuit 10, \spadesuit 15, \spadesuit 20$ (in general, $\spadesuit 5i$ for each $i \in \{1, 2, \dots, n\}$) are dummy; they are discarded to the foundation.

Cards $\clubsuit 23, \clubsuit 24, \clubsuit 25, \clubsuit 26$ (in general, $\clubsuit 5n + 3, \clubsuit 5n + 4, \dots, \clubsuit 5n + m + 2$) are stored in the $2n$ empty boxes (see empty boxes labeled with Y and Z in Fig. 2). Since the number m of clauses satisfies $m \leq 3n/2$ ($< 2n$), those m cards can be stored in the $2n$ empty boxes. Green card $\clubsuit 21$ in the stock and $\clubsuit 22$ in the sixth layer are used for removing all green cards in the fifth and sixth layers. (In Fig. 4, no green cards exist in the sixth layer).

The number of the remaining empty boxes is $2n - m$ ($= 4$ when $n = m = 4$, see Fig. 4). The value of $2n - m$ is divisible by four, since we assumed that n is even and m is divisible by four. Let r be an integer such that $2n - m = 4r$ ($= 4 \cdot 1$ when $n = m = 4$). The bottom four cards of the stock are $\spadesuit 31, \diamondsuit 31, \heartsuit 31, \clubsuit 31$ (in general, $\spadesuit 6n + m + 3, \diamondsuit 6n + m + 3, \heartsuit 6n + m + 3, \clubsuit 6n + m + 3$). The $2n - m$ empty boxes

in the rectangular layout are filled by

$$\begin{aligned} &\spadesuit 6n + m + 4, \spadesuit 6n + m + 5, \dots, \spadesuit 6n + m + r + 3, \\ &\heartsuit 6n + m + 4, \heartsuit 6n + m + 5, \dots, \heartsuit 6n + m + r + 3, \\ &\diamondsuit 6n + m + 4, \diamondsuit 6n + m + 5, \dots, \diamondsuit 6n + m + r + 3, \\ &\clubsuit 6n + m + 4, \clubsuit 6n + m + 5, \dots, \clubsuit 6n + m + r + 3. \end{aligned}$$

The total number $4k$ of cards of the initial layout of Golf Solitaire is $4k = 6 \times (3n + n) + (6n + 8) = 30n + 8$.

2.2 NP-Completeness of Generalized Golf Solitaire

In this section, we will show that the instance C of 3SAT is satisfiable if and only if all the cards of the rectangular layout and the stock can be moved to the foundation.

Assume that the instance C of 3SAT is satisfiable. When card $\spadesuit 5i - 3$ is placed on the top of the foundation for every $i \in \{1, 2, \dots, n\}$ (see cards $\spadesuit 2, \spadesuit 7, \spadesuit 12, \spadesuit 17$ in Fig. 5), either $\{\spadesuit 5i - 4\}$ or $\{\spadesuit 5i - 2, \spadesuit 5i - 1\}$ is moved from the first layer to the foundation. If $\spadesuit 5i - 4$ (resp. $\spadesuit 5i - 2, \spadesuit 5i - 1$) is moved, then a card with label c_{j_1} (resp. c_{j_2}, c_{j_3}) is exposed, if x_i appears in c_{j_1} positively and in c_{j_2}, c_{j_3} negatively.

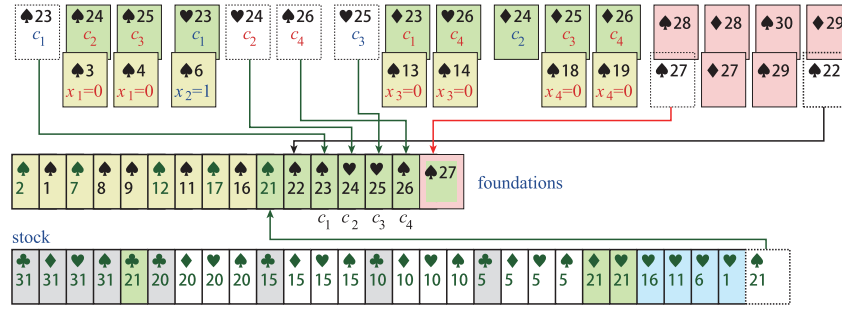


Fig. 6 All clauses c_1, c_2, \dots, c_m are satisfied if and only if at least one of the literals of every clause $c_j \in \{c_1, c_2, \dots, c_m\}$ has value 1.

Since C is satisfiable, we can choose $\{\spadesuit 5i - 4\}$ or $\{\spadesuit 5i - 2, \spadesuit 5i - 1\}$ so that at least one of the three c_j -cards of rank $5n + j + 2$ (in the second layer) is exposed for every $j \in \{1, 2, \dots, m\}$ (see Fig. 5). Therefore, by placing cards $\spadesuit 5n + 1$ and $\spadesuit 5n + 2$ on the foundation (see $\spadesuit 21$ and $\spadesuit 22$ in Fig. 6), we can move cards of ranks $5n + 3, 5n + 4, \dots, 5n + m + 2$ (labeled with c_1, c_2, \dots, c_m) and the target card of rank $5n + m + 3$ to the foundation (see $\spadesuit 23, \heartsuit 24, \heartsuit 25, \spadesuit 26$ and $\spadesuit 27$ in Fig. 6).

Once the target card is moved, then all red cards, all blue cards, and all of the remaining yellow cards of Fig. 4 can trivially be moved to the foundation. Then, all green cards, all white cards, and all grey cards can also be moved to the foundation. Hence, if the instance C of 3SAT is satisfiable, then all the cards of the rectangular layout and the stock can be moved to the foundation.

Assume the player can move all the cards of the rectangular layout and stock to the foundation. Consider a card of rank $5n + m + 3$ (see $\spadesuit 27, \heartsuit 27$ in the first layer in Fig. 4). This card can be moved to the foundation only if a card of rank $5n + m + 2$ or $5n + m + 4$ (rank 26 or 28) is on the top of the foundation. However, all of the four cards of rank $5n + m + 4$ ($= 28$) are located beneath cards of rank $5n + m + 3$ ($= 27$) (see red cards in Fig. 4). Thus, a card of rank $5n + m + 3$ can be moved to the foundation only if a card of rank $5n + m + 2$ ($= 26$) is placed on the top of the foundation.

As we explained in Fig. 6, a card of rank $5n + m + 2$ ($= 26$) can be placed on the top of the foundation only if a card of rank $5n + j + 2$ (labeled with c_j) is exposed for every $j \in \{1, 2, \dots, m\}$ in the second layer (see $\spadesuit 23, \heartsuit 24, \heartsuit 25, \spadesuit 26$ in Figs. 5 and 6).

In the first layer, any of the three yellow cards $\spadesuit 5i - 4, \spadesuit 5i - 2, \spadesuit 5i - 1$ (see Fig. 2) can be removed only if a card of rank $5i - 3$ is placed on the foundation, for every $i \in \{1, 2, \dots, n\}$. (Note that all cards of rank $5i$ are buried deep within the stock. See the $4n$ cards of ranks 5, 10, 15, 20 in the stock in Fig. 4.) Card $\spadesuit 2$ ($= \spadesuit 5i - 3$ when $i = 1$) is initially placed on the table as a foundation, and cards $\spadesuit 5i - 3 \in \{\spadesuit 7, \spadesuit 12, \dots, \spadesuit 5n - 3\}$ are the topmost $n - 1$ cards of the initial stock (see Fig. 4).

Thus, the set of yellow cards removed from the first layer in Fig. 5 indicates the truth assignment satisfying

all clauses of C . (From Figs. 5 and 6, one can see that $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfy all the clauses.) Hence, if the player can move all the cards of the rectangular layout and the stock to the foundation, then C is satisfiable.

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