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Cellular Automata Associated with Σ-Algebras

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SUMMARY In this paper we present a novel treatment of cellular automata (CA) from an algebraic point of view. CA on monoids associated with Σ-algebras are introduced. Then an extension of Hedlund's theorem which connects CA associated with Σ-algebras and continuous functions between prodiscrete topological spaces on the set of configurations are discussed.

key words: *Cellular Automata, Σ-algebras, Hedlund's theorem*

1. Introduction

Theory of cellular automata (CA) due to von Neumann and Ulam has been recognized its importance in many fields of science [10]. A fascinating aspect of CA is to allow various researches by simulation and visual representation with computers as well as mathematical study. This is originated from a fact that main feature of CA consists of configurations, i.e., arrays of states on cells, and CA transform the configurations to new ones according to given local rules. The game of life is one of the most famous CA. Recently CA often mean CA on groups, and mathematical theory of CA has been developed subject to theory of groups [1].

Motivated by [5] and [6], the paper presents three different points of view from traditional researches on CA.

- The first point is to treat with CA on monoids. The monoids are weaker algebraic systems than groups, and CA on groups certainly provide richer results. However, our formulation of CA on monoids might serve as a start to seek new properties of CA on monoids.
- The second point of the paper is concerned with a local rule for CA, usually defined as a function from a finite set (called a neighborhood) of cells into the state set. To seek reversible CA we need the representation of composition of transition functions of CA. However, the usual definition of local rules definition has less power [2] to express the composition of transition functions. To improve this we re-formulate local rules as terms of Σ-algebras (or formulae [5], [6] in propositional logic). Then the composition of transition functions of CA can be simply described by the multiplication of terms, defined with making use of monoid action.
- The third intention in the paper gives a wider view of configurations of CA as valuations for variables (or atoms) in propositional logic or Σ-algebras. This enables us to define CA associated with Σ-algebras.

The paper is organized as follows. In Sect. 2, as a preparation of the definition of transition functions of CA, the formal definitions of Σ-terms and Σ-algebras are reviewed and a valuation of Σ-terms is defined. In Sect. 3 we define shifted configurations and shifted terms as monoid actions. In Sect. 4 we state a new definition of transition functions of CA associated with Σ-algebras by using monoid actions on configurations and the valuation of Σ-term over monoids. In Sect. 5 we define the multiplication of Σ-terms, and prove that the composition of transition functions can be expressed by the multiplication. In Sect. 6 we study restricted transition functions which will be useful in the next section. In Sect. 7 we recall the prodiscrete topology [1] on the configuration space and prove that all transition functions of CA associated with Σ-algebras are continuous, which is an extension of Hedlund's theorem [3, Theorem 3.1] that every transition function of classical cellular automata is continuous and commutes with the shift.

2. Σ-Algebras

In the section we will review signatures as sets of operator symbols and terms constructed as appropriate strings of variables and operator symbols.

Definition 2.1: Let $\mathbb{N}$ be the set of all naturals. A pair (Σ, a) of a (finite) set Σ and a function $a : \Sigma \to \mathbb{N}$ is called the *signature*. The function a is called the arity function for Σ. □

In the following of the paper a signature (Σ, a) is written Σ for short. Let Σ be a signature and n a natural. An element $\sigma \in \Sigma$ such that $a(\sigma) = n$ is called an *n-ary operator symbol* and Σ_n denotes the set of all n-ary operator symbols in Σ. In particular, an element $c \in \Sigma_0$ is called a constant (symbol).

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Example 2.2: (1) Let $\Sigma = \{\bot, \to\}$ with $a(\bot) = 0$ and $a(\to) = 2$. Then the terms of the signature are formulae in propositional logic.
(2) Let $\Sigma = \{0, +\}$ with $a(0) = 0$ and $a(+) = 2$. Then the signature is used to formulate algebraic systems with a constant and a binary operation, such as monoids and groups. □

Terms constructed as concatenations of variables and operator symbols form a Σ-algebra of a given signature. The terms for the signature are formally defined as follows.

Definition 2.3: Let Σ be a signature and X a set. The Σ-*terms* over X is inductively defined as follows:

(a) Each $x \in X$ is a Σ-term over X,
(b) Each constant $c \in \Sigma_0$ is a Σ-term over X,
(c) If $\sigma \in \Sigma_n$ $(n > 0)$ and $A_1, \dots, A_n$ are Σ-terms over X, then the string $\sigma A_1 \cdots A_n$ is a Σ-term over X. □

The set of all Σ-terms over X will be denoted by $\Sigma(X)$.

For a Σ-term A over X the set $N(A)$ of variables in A is defined as follows:

(a) $N(x) = \{x\}$ for $x \in X$,
(b) $N(c) = \emptyset$ for $c \in \Sigma_0$,
(c) $N(\sigma A_1 \cdots A_n) = \cup_{j=1}^{n} N(A_j)$ for $\sigma \in \Sigma_n$ $(n > 0)$ and $A_1, \dots, A_n \in \Sigma(X)$.

By the definition $N(A)$ is a finite subset of X.

Let Σ be a signature in what follows. For a set Q and a natural $n \in \mathbb{N}$ the n-th product Q^n of Q is defined by $Q^0 = \{*\}$ (a singleton set) and $Q^{n+1} = Q^n \times Q$.

Definition 2.4: A set Q is a Σ-*algebra* if Q is equipped with an element $c_\diamond \in Q$ for each constant $c \in \Sigma_0$ and a function $\sigma_\diamond : Q^n \to Q$ for each operator symbol $\sigma \in \Sigma_n$ $(n > 0)$. □

Many algebraic structures may be formulated as Σ-algebras. We show two typical examples of Σ-algebras defined on a set $Q = \{0, 1\}$.

Example 2.5: (1) Let $\Sigma = \{\bot, \to\}$ be a signature in 2.2 (1). Then $Q = \{0, 1\}$ is a Σ-algebra (algebra of classical truth values) by $\bot_\diamond = 0$, $0 \to_\diamond 0 = 0 \to_\diamond 1 = 1 \to_\diamond 1 = 1$ and $1 \to_\diamond 0 = 0$.
(2) Let $\Sigma = \{0, +\}$ be a signature in 2.2 (2). Then $Q = \{0, 1\}$ is a Σ-algebra (commutative group) by $0_\diamond = 0$, $0 +_\diamond 0 = 1 +_\diamond 1 = 0$ and $0 +_\diamond 1 = 1 +_\diamond 0 = 1$. □

The valuation of terms is a natural extension of a function of variables into a Σ-algebra. Formally we define the valuation of terms as follows.

Definition 2.6: Let Q be a Σ-algebra and $A \in \Sigma(X)$. For a function $m : X \to Q$ the *value* $m[\![A]\!] \in Q$ of A by m is inductively defined as follows.

(a) $m[\![x]\!] = m(x)$ for all $x \in X$.
(b) $m[\![c]\!] = c_\diamond$ for all constants $c \in \Sigma_0$.
(c) $m[\![\sigma A_1 \cdots A_n]\!] = \sigma_\diamond(m[\![A_1]\!], \dots, m[\![A_n]\!])$ for $\sigma \in \Sigma_n$ $(n > 0)$ and $A_1, \dots, A_n \in \Sigma(X)$. □

Strictly speaking the valuation $m[\![A]\!]$ defined above depends on Σ-algebra Q. However we use the notation for the sake of simplicity.

For two Σ-terms A and B over X we write as $A \equiv B$, if $m[\![A]\!] = m[\![B]\!]$ for all functions $m : X \to Q$.

In the paper a function $m : X \to Q$ will be called a *configuration* of Q over X according to a context of CA theory, and the set of all configurations $m : X \to Q$ is denoted by Q^X. In other words, the set Q^X presents the configuration space of Q over X. Remark that a function $m : X \to Q$ is often called a valuation (or interpretation) of X into Q in a context of logic and universal algebras.

Projections are basic functions for a product space Q^X. Let $x \in X$ and $V \subseteq W \subseteq X$. A projection $p_x : Q^X \to Q$ is a function such that $\forall m \in Q^X.\ p_x(m) = m(x)$. An extended projection $p^W_V : Q^W \to Q^V$ is defined by $\forall q \in Q^W\, \forall x \in V.\ p^W_V(q)(x) = q(x)$. Note that p^X_V will be written as p_V, for short. It is obvious that $p^W_V \circ p_W = p_V$ (if $V \subseteq W$), where the composition $g \circ f$ of a function $f : X \to Y$ followed by a function $g : Y \to Z$ is defined as usual:

$$\forall x \in X.\ (g \circ f)(x) = g(f(x)).$$

In the following sections, suppose that M is a monoid with a unit e and Q is a finite Σ-algebra.

3. Monoid Actions

The ordinary definition [1] of CA on groups makes use of group action on configurations. In this section we introduce monoid actions on configurations and terms to redefine CA using terms over monoid. (Cf. [5], [6])

The multiplication of subsets V and W of M denotes the subset

$$VW = \{xy \in M \mid x \in V \wedge y \in W\}$$

of M. A multiplication $\{x\}W$ will be written as xW for short. The monoid action naturally defines shift functions on the configuration space Q^M.

Definition 3.1: For each $x \in M$ two functions $x^*, x_* : Q^M \to Q^M$ are defined as follows:

$$x^*(m)(y) = m(xy) \quad \text{and} \quad x_*(m)(y) = m(yx)$$

for all $m \in Q^M$ and $y \in M$. We call x^* and x_* the *left shift* and the *right shift* (functions) by $x \in M$, respectively □

It is trivial that $x^* = x_*$ if M is a commutative monoid.

Example 3.2: Let $\mathbb{N}$ be the additive monoid of all naturals and $Q = \{0, 1\}$ the Σ-algebra in 2.5 (2). Take a configuration $m \in Q^{\mathbb{N}}$ defined by $m(n) = 1$ if n is a square number, and $m(n) = 0$ otherwise. Also set $x = 2$. Then two configurations m and $x^*(m)$ are illustrated in the table below.

$\mathbb{N}$	$e = 0$	1	2	3	4	5	6	7	8	9	10	$\cdots$
m	1	1	0	0	1	0	0	0	0	1	0	$\cdots$
$x^*(m)$	0	0	1	0	0	0	0	1	0	0	0	$\cdots$

In what follows we will write $x^*(m)$ and $x_*(m)$ as x^*m and x_*m, respectively. The following states the basic properties of shift functions.

Proposition 3.3: Let $x, y \in M$ and $m \in Q^M$. Then

(a) $e^*m = m, \quad e_*m = m$
(b) $(xy)^*m = y^*(x^*m), \quad (xy)_*m = x_*(y_*m)$,
(c) $p_y \circ x^* = p_{xy}, \quad p_y \circ x_* = p_{yx}$.

Proof. (a*) $e^*m = m$:

$$\begin{aligned}\forall z \in M.\ (e^*m)(z) &= m(ez) && \{\ 3.1\ \}\\ &= m(z). && \{\ e : \text{unit}\ \}\end{aligned}$$

(a$_*$) $e_*m = m$:

$$\begin{aligned}\forall z \in M.\ (e_*m)(z) &= m(ze) && \{\ 3.1\ \}\\ &= m(z). && \{\ e : \text{unit}\ \}\end{aligned}$$

(b*) $(xy)^*m = y^*(x^*m)$:

$$\begin{aligned}\forall z \in M.\ ((xy)^*m)(z) &= m((xy)z) && \{\ 3.1\ \}\\ &= m(x(yz)) && \{\text{associative law}\ \}\\ &= (x^*m)(yz) && \{\ 3.1\ \}\\ &= (y^*(x^*m))(z). && \{\ 3.1\ \}\end{aligned}$$

(b$_*$) $(xy)_*m = x_*(y_*m)$:

$$\begin{aligned}\forall z \in M.\ ((xy)_*m)(z) &= m(z(xy)) && \{\ 3.1\ \}\\ &= m((zx)y) && \{\text{associative law}\ \}\\ &= (y_*m)(zx) && \{\ 3.1\ \}\\ &= (x_*(y_*m))(z). && \{\ 3.1\ \}\end{aligned}$$

(c*) $p_y \circ x^* = p_{xy}$:

$$\begin{aligned}\forall m \in Q^M.\ (p_y \circ x^*)(m) &= p_y(x^*m)\\ &= (x^*m)(y)\\ &= m(xy)\\ &= p_{xy}(m).\end{aligned}$$

(c$_*$) $p_y \circ x_* = p_{yx}$:

$$\begin{aligned}\forall m \in Q^M.\ (p_y \circ x_*)(m) &= p_y(x_*m)\\ &= (x_*m)(y)\\ &= m(yx)\\ &= p_{yx}(m).\end{aligned}$$

□

We now define the monoid action on the set $\Sigma(M)$ of Σ-terms over M.

Definition 3.4: For a term $A \in \Sigma(M)$ and $x \in M$, the *shifted term* $xA \in \Sigma(M)$ is inductively defined as follows:

(a) The shifted term xA for $A \in M$ is the monoid multiplication xA in M,
(b) $xA = A$ for $A \in \Sigma_0$,
(c) $xA = \sigma(xA_1)\cdots(xA_n)$ for $A = \sigma A_1 \cdots A_n$ where $\sigma \in \Sigma_n$ $(n > 0)$ and $A_1, \ldots, A_n \in \Sigma(M)$. □

The following shows the basic properties of the monoid action on $\Sigma(M)$.

Proposition 3.5: Let $A, B \in \Sigma(M)$, $x, y \in M$ and $m \in Q^M$. Then

(a) $eA = A$,
(b) $(xy)A = x(yA)$,
(c) $N(xA) = xN(A)$.
(d) $(x^*m)[\![A]\!] = m[\![xA]\!]$,
(e) $A \equiv B$ implies $xA \equiv xB$.

Proof. (a) $eA = A$:

$$ex = x, \quad \{\ e : \text{unit}\ \}$$

$$ec = c, \quad \{\ 3.4\,(\text{b})\ \}$$

$$\begin{aligned}e(\sigma A_1 \cdots A_n) &= \sigma(eA_1)\cdots(eA_n) && \{\ 3.4\,(\text{c})\ \}\\ &= \sigma A_1 \cdots A_n. && \{\ \text{IH (Induction hypothesis)}\ \}\end{aligned}$$

(b) $(xy)A = x(yA)$:

$$(xy)z = x(yz), \quad \{\ \text{associativity}\ \}$$

$$\begin{aligned}(xy)c &= c && \{\ 3.4\,(\text{b})\ \}\\ &= xc && \{\ 3.4\,(\text{b})\ \}\\ &= x(yc), && \{\ 3.4\,(\text{b})\ \}\end{aligned}$$

$$\begin{aligned}&(xy)(\sigma A_1 \cdots A_n)\\ = {}& \sigma((xy)A_1)\cdots((xy)A_n) && \{\ 3.4\,(\text{c})\ \}\\ = {}& \sigma(x(yA_1))\cdots(x(yA_n)) && \{\ \text{IH}\ \}\\ = {}& x(\sigma(yA_1)\cdots(yA_n)) && \{\ 3.4\,(\text{c})\ \}\\ = {}& x(y(\sigma A_1 \cdots A_n)). && \{\ 3.4\,(\text{c})\ \}\end{aligned}$$

(c) $N(xA) = xN(A)$:

$$\begin{aligned}N(xy) &= \{xy\}\\ &= xN(y),\end{aligned}$$

$$\begin{aligned}N(xc) &= N(c) && \{\ 3.4\,(\text{b})\ \}\\ &= \emptyset\\ &= xN(c), && \{\ N(c) = \emptyset\ \}\end{aligned}$$

$$\begin{aligned}N(x(\sigma A_1 \ldots A_n)) &= N(\sigma(xA_1)\ldots(xA_n))\\ &= \cup_{j=1}^{n} N(xA_j)\\ &= \cup_{j=1}^{n} xN(A_j) && \{\ \text{IH}\ \}\\ &= x(\cup_{j=1}^{n} N(A_j))\\ &= xN(A).\end{aligned}$$

(d) $(x^*m)[\![A]\!] = m[\![xA]\!]$:
Induction on A.

$$\begin{array}{rcll} \forall z \in M.\ (x^*m)[\![z]\!] & = & (x^*m)(z) & \{\, 2.6\,(a)\,\} \\ & = & m(xz) & \{\, 3.1\,\} \\ & = & m[\![xz]\!], & \{\, 2.6\,(a)\,\} \end{array}$$

$$\begin{array}{rcll} (x^*m)[\![c]\!] & = & c_\diamond & \{\, 2.6\,(b)\,\} \\ & = & m[\![c]\!] & \{\, 2.6\,(b)\,\} \\ & = & m[\![xc]\!], & \{\, 3.1\,(b)\,\} \end{array}$$

$$\begin{array}{cll} \multicolumn{3}{l}{(x^*m)[\![\sigma A_1 \cdots A_n]\!]} \\ = & \sigma_\diamond((x^*m)[\![A_1]\!], \ldots, (x^*m)[\![A_n]\!]) & \{\, 2.6\,(c)\,\} \\ = & \sigma_\diamond(m[\![xA_1]\!], \ldots, m[\![xA_n]\!]) & \{\, \mathrm{IH}\,\} \\ = & m[\![\sigma(xA_1)\cdots(xA_n)]\!]. & \{\, 2.6\,(c)\,\} \\ = & m[\![x(\sigma A_1 \cdots A_n)]\!]. & \{\, 3.1\,(c)\,\} \end{array}$$

(e) Assume $A \equiv B$. Then

$$\begin{array}{rcll} m[\![xA]\!] & = & (x^*m)[\![A]\!] & \{\, (c)\,\} \\ & = & (x^*m)[\![B]\!] & \{\, A \equiv B\,\} \\ & = & m[\![xB]\!]. & \{\, (c)\,\} \end{array}$$

□

4. Transition Functions of CA

Transition functions of CA on a monoid are formulated [5] by using monoid actions on configuration defined in Sect. 3 and the valuation of Σ-terms over a monoid reviewed in Sect. 2.

Definition 4.1: For a term $A \in \Sigma(M)$ define a function $T_A : Q^M \to Q^M$ by

$$T_A(m)(x) = m[\![xA]\!]$$

for all $m \in Q^M$ and $x \in M$. The function T_A is called the *transition* function (global map or parallel map) defined by A. □

The transition system (Q^M, T_A) is called a CA defined by A. The following example suggests that CA defined above contain ordinary CA [1] defined with local functions on neighborhoods.

Example 4.2: Let $\Sigma = \{\sigma\}$ be a signature consisting of only one operator symbol σ with arity n, and $N = \{x_1, \ldots, x_n\}$ a finite subset of M. Also suppose that Q is a Σ-algebra with a structure function $\sigma_\diamond : Q^n \to Q$. The transition function $T_A : Q^M \to Q^M$ defined by a Σ-term $A = \sigma x_1 \cdots x_n$ on M is nothing else the transition function of a CA usually defined with a local rule $\sigma_\diamond$.

$$\begin{array}{cll} & T_A(m)(x) & \\ = & m[\![xA]\!] & \\ = & m[\![\sigma(xx_1)\cdots(xx_n)]\!] & \{\, A = \sigma x_1 \cdots x_n\,\} \\ = & \sigma_\diamond(m[\![xx_1]\!], \ldots, m[\![xx_n]\!]) & \\ = & \sigma_\diamond(m(xx_1), \ldots, m(xx_n)) & \{\, xx_i \in M\,\} \\ = & \sigma_\diamond((x^*m)(x_1), \ldots, (x^*m)(x_n)) & \\ \doteq & \sigma_\diamond(p_N(x^*m)). & \{\, x_i \in N\,\} \end{array}$$

The last identity $\doteq$ is given by an isomorphism $Q^n \cong Q^N$. □

The following example shows that any elementary CA can be represented in the formulation of CA defined in the paper.

Example 4.3: Let M be the additive monoid of all integers, $\Sigma = \{\sigma\}$ a signature of one operator symbol σ with arity 3, and $N = \{x_1, x_2, x_3\}$ a subset consisting of three elements of M. Also suppose that $Q = \{0, 1\}$ is a Σ-algebra with a structure function $\sigma_\diamond : Q^3 \to Q$. The transition system (Q^M, T_A) defined by a Σ-term $A = \sigma x_1 x_2 x_3$ and it is a 1-dimensional CA with a triplet local rule on neighborhood N. Moreover, if $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $\sigma_\diamond : Q^3 \to Q$ is defined by

$$\sigma_\diamond(a, b, c) = a + b + c \pmod 2$$

then the transition system (Q^M, T_A) is the elementary CA with Wolfram rule 150. □

For $q \in Q$ the constant configuration $\hat{q} \in Q^M$ is defined by $\hat{q}(x) = q$ for all $x \in M$. The following states the basic properties of transition functions and (d) is an extension of commutability with the shift in Hedlund's theorem.

Proposition 4.4: Let $x, y \in M$, $m \in Q^M$, $A, B, A_1, \ldots, A_n \in \Sigma(M)$, $c \in \Sigma_0$ and $\sigma \in \Sigma_n$ $(n > 0)$. Then

(a) $T_x(m)(y) = m(yx)$. In particular $T_e = \mathrm{id}_{Q^M}$ (the identity function on Q^M).
(b) $T_c(m) = \hat{c}_\diamond$,
(c) $T_{\sigma A_1 \cdots A_n}(m)(x) = \sigma_\diamond(T_{A_1}(m)(x), \ldots, T_{A_n}(m)(x))$,
(d) $T_A(x^*m) = x^*(T_A(m))$,
(e) $T_A = T_B$ iff $A \equiv B$.

Proof. (a) First we have $T_x(m)(y) = m[\![yx]\!] = m(yx)$. In particular, $T_e(m)(y) = m(ye) = m(y)$ and so $T_e = \mathrm{id}_{Q^M}$.
(b) $T_c(m)(y) = m[\![yc]\!] = m[\![c]\!] = c_\diamond$ for all $y \in M$. Hence $T_c(m) = \hat{c}_\diamond$.
(c) $T_{\sigma A_1 \cdots A_n}(m)(x) = \sigma_\diamond(T_{A_1}(m)(x), \ldots, T_{A_1}(m)(x))$:

$$\begin{array}{cll} & T_{\sigma A_1 \cdots A_n}(m)(x) & \\ = & m[\![x(\sigma A_1 \cdots A_n)]\!] & \{\, 4.1\,\} \\ = & m[\![\sigma(xA_1)\cdots(xA_n)]\!] & \{\, 3.4\,(c)\,\} \\ = & \sigma_\diamond(m[\![xA_1]\!], \ldots, m[\![xA_n]\!]) & \{\, 2.6\,(c)\,\} \\ = & \sigma_\diamond(T_{A_1}(m)(x), \ldots, T_{A_n}(m)(x)). & \{\, 4.1\,\} \end{array}$$

(d) $T_A(x^*m) = x^*(T_A(m))$:

$$\begin{array}{rcll} T_A(x^*m)(z) & = & (x^*m)[\![zA]\!] & \{\, 4.1\,\} \\ & = & m[\![x(zA)]\!] & \{\, 3.3\,(c)\,\} \\ & = & m[\![(xz)A]\!] & \{\, 3.5\,(b)\,\} \\ & = & T_A(m)(xz) & \{\, 4.1\,\} \\ & = & (x^*(T_A(m)))(z). & \{\, 3.1\,\} \end{array}$$

(e) $T_A = T_B$ iff $A \equiv B$:
First assume $T_A = T_B$. Then for all $m \in Q^M$ we have

$$\begin{array}{rcll} m[\![A]\!] & = & T_A(m)(e) & \{\, 4.1\,\} \\ & = & T_B(m)(e) & \{\, T_A = T_B\,\} \\ & = & m[\![B]\!], & \{\, 4.1\,\} \end{array}$$

which implies $A \equiv B$. Conversely assume $A \equiv B$. Then

$$\begin{array}{rcll} T_A(m)(x) & = & m[\![xA]\!] & \\ & = & m[\![xB]\!] & \{ A \equiv B,\ 3.3\,(\mathrm{d}) \} \\ & = & T_B(m)(x). & \end{array}$$

□

The statement 4.4 (d) claims that a transition function T_A commutes with shifts, that is, $x^* \circ T_A = T_A \circ x^*$ for all $x \in M$.

5. Multiplication of Terms

Using the monoid action the multiplication of terms can be defined as well as shifted terms. Consequently it turns out that the multiplication of terms dominates the composition of transition functions of CA on monoids.

Definition 5.1: Let A and B be terms over a monoid M. The multiplication $AB \in \Sigma(M)$ of A and B is defined by induction on A.

(a) AB is the shifted term for $A \in M$,
(b) $AB = A$ for $A \in \Sigma_0$,
(c) $AB = \sigma(A_1B)\cdots(A_nB)$ for $A = \sigma A_1 \cdots A_n$ where $\sigma \in \Sigma_n$ $(n > 0)$ and $A_1,\ldots, A_n \in \Sigma(X)$. □

The following states the basic properties of the multiplication of terms.

Proposition 5.2: Let $A, B, C \in \Sigma(M)$, $x \in M$ and $m \in Q^M$. Then

(a) $Ae = A$,
(b) $(AB)C = A(BC)$,
(c) $N(AB) = N(A)N(B)$,
(d) $(x_*m)[\![A]\!] = m[\![Ax]\!]$.

Proof. (a) $Ae = A$: By induction.

$$xe = x, \quad \{ e : \text{unit} \}$$

$$ce = c, \quad \{ 5.1\,(\mathrm{b}) \}$$

$$\begin{array}{rcll} (\sigma A_1 \cdots A_n)e & = & \sigma(A_1e)\cdots(A_ne) & \{ 5.1\,(\mathrm{c}) \} \\ & = & \sigma A_1 \cdots A_n. & \{ \mathrm{IH} \} \end{array}$$

(b) $(AB)C = A(BC)$: By induction.
$(xB)C = x(BC)$:

$$(xy)C = x(yC), \quad \{ 3.5\,(\mathrm{b}) \}$$

$$\begin{array}{rcll} (xc)C & = & cC & \{ 3.4\,(\mathrm{b}) \} \\ & = & c & \{ 5.1\,(\mathrm{b}) \} \\ & = & xc & \{ 3.4\,(\mathrm{b}) \} \\ & = & x(cC), & \{ 5.1\,(\mathrm{b}) \} \end{array}$$

$$\begin{array}{cll} & (x(\sigma B_1 \cdots B_n))C & \\ = & (\sigma(xB_1)\cdots(xB_n))C & \{ 3.4\,(\mathrm{c}) \} \\ = & \sigma((xB_1)C)\cdots((xB_n)C) & \{ 5.1\,(\mathrm{c}) \} \\ = & \sigma(x(B_1C)\cdots(x(B_nC)) & \{ \mathrm{IH} \} \\ = & x(\sigma(B_1C)\cdots(B_nC)) & \{ 3.4\,(\mathrm{c}) \} \\ = & x((\sigma B_1 \cdots B_n)C). & \{ 5.1\,(\mathrm{c}) \} \end{array}$$

$(AB)C = A(BC)$:

$$(xB)C = x(BC) \quad \{ \text{above} \}$$

$$(cB)C = cC = c = c(BC) \quad \{ 5.1\,(\mathrm{b}) \}$$

$$\begin{array}{cll} & ((\sigma A_1 \cdots A_n)B)C & \\ = & ((\sigma(A_1B)\cdots(A_nB))C & \{ 5.1\,(\mathrm{c}) \} \\ = & \sigma((A_1B)C)\cdots((A_nB)C) & \{ 5.1\,(\mathrm{c}) \} \\ = & \sigma(A_1(BC))\cdots(A_n(BC)) & \{ \mathrm{IH} \} \\ = & (\sigma A_1 \cdots A_n)(BC). & \{ 5.1\,(\mathrm{c}) \} \end{array}$$

(c) $N(AB) = N(A)N(B)$: By induction on A.

$$\begin{array}{rcll} N(xB) & = & xN(B) & \{ 3.5\,(\mathrm{c}) \} \\ & = & N(x)N(B), & \end{array}$$

$$\begin{array}{rcll} N(cB) & = & \emptyset & \{ cB = c \} \\ & = & N(c)N(B), & \end{array}$$

$$\begin{array}{cll} & N((\sigma A_1 \ldots A_n)B) & \\ = & N(\sigma(A_1B)\ldots(A_nB)) & \{ 5.1\,(\mathrm{c}) \} \\ = & \cup_j N(A_jB) & \\ = & \cup_j N(A_j)N(B) & \{ \mathrm{IH} \} \\ = & (\cup_j N(A_j))N(B) & \{ \mathrm{IH} \} \\ = & N(\sigma A_1 \ldots A_n)N(B). & \end{array}$$

(d) $(x_*m)[\![A]\!] = m[\![Ax]\!]$: By induction on A.

$$\begin{array}{rcll} \forall z \in M.\ (x_*m)[\![z]\!] & = & (x_*m)(z) & \{ 2.6\,(\mathrm{a}) \} \\ & = & m(zx) & \{ 3.1 \} \\ & = & m[\![zx]\!], & \{ 2.6\,(\mathrm{a}) \} \end{array}$$

$$\begin{array}{rcll} (x_*m)[\![c]\!] & = & c_\diamond & \{ 2.6\,(\mathrm{b}) \} \\ & = & m[\![c]\!] & \{ 2.6\,(\mathrm{b}) \} \\ & = & m[\![cx]\!], & \{ 5.1\,(\mathrm{b}) \} \end{array}$$

$$\begin{array}{cll} & (x_*m)[\![\sigma A_1 \cdots A_n]\!] & \\ = & \sigma_\diamond((x_*m)[\![A_1]\!],\ldots,(x_*m)[\![A_n]\!]) & \{ 2.6\,(\mathrm{c}) \} \\ = & \sigma_\diamond(m[\![A_1x]\!],\ldots,m[\![A_nx]\!]) & \{ \mathrm{IH} \} \\ = & m[\![\sigma(A_1x)\cdots(A_nx)]\!]. & \{ 2.6\,(\mathrm{c}) \} \\ = & m[\![(\sigma A_1 \cdots A_n)x]\!]. & \{ 5.1\,(\mathrm{c}) \} \end{array}$$

□

Remark. It is clear that $xA = Ax$ if M is a commutative monoid. However a transition function T_A need not commute with all right shifts x_*, because $(T_A \circ x_*)(m)(y) = m[\![yAx]\!]$ by 5.2 (d) and $(x_* \circ T_A)(m)(y) = m[\![yxA]\!]$.

Ordinarily transition functions [1] of CA are defined with local rules $r : Q^N \to Q$. The composition [2] of local rules which corresponds to the composition of transition functions has been extensively studied by [4]. However the multiplication of Σ-terms directly represents the composition of transition functions of CA.

Theorem 5.3: For all terms $A, B \in \Sigma(M)$ the identity

$$T_A \circ T_B = T_{AB}$$

holds.

Proof. We need to show the following.

(a) $T_x \circ T_B = T_{xB}$ for $x \in M$,

(b) $T_c \circ T_B = T_{cB}$ for $c \in \Sigma_0$,
(c) $T_{\sigma A_1 \cdots A_n} \circ T_B = T_{(\sigma A_1 \cdots A_n)B}$ for $\sigma \in \Sigma_n$ ($n > 0$) and $A_1, \ldots, A_n \in \Sigma(M)$.

(a) $T_x \circ T_B = T_{xB}$:

$$\begin{aligned}(T_x \circ T_B)(m)(y) &= T_x(T_B(m))(y) \\ &= T_B(m)(yx) && \{\, 4.4\,(a) \,\} \\ &= m[\![(yx)B]\!] && \{\, 4.1 \,\} \\ &= m[\![y(xB)]\!] && \{\, 5.2\,(b) \,\} \\ &= T_{xB}(m)(y). && \{\, 4.1 \,\}\end{aligned}$$

(b) $T_c \circ T_B = T_c$:

$$\begin{aligned}(T_c \circ T_B)(m) &= T_c(T_B(m)) \\ &= \hat{c}_\diamond && \{\, 4.4\,(b) \,\} \\ &= T_c(m). && \{\, 4.4\,(b) \,\}\end{aligned}$$

(c) $T_{\sigma A_1 \cdots A_n} \circ T_B = T_{(\sigma A_1 \cdots A_n)B}$:

$$\begin{aligned}& (T_{\sigma A_1 \cdots A_n} \circ T_B)(m)(x) \\ =\ & T_{\sigma A_1 \cdots A_n}(T_B(m))(x) \\ =\ & \sigma_\diamond(T_{A_1}(T_B(m))(x), \ldots, T_{A_n}(T_C(m))(x)) && \{\, 4.4\,(f) \,\} \\ =\ & \sigma_\diamond((T_{A_1} \circ T_B)(m)(x), \ldots, (T_{A_n} \circ T_B)(m)(x)) \\ =\ & \sigma_\diamond(T_{A_1 B}(m)(x), \ldots, T_{A_1 B}(m)(x)) && \{\, \text{IH} \,\} \\ =\ & \sigma_\diamond(m[\![x(A_1 B)]\!], \ldots, m[\![x(A_n B)]\!]) && \{\, 4.1 \,\} \\ =\ & m[\![\sigma(x(A_1 B)) \cdots (x(A_n B))]\!] && \{\, 2.6\,(c) \,\} \\ =\ & m[\![x(\sigma(A_1 B) \cdots (A_n B))]\!] && \{\, 3.4\,(c) \,\} \\ =\ & T_{\sigma(A_1 B) \cdots (A_n B)}(m)(x) && \{\, 4.1 \,\} \\ =\ & T_{(\sigma A_1 \cdots A_n)B}(m)(x). && \{\, 5.1\,(c) \,\}\end{aligned}$$

This completes the proof. □

Remark. By 5.2 (c) it holds that

$$\begin{aligned}T_{(AB)C} &= T_{AB} \circ T_C && \{\, 5.3 \,\} \\ &= (T_A \circ T_B) \circ T_C && \{\, 5.3 \,\} \\ &= T_A \circ (T_B \circ T_C) && \{\, \text{associative law} \,\} \\ &= T_A \circ T_{BC} && \{\, 5.3 \,\} \\ &= T_{A(BC)}, && \{\, 5.3 \,\}\end{aligned}$$

which implies $(AB)C \equiv A(BC)$ from 4.4 (e).

Let A be a term on a monoid M. A transition function $T_A : Q^M \to Q^M$ is called *reversible* if it is bijective and $T_A^{-1} = T_B$ for some term B. Also A is *reversible* if there exists a term B on M such that $BA \equiv e$ and $AB \equiv e$. By the virtue of 4.4 (a), (e) and the last theorem 5.3, T_A is reversible iff A is reversible.

Corollary 5.4: Let $A, A', B, B' \in \Sigma(M)$ and $m \in Q^M$. Then the following holds.

(a) $T_A(m)[\![B]\!] = m[\![BA]\!]$,
(b) $A \equiv A'$ and $B \equiv B'$ imply $AB \equiv A'B'$.

Proof. (a) $T_A(m)[\![B]\!] = m[\![BA]\!]$:

$$\begin{aligned}m[\![BA]\!] &= m[\![e(BA)]\!] && \{\, 3.5\,(a) \,\} \\ &= T_{BA}(m)(e) && \{\, 4.1 \,\} \\ &= (T_B \circ T_A)(m)(e) && \{\, 5.3 \,\} \\ &= T_B(T_A(m))(e) \\ &= T_A(m)[\![eB]\!] && \{\, 4.1 \,\} \\ &= T_A(m)[\![B]\!]. && \{\, 3.5\,(a) \,\}\end{aligned}$$

(b) Assume $A \equiv A'$ and $B \equiv B'$. Then

$$\begin{aligned}m[\![AB]\!] &= T_B(m)[\![A]\!] && \{\, (a) \,\} \\ &= T_{B'}(m)[\![A]\!] && \{\, B \equiv B',\ 4.4\,(i) \,\} \\ &= T_{B'}(m)[\![A']\!] && \{\, A \equiv A' \,\} \\ &= m[\![A'B']\!]. && \{\, (a) \,\}\end{aligned}$$

$$\begin{aligned}T_{AB} &= T_A \circ T_B \\ &= T_{A'} \circ T_{B'} \\ &= T_{A'B'},\end{aligned}$$

which proves $AB \equiv A'B'$. □

6. Restricted Transition Functions

In this section we will formulate restricted valuations of terms by partial configurations, called patterns, and restricted transition functions defined on suitable patterns. In fact the valuation $m[\![A]\!]$ of a term $A \in \Sigma(M)$ by a configuration $m \in Q^M$ depends on values $m(x)$ only for $x \in N(A)$. This suggests that terms may be evaluated by patterns.

Definition 6.1: Let $A \in \Sigma(M)$ and V a subset of M such that $N(A) \subseteq V$. For a pattern $d \in Q^V$ the value $d[\![A]\!]_V \in Q$ of A by d can be defined by the analogous way with 2.6:

(a) $d[\![x]\!]_V = d(x)$ for all $x \in V$.
(b) $d[\![c]\!]_V = c_\diamond$ for all constants $c \in \Sigma_0$.
(c) $d[\![\sigma A_1 \cdots A_n]\!]_V = \sigma_\diamond(d[\![A_1]\!]_V, \ldots, d[\![A_n]\!]_V)$ if $N(\sigma A_1 \cdots A_n) \subseteq V$. □

The following proposition states the basic property of restricted valuations.

Proposition 6.2: Let $m, m' \in Q^M$, $V \subseteq M$ and $A \in \Sigma(M)$ such that $N(A) \subseteq V$. Then

(a) $p_V(m)[\![A]\!]_V = m[\![A]\!]$,
(b) If $p_V(m) = p_V(m')$, then $m[\![A]\!] = m'[\![A]\!]$.

Proof. (a) It is clear from the definition.
(b)

$$\begin{aligned}m[\![A]\!] &= p_V(m)[\![A]\!] && \{\, (a) \,\} \\ &= p_V(m')[\![A]\!] && \{\, p_V(m) = p_V(m') \,\} \\ &= m'[\![A]\!]. && \{\, (a) \,\}\end{aligned}$$

□

The transition function $T_A : Q^M \to Q^M$ can be restricted as follows: Let $V \subseteq M$ and set $N = N(A)$. The *restricted* transition function $R_V : Q^{VN} \to Q^V$ is defined by

$$R_V(d)(x) = d[\![xA]\!]_V$$

for all $d \in Q^{VN}$ and $x \in V$. Remark that $d[\![xA]\!]_V$ is well-defined since $N(xA) = xN \subseteq VN$.

The following lemma will be useful to show the continuity of T_A in the next section.

Lemma 6.3: Let $A \in \Sigma(M)$, $N = N(A)$ and $V \subseteq M$. Then the square

$$\begin{tikzcd} Q^M \arrow[r, "T_A"] \arrow[d, "p_{VN}"'] & Q^M \arrow[d, "p_V"] \\ Q^{VN} \arrow[r, "R_V"'] & Q^V \end{tikzcd}$$

commutes, that is, $R_V \circ p_{VN} = p_V \circ T_A$.

Proof. For $m \in Q^M$ and $x \in V$ we have

$$\begin{array}{lll} & (R_V \circ p_{VN})(m)(x) & \\ = & R_V(p_{VN}(m))(x) & \\ = & p_{VN}(m)[\![xA]\!]_{VN} & \{ N(xA) \subseteq VN \} \\ = & m[\![xA]\!] & \{ 6.2 \text{ (a) } \} \\ = & T_A(m)(x) & \{ 4.1 \} \\ = & (p_V(T_A(m))(x) & \{ x \in V \} \\ = & (p_V \circ T_A)(m)(x). & \end{array}$$

□

7. Prodiscrete Topology

In this section we will review the prodiscrete topology [1] on the configuration space Q^M and extend so-called Hedlund's theorem [3] that transition functions of CA are continuous.

First we recall two fundamental operations of families of sets introduced by a function $f : X \to Y$ of a set X into a set Y. For a subset $\mathcal{Y}$ of the power set $\wp(Y)$ define a subset $f^{-1}(\mathcal{Y})$ of $\wp(X)$ by

$$A \in f^{-1}(\mathcal{Y}) \leftrightarrow \exists B \in \mathcal{Y}.\ A = f^{-1}(B).$$

For a subset $\mathcal{X}$ of $\wp(X)$ define a subset $f_*(\mathcal{X})$ of $\wp(Y)$ by

$$B \in f_*(\mathcal{X}) \leftrightarrow f^{-1}(B) \in \mathcal{X}.$$

Proposition 7.1: Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $\mathcal{X} \subseteq \wp(X)$, $\mathcal{Y} \subseteq \wp(Y)$ and $\mathcal{Z} \subseteq \wp(Z)$. Then

(a) If $\mathcal{Y}$ is a topology on Y, then $f^{-1}(\mathcal{Y})$ is a topology on X,
(b) If $\mathcal{X}$ is a topology on X, then $f_*(\mathcal{X})$ is a topology on Y,
(c) $(g \circ f)^{-1}(\mathcal{Z}) = f^{-1}(g^{-1}(\mathcal{Z}))$,
(d) $f^{-1}(\mathcal{Y}) \subseteq \mathcal{X} \leftrightarrow \mathcal{Y} \subseteq f_*(\mathcal{X})$.

Proof. (d) ($\to$) Assume $f^{-1}(\mathcal{Y}) \subseteq \mathcal{X}$. Then

$$\begin{array}{llll} B \in \mathcal{Y} & \to & f^{-1}(B) \in f^{-1}(\mathcal{Y}) & \\ & \to & f^{-1}(B) \in \mathcal{X} & \{ f^{-1}(\mathcal{Y}) \subseteq \mathcal{X} \} \\ & \to & B \in f_*(\mathcal{X}), & \end{array}$$

which shows $\mathcal{Y} \subseteq f_*(\mathcal{X})$.
($\leftarrow$) Assume $\mathcal{Y} \subseteq f_*(\mathcal{X})$. Then

$$\begin{array}{lll} & A \in f^{-1}(\mathcal{Y}) & \\ \to & \exists B \in \mathcal{Y}.\ A = f^{-1}(B) & \\ \to & B \in f_*(\mathcal{X}) \wedge A = f^{-1}(B) & \{ \mathcal{Y} \subseteq f_*(\mathcal{X}) \} \\ \to & f^{-1}(B) \in \mathcal{X} \wedge A = f^{-1}(B) & \\ \to & A \in \mathcal{X}, & \end{array}$$

which shows $f^{-1}(\mathcal{Y}) \subseteq \mathcal{X}$. □

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be topological spaces, that is, $\mathcal{X}$ and $\mathcal{Y}$ are topologies on X and Y, respectively. Recall that a function $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ is continuous iff $f^{-1}(\mathcal{Y}) \subseteq \mathcal{X}$.

The *prodiscrete topology* $\mathcal{T}_0$ on Q^X is the least topology such that for all $x \in X$ the projection $p_x : Q^X \to Q$ is continuous, where the set Q has the discrete topology $\wp(Q)$. Formally we define the prodiscrete topology as follows:

Definition 7.2: The *prodiscrete topology* $\mathcal{T}_0$ on Q^X is a topology such that for all topologies $\mathcal{T}$ on Q^X the following equivalence holds:

$$\mathcal{T}_0 \subseteq \mathcal{T} \quad \leftrightarrow \quad \forall x \in X.\ p_x^{-1}(\wp(Q)) \subseteq \mathcal{T}.$$

□

Since Q is finite, Q is compact. Thus the product space Q^X with the prodiscrete topology $\mathcal{T}_0$ is compact Hausdorff and totally disconnected [1].

Proposition 7.3: If X is a finite set, then the prodiscrete topology on Q^X is the discrete topology, that is, $\mathcal{T}_0 = \wp(Q^X)$.

Proof. We need to see that every point $c \in Q^X$ is open. It is well-known that the identity $c = \cap_{x \in X} p_x^{-1}(p_x(c))$. Hence c is open, because Q is discrete and all projections p_x are continuous and X is finite. □

Note. $c = \cap_{x \in X} p_x^{-1}(p_x(c))$:

$$\begin{array}{lll} c' \in \cap_{x \in X} p_x^{-1}(p_x(c)) & \leftrightarrow & \forall x \in X.\ c' \in p_x^{-1}(p_x(c)) \\ & \leftrightarrow & \forall x \in X.\ p_x(c') = p_x(c) \\ & \leftrightarrow & \forall x \in X.\ c'(x) = c(x) \\ & \leftrightarrow & c' = c. \end{array}$$

□

The following is a well-known property of product topologies.

Proposition 7.4: A function $f : (Z, \mathcal{O}) \to (Q^X, \mathcal{T}_0)$ is continuous iff for all $x \in X$ the composite $p_x \circ f : (Z, \mathcal{O}) \to (Q, \wp(Q))$ is continuous.

Proof.

$$\begin{array}{lll} & f^{-1}(\mathcal{T}_0) \subseteq \mathcal{O} & \\ \leftrightarrow & \mathcal{T}_0 \subseteq f_*(\mathcal{O}) & \{ 7.1 \text{ (d) } \} \\ \leftrightarrow & \forall x \in X.\ p_x^{-1}(\wp(Q)) \subseteq f_*(\mathcal{O}) & \{ 7.1 \text{ (b)}, 7.2 \} \\ \leftrightarrow & \forall x \in X.\ f^{-1}(p_x^{-1}(\wp(Q))) \subseteq \mathcal{O} & \{ 7.1 \text{ (d) } \} \\ \leftrightarrow & \forall x \in X.\ (p_x \circ f)^{-1}(\wp(Q)) \subseteq \mathcal{O}. & \{ 7.1 \text{ (b) } \} \end{array}$$

□

We now prove an extension of a fundamental theorem for CA due to Hedlund [3, Theorem 3.1] that every transition function of CA is continuous.

Theorem 7.5: Let $V \subseteq M$, $x \in M$ and $A \in \Sigma(M)$. Then

(a) The projection $p_V : Q^M \to Q^V$ is continuous,
(b) The shift functions x^* and $x_* : Q^M \to Q^M$ are continuous,
(c) The transition function $T_A : Q^M \to Q^M$ is continuous.

Proof. (a)

$$\begin{array}{lll} \to & \forall x \in M.\ p_x : \text{conti} & \{\ 7.2\ \} \\ \to & \forall x \in V.\ p_x : \text{conti} & \{\ V \subseteq M\ \} \\ \to & \forall x \in V.\ p^V_x \circ p_V : \text{conti} & \{\ p^V_x \circ p_V = p_x\ \} \\ \to & p_V : \text{conti.} & \{\ 7.4\ \} \end{array}$$

(b^*)

$$\begin{array}{lll} \to & \forall y \in M.\ p_y : \text{conti} & \{\ 7.2\ \} \\ \to & \forall y \in M.\ p_{xy} : \text{conti} & \{\ xy \in M\ \} \\ \to & \forall y \in M.\ p_y \circ x^* : \text{conti} & \\ & & \{\ 3.3\,(c)\ p_y \circ x^* = p_{xy}\ \} \\ \to & x^* : \text{conti.} & \{\ 7.4\ \} \end{array}$$

(b_*)

$$\begin{array}{lll} \to & \forall y \in M.\ p_y : \text{conti} & \{\ 7.2\ \} \\ \to & \forall y \in M.\ p_{yx} : \text{conti} & \{\ yx \in M\ \} \\ \to & \forall y \in M.\ p_y \circ x_* : \text{conti} & \\ & & \{\ 3.3\,(c)\ p_y \circ x_* = p_{yx}\ \} \\ \to & x_* : \text{conti.} & \{\ 7.4\ \} \end{array}$$

(c) Set $V = x$ in 6.3. Then we have a commutative square

$$\begin{tikzcd} Q^M \arrow[r, "T_A"] \arrow[d, "p_{xN}"'] & Q^M \arrow[d, "p_x"] \\ Q^{xN} \arrow[r, "R_x"'] & Q. \end{tikzcd}$$

Also the projection p_{xN} is continuous by (a) and the restricted transition function $R_x : Q^{xN} \to Q$ is continuous, because xN is a finite subset of M and so Q^{xN} is a discrete space by 7.3. Hence

$$\begin{array}{lll} \to & \forall x \in M.\ p_{xN} : \text{conti} \ \wedge\ R_x : \text{conti} & \\ \to & \forall x \in M.\ R_x \circ p_{xN} : \text{conti} & \{\ 7.4\ \} \\ \to & \forall x \in M.\ p_x \circ T_A : \text{conti} & \\ & & \{\ R_x \circ p_{xN} = p_x \circ T_A\ \} \\ \to & T_A : \text{conti.} & \{\ 7.4\ \} \end{array}$$

□

The converse statement of 7.5 (c) does not always hold for CA associated with (fixed) Σ-algebras. More precisely, for a continuous function $T : Q^M \to Q^M$ which commutes with all shifts x^*, there may exist no Σ-term A such that $T = T_A$. We give a counter example.

Example 7.6: Let $\Sigma = \{0, +\}$ be a signature in 2.2 (2) and $Q = \{0, 1\}$ a Σ-algebra in 2.5 (2). It is readily checked (by induction) that $\hat{0}[\![A]\!] = 0$ and so $T_A(\hat{0}) = \hat{0}$ for all $A \in \Sigma(M)$. Consider the constant function $T : Q^M \to Q^M$ such that $T(m) = \hat{1}$ for all $m \in Q^M$. Then T is continuous and commutes with shifts x^*, but there is no Σ-term A such that $T = T_A$. □

Finally we discuss a particular case that a continuous functions on Q^M commuting with shifts is a transition function of CA associated with Σ-algebras.

We will denote a finite subset V of M by the notation $V \subseteq_* M$. A subset $A \subseteq Q^M$ is called a *basic* subset of Q^M if there exist $V \subseteq_* M$ and $D \subseteq Q^V$ such that $A = p_V^{-1}(D)$. Remark that Q^V is a finite set. The set of all basic subsets of Q^M will be denoted by $\mathcal{B}$. Clearly $p_x^{-1}(\wp(Q)) \subseteq \mathcal{B}$ for all $x \in M$. Note that $\mathcal{B} \subseteq \mathcal{T}_0$: For $V \subseteq_* M$ and $D \subseteq Q^V$ we have

$$\begin{array}{lll} p_V^{-1}(D) & = & p_V^{-1}(\cup_{d\in D} d) \qquad \{\ D = \cup_{d\in D} d\ \} \\ & = & \cup_{d\in D} p_V^{-1}(d) \\ & = & \cup_{d\in D}(\cap_{x\in V} p_x^{-1}(p_x^V(d))) \\ & & \qquad \{\ d = \cap_{x\in V}(p_x^V)^{-1}(p_x^V(d))\ \} \\ & \in & \mathcal{T}_0. \qquad\qquad \{\ p_x^{-1}(\wp(Q)) \subseteq \mathcal{T}_0\ \} \end{array}$$

Also define a subset $\mathcal{T}_1$ of $\wp(Q^M)$ as follows:

$$C \in \mathcal{T}_1 \leftrightarrow \forall c \in C\, \exists B \in \mathcal{B}.\ c \in B \subseteq C.$$

It is easy to verify that $\mathcal{T}_1$ is a topology on Q^M. The next proposition suggests that $\mathcal{B}$ is a basis of the prodiscrete topology $\mathcal{T}_0$.

Proposition 7.7: $\mathcal{T}_0 = \mathcal{T}_1$.

Proof. $\mathcal{T}_0 \subseteq \mathcal{T}_1$:

$$\begin{array}{lll} \to & \mathcal{B} \subseteq \mathcal{T}_1 & \\ \to & \forall x \in M.\ p_x^{-1}(\wp(Q)) \subseteq \mathcal{T}_1 & \{\ p_x^{-1}(\wp(Q)) \subseteq \mathcal{B}\ \} \\ \to & \mathcal{T}_0 \subseteq \mathcal{T}_1. & \{\ 7.2\ \} \end{array}$$

$\mathcal{T}_1 \subseteq \mathcal{T}_0$:

$$\begin{array}{llll} C \in \mathcal{T}_1 & \to & C = \cup_k p_{V_k}^{-1}(D_k) & \{\ \text{Def. of } \mathcal{T}_1\ \} \\ & \to & C \in \mathcal{T}_0. & \{\ \mathcal{B} \subseteq \mathcal{T}_0\ \} \end{array}$$

□

The following proposition is a well-known property of continuous functions from prodiscrete spaces into discrete spaces.

Proposition 7.8: A function $t : Q^M \to Q$ is continuous iff there exist a finite subset $V \subseteq_* M$ and a function $f : Q^V \to Q$ such that $t = f \circ p_V$.

Proof. (←) Let $V \subseteq_* M$ and $f : Q^V \to Q$ a function. The projection p_V is continuous by 7.5 (a). Since V is finite, Q^V is discrete. Hence f is continuous and the composite $f \circ p_V$ is continuous.
(→) Assume that a function $t : Q^M \to Q$ is continuous. For all $m \in Q^M$ it is obvious that $m \in t^{-1}(t(m))$ and $t^{-1}(t(m))$ is open in Q^M, since Q is discrete. By 7.7 there exist a finite subset $V_m \subseteq_* M$ and a subset $D_m \subseteq Q^{V_m}$ such that $m \in p_{V_m}^{-1}(D_m) \subseteq t^{-1}(t(m))$. The collection of all open sets $p_{V_m}^{-1}(D_m)$ is an open covering of Q^M. Since Q^M is compact,

the covering has a finite subcovering, that is,

$$\exists m_1, \ldots, m_k \in Q^M. \ \cup_{j=1}^k p_{V_{m_j}}^{-1}(D_{m_j}) = Q^M.$$

Set $V = \cup_{j=1}^k V_{m_j}$. Then V is also finite. For all $m \in Q^M$ there is a j $(1 \le j \le k)$ such that $m \in p_{V_{m_j}}^{-1}(D_{m_j}) \subseteq t^{-1}(t(m_j))$. Hence we have

$$\begin{array}{lll} & p_V^{-1}(p_V(m)) & \\ \subseteq & p_{V_{m_j}}^{-1}(p_{V_{m_j}}(m)) & \{ V_{m_j} \subseteq V \} \\ \subseteq & p_{V_{m_j}}^{-1}(D_{m_j}) & \{ m \in p_{V_{m_j}}^{-1}(D_{m_j}) \} \\ \subseteq & t^{-1}(t(m_j)) & \{ p_{V_{m_j}}^{-1}(D_{m_j}) \subseteq t^{-1}(t(m_j)) \} \\ = & t^{-1}(t(m)), & \{ m \in t^{-1}(t(m_j)) \} \end{array}$$

which means that $p_V(m') = p_V(m)$ implies $t(m') = t(m)$. Of course the projection p_V is a surjection. Therefore there exists a unique function $f : Q^V \to Q$ such that $t = f \circ p_V$. □

Let $T : Q^M \to Q^M$ be a continuous function such that $x^* \circ T = T \circ x^*$ for all $x \in M$. Then by the virtue of the last proposition $\exists V \subseteq_* M \, \exists A \in \Sigma(M). \ p_e \circ T = f \circ p_V$. Moreover, we assume that $\forall d \in Q^V. \ f(d) = d[\![A]\!]_V$ for some Σ-term A with $N(A) \subseteq V$. Then $T = T_A$ holds, because

$$\begin{array}{rcll} T(m)(x) & = & (p_x \circ T)(m) & \{ p_x(m) = m(x) \} \\ & = & (p_e \circ x^* \circ T)(m) & \{ p_x = p_e \circ x^* \} \\ & = & (p_e \circ T \circ x^*)(m) & \{ x^* \circ T = T \circ x^* \} \\ & = & (f \circ p_V \circ x^*)(m) & \{ p_e \circ T = f \circ p_V \} \\ & = & f(p_V(x^* m)) & \\ & = & p_V(x^* m)[\![A]\!]_V & \{ f(d) = d[\![A]\!]_V \} \\ & = & (x^* m)[\![A]\!] & \{ 6.2\,(a) \} \\ & = & m[\![xA]\!] & \{ 3.5\,(d) \} \\ & = & T_A(m)(x). & \{ 4.1 \} \end{array}$$

8. Conclusion

The paper presented a novel treatment of CA: cells are elements of monoids rather than of groups, local rules are terms of Σ-algebras rather than functions from neighborhoods to state sets, and configurations are valuations for variables in Σ-algebras. And then Hedlund's theorem for CA associated with Σ-algebras was proved.

CA treated in the paper is not on groups but on monoids. Monoids are weaker algebraic systems than groups because of loss of existence of inverse elements. Since a group is also a monoid, the class of CA on monoids might include CA on groups. Consequently the paper provides a wider overview for CA.

Cell spaces on groups are invariant in a sense that group action (shift) translates the whole cell space onto itself by the existence of inverse elements. On the other hand, cell spaces on monoids are not always invariant under translation and have a certain difficulty to manipulate. Moore's theorem [7] for CA on the additive monoid $\mathbb{N}$ of all naturals fails by the reason why monoid action does not preserve erasable patterns, but Myhill's theorem [8] holds for CA on $\mathbb{N}$. Richardson [9] studied nondeterministic CA on Euclidean lattices $\mathbb{Z}^k$ and extended Hedlund's theorem to those CA. We have left to study nondeterministic CA on monoids as a future work. For example studying an extension of Richardson's theorem which connects CA and continuous relations between prodiscrete topological spaces on the set of configurations might be one of interesting research candidates. Also, it is open if there exists a universal CA on monoids like the game of life or CA-110.

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