PAPER Special Section on Foundations of Computer Science—Developments of the Theory of Algorithms and Computation—

Reconfiguration of Vertex Covers in a Graph*

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SUMMARY Suppose that we are given two vertex covers C_0 and C_t of a graph G, together with an integer threshold $k \geq \max\{|C_0|, |C_t|\}$. Then, the VERTEX COVER RECONFIGURATION problem is to determine whether there exists a sequence of vertex covers of G which transforms C_0 into C_t such that each vertex cover in the sequence is of cardinality at most k and is obtained from the previous one by either adding or deleting exactly one vertex. This problem is PSPACE-complete even for planar graphs. In this paper, we first give a linear-time algorithm to solve the problem for even-hole-free graphs, which include several well-known graphs, such as trees, interval graphs and chordal graphs. We then give an upper bound on k for which any pair of vertex covers in a graph G has a desired sequence. Our upper bound is best possible in some sense.
key words: *combinatorial reconfiguration, even-hole-free graph, graph algorithm, vertex cover*

1. Introduction

A *vertex cover* C of a graph G is a vertex subset of G which contains at least one of the two endpoints of every edge in G. (See Fig. 1 which depicts six different vertex covers of the same graph.) Then, the VERTEX COVER problem is a well-known NP-complete problem [7], defined as follows: Given a graph G and an integer k, it determines whether G has a vertex cover of cardinality at most k.

The VERTEX COVER problem has several applications [15], such as in the SNP assembly problem in computational biochemistry and in a computer network security problem. In the computer network security problem, each vertex corresponds to a router and each edge corresponds to a link in a computer network, and we wish to pick a subset of routers for monitoring packets flowing on the links; such a subset forms a vertex cover of the corresponding graph.

However, a practical issue in computer network security requires that the formulation be considered in more dynamic situations: in order to maintain routers, we sometimes need to change the current subset of routers to another subset. Of course, we wish to keep monitoring all links even during the transformation. This situation can be formulated by the concept of reconfiguration problems that have been extensively studied in recent literature [2], [4], [8]–[10], [12], [14].

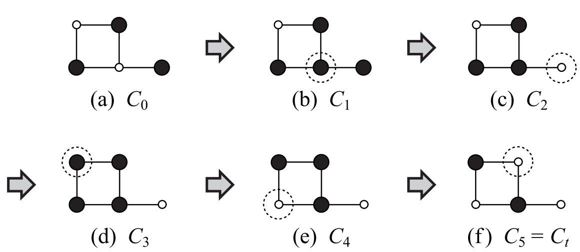

Fig. 1 A sequence $\langle C_0, C_1, \ldots, C_5\rangle$ of vertex covers of the same graph, where the vertices in vertex covers are depicted by large black circles.

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1.1 Our Problems

Suppose that we are given two vertex covers C_0 and C_t of a graph $G = (V, E)$, together with an integer threshold $k \geq \max\{|C_0|, |C_t|\}$. Then, the VERTEX COVER RECONFIGURATION problem is to determine whether there exists a sequence $\langle C_0, C_1, \ldots, C_\ell\rangle$ of vertex covers of G such that

(a) $C_\ell = C_t$, and $|C_i| \leq k$ for all i, $0 \leq i \leq \ell$; and
(b) for each index i, $1 \leq i \leq \ell$, the vertex cover C_i of G is obtained from the previous one C_{i-1} by either deleting or adding a single vertex $u \in V$, that is, $C_{i-1} \vartriangle C_i = (C_{i-1} \setminus C_i) \cup (C_i \setminus C_{i-1}) = \{u\}$.

Figure 1 illustrates a sequence $\langle C_0, C_1, \ldots, C_5\rangle$ of vertex covers of the same graph which transforms C_0 into $C_t = C_5$, where the vertex which is deleted from (or added to) the previous vertex cover is surrounded by a dotted circle.

The existence of such a transformation clearly depends on the value of a given threshold k. For example, if $k \geq 4$, then the instance of the two vertex covers C_0 and C_t in Fig. 1 is a yes-instance, because all vertex covers in Fig. 1 have cardinality at most four. On the other hand, if $k \leq 3$, then the instance in Fig. 1 is a no-instance, because there is no transformation between C_0 and C_t that consists only of vertex covers of cardinality at most three.

Therefore, we can get a natural minimization problem, called the MINMAX VERTEX COVER RECONFIGURATION problem, in which we wish to minimize the maximum cardinality of any vertex cover in a transformation for two given vertex covers C_0 and C_t of a graph G; we denote by $f^*_G(C_0, C_t)$ the optimal value. Then, the answer to VERTEX COVER RECONFIGURATION is "yes" if $k \geq f^*_G(C_0, C_t)$; otherwise "no." (A formal definition will be given in Sect. 2.)

1.2 Related and Known Results

Recently, this type of problems has been studied extensively in the framework of *reconfiguration problems* [10], which arise when we wish to find a step-by-step transformation between two feasible solutions of a search problem such that all intermediate solutions are also feasible and each step abides by a prescribed reconfiguration rule (i.e., an adjacency relation defined on feasible solutions of the original search problem). For example, in VERTEX COVER RECONFIGURATION, feasible solutions are defined to be all vertex covers of a graph with cardinality at most a given threshold k; and the reconfiguration rule is defined to be the condition (b) in Sect. 1.1. (We simply say a reconfiguration problem *on* a search problem A if the reconfiguration problem considers feasible solutions of A as the intermediate solutions.) This reconfiguration framework has been applied to several well-known problems, including INDEPENDENT SET [9], [10], [12], [14], SATISFIABILITY [8], CLIQUE, MATCHING [10], VERTEX-COLORING [2], [4], etc.

Both VERTEX COVER RECONFIGURATION and MINMAX VERTEX COVER RECONFIGURATION are known to be PSPACE-complete for planar graphs of maximum degree three [10], and hence it is very unlikely that they are solvable in polynomial time even for planar graphs.

From the viewpoint of approximation, it is known that the optimal value $f^*_G(C_0, C_t)$ can be approximated within a factor 2 in linear time; indeed, this approximation result can be obtained from a linear-time 2-approximation algorithm for the reconfiguration problem on SET COVER [10, Theorem 6]. However, as far as we know, only this 2-approximation is known for the reconfiguration problem on SET COVER, and hence it is desired to investigate further algorithmic (positive) results for the reconfiguration problems on VERTEX COVER.

One may think that some known results for the reconfiguration problem on INDEPENDENT SET [9], [10], [12], [14] can be converted into our VERTEX COVER RECONFIGURATION; because, if a vertex subset C is a vertex cover of a graph $G = (V, E)$, then $V \setminus C$ forms an independent set of G, and *vice versa*. There are three types of reconfiguration problems on INDEPENDENT SET which employ different reconfiguration rules. Although one of the three rules corresponds to the one for our VERTEX COVER RECONFIGURATION, almost all (positive) results are given to the other reconfiguration rules; this will be discussed in Sect. 3.

1.3 Our Contribution

In this paper, we investigate algorithmic results for the VERTEX COVER RECONFIGURATION and MINMAX VERTEX COVER RECONFIGURATION problems. (An extended abstract of this paper has been presented in [11].)

We first show that both reconfiguration problems can be solved in linear time for even-hole-free graphs. We will define the class of even-hole-free graphs later, but we here note that this graph class contains several well-known graph classes, such as trees, interval graphs and chordal graphs.

We then give an upper bound on the optimal value $f^*_G(C_0, C_t)$ for two vertex covers C_0 and C_t of a graph G. Our upper bound holds for any graph; as a corollary, we have $f^*_G(C_0, C_t) \leq \max\{|C_0|, |C_t|\} + 1$ if G is a tree; and $f^*_G(C_0, C_t) \leq \max\{|C_0|, |C_t|\} + 2$ if G is a cactus. We note that our upper bound is best possible in the following sense: there are instances of cacti such that $f^*_G(C_0, C_t) = \max\{|C_0|, |C_t|\} + 2$. (See Sect. 4.2 for details.)

We finally note that our second result gives an approximation for $f^*_G(C_0, C_t)$ with absolute performance guarantee. For an instance of MINMAX VERTEX COVER RECONFIGURATION, let $\mathsf{ap}_G(C_0, C_t)$ be an objective value computed by an algorithm. Then, for two integers $\rho \geq 1$ and $c \geq 0$, we say that the algorithm is a (ρ, c)*-approximation* if $\mathsf{ap}_G(C_0, C_t) \leq \rho \cdot f^*_G(C_0, C_t) + c$ holds for any instance. As we have mentioned above, there is a linear-time $(2, 0)$-approximation algorithm for $f^*_G(C_0, C_t)$, which follows from the 2-approximation for the reconfiguration problem on SET COVER [10]. On the other hand, our second result gives a $(1, \alpha)$-approximation for $f^*_G(C_0, C_t)$, where α is some integer defined later. Although this integer α depends on an input graph G, it is remarkable that α can be obtained only by a *local computation*: we just focus on a transformation restricted on 2-connected subgraphs of G, and extend it to a transformation on the whole graph G.

2. Preliminaries

In this section, we define some terms which will be used throughout the paper.

2.1 Graph Notation

In this paper, we may assume without loss of generality that graphs are simple and undirected. For a graph G, we sometimes denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For a vertex subset V' of a graph G, we denote by $G[V']$ the subgraph of G induced by V'.

A vertex u in a connected graph $G = (V, E)$ is called a *cut vertex* of G if the induced subgraph $G[V \setminus \{u\}]$ is disconnected. A connected graph G is said to be 2*-connected* if G has no cut vertex.

A vertex subset C of a graph G is called a *vertex cover* of G if at least one of $v \in C$ and $w \in C$ holds for every edge $vw \in E(G)$. We say that an edge $vw \in E(G)$ is *covered* by v if $v \in C$.

2.2 Definitions for VERTEX COVER RECONFIGURATION

Let C_i and C_j be two vertex covers of a graph G. We say that C_i and C_j are *adjacent* if their symmetric difference $C_i \vartriangle C_j$ consists of exactly one vertex u, that is, C_j can be obtained from C_i by either *deleting* or *adding* the vertex u. A *reconfiguration sequence* between two vertex covers C and C' of G is a sequence $\mathcal{C} = \langle C_1, C_2, \ldots, C_\ell \rangle$ of vertex covers of G

such that $C_1 = C$, $C_\ell = C'$, and C_{i-1} and C_i are adjacent for $i = 2, 3, \ldots, \ell$. For notational convenience, we write $C_i \in \mathcal{C}$ if a vertex cover C_i appears in a reconfiguration sequence $\mathcal{C}$. Observe that any reconfiguration sequence is *reversible*, that is, $\langle C_\ell, C_{\ell-1}, \ldots, C_1 \rangle$ is a reconfiguration sequence from $C_\ell = C'$ to $C_1 = C$, because the adjacency relation on vertex covers of G is symmetric.

For a reconfiguration sequence $\mathcal{C}$, let

$$f(\mathcal{C}) = \max\{|C_i| : C_i \in \mathcal{C}\},$$

that is, the maximum cardinality of a vertex cover that appears in $\mathcal{C}$. For two vertex covers C and C' of a graph G, we define the *reconfiguration index* $f^*_G(C, C')$, as follows:

$$f^*_G(C, C') = \min\{f(\mathcal{C}) : \mathcal{C} \text{ is a reconfiguration sequence between } C \text{ and } C'\}.$$

It should be noted that the reconfiguration index $f^*_G(C, C')$ is well defined, because any pair of vertex covers C and C' of G has a trivial reconfiguration sequence $\mathcal{C}$ such that $f(\mathcal{C}) = |C \cup C'|$, as follows: we add to C the vertices in $C' \setminus C$ one by one, and obtain the vertex cover $C \cup C'$ of G; and we delete from $C \cup C'$ the vertices in $C \setminus C'$ one by one. We note in passing that this trivial reconfiguration sequence $\mathcal{C}$ gives a 2-approximation for $f^*_G(C, C')$ as shown in [10, Theorem 6], because we clearly have

$$f^*_G(C, C') \geq \max\{|C|, |C'|\} \tag{1}$$

and $f^*_G(C, C') \leq |C \cup C'| \leq 2 \cdot \max\{|C|, |C'|\}$.

Given two vertex covers C_0 and C_t of a graph G and a positive integer k, the VERTEX COVER RECONFIGURATION problem is to determine whether $f^*_G(C_0, C_t) \leq k$; while the MINMAX VERTEX COVER RECONFIGURATION problem is to compute the reconfiguration index $f^*_G(C_0, C_t)$. Note that both problems do not ask for an actual reconfiguration sequence between C_0 and C_t. We always denote by C_0 and C_t the *initial* and *target* vertex covers of G, respectively.

3. Even-Hole-Free Graphs

A graph G is *even-hole free* if any induced subgraph of G is not a cycle consisting of an even number of vertices [6]. This graph class includes several well-known classes, such as trees, interval graphs and chordal graphs.

Theorem 1: Both VERTEX COVER RECONFIGURATION and MINMAX VERTEX COVER RECONFIGURATION can be solved in linear time for even-hole-free graphs.

As a proof of Theorem 1, it suffices to give a linear-time algorithm which computes $f^*_G(C_0, C_t)$ for any pair of vertex covers C_0 and C_t of an even-hole-free graph G. Our algorithm employs a nice property of reconfiguration sequences of independent sets, given by Kamiński et al. [12]. We first explain this property in Sect. 3.1, and then give our algorithm in Sect. 3.2.

3.1 Reconfiguration of Independent Sets

Kamiński et al. [12] deeply studied three types of reconfiguration problems on INDEPENDENT SET in a graph. In this subsection, we define and explain only one type used for our algorithm, which they call "Token Addition and Removal (TAR) model" [12].

In the TAR-model, two independent sets I_i and I_j of a graph G are *adjacent* if their symmetric difference $I_i \triangle I_j$ consists of a single vertex u, that is, I_j can be obtained from I_i by either *removing* or *adding* the vertex u. Similarly as for vertex covers, a *reconfiguration sequence* between two independent sets I and I' of G is a sequence $\langle I_1, I_2, \ldots, I_\ell \rangle$ of independent sets of G such that $I_1 = I$, $I_\ell = I'$, and I_{i-1} and I_i are adjacent for $i = 2, 3, \ldots, \ell$. Kamiński et al. [12] gave the following lemma for even-hole-free graphs.

Lemma 1 ([12]): Let I_0 and I_t be any pair of independent sets of an even-hole-free graph G such that $|I_0| = |I_t|$. Then, there exists a reconfiguration sequence $\mathcal{I}$ between I_0 and I_t such that $|I_i| \geq |I_0| - 1$ for all independent sets $I_i \in \mathcal{I}$.

Based on Lemma 1, we give the following lemma. In contrast to Lemma 1 in which $|I_0| = |I_t|$ always holds for two independent sets I_0 and I_t, note that two vertex covers C_0 and C_t in Lemma 2 do not necessarily have the same cardinality.

Lemma 2: Let C_0 and C_t be any pair of vertex covers of an even-hole-free graph G. Then,

$$f^*_G(C_0, C_t) \leq \max\{|C_0|, |C_t|\} + 1.$$

Proof. We may assume without loss of generality that $|C_0| \geq |C_t|$; recall that any reconfiguration sequence is reversible. Then, we construct a vertex cover C'_t of G such that $|C'_t| = |C_0|$ by adding to C_t exactly $|C_0| - |C_t|$ (≥ 0) vertices chosen arbitrarily from $V(G) \setminus C_t$. It suffices to show that there is a reconfiguration sequence $\mathcal{C}$ between C_0 and C'_t such that $f(\mathcal{C}) \leq \max\{|C_0|, |C'_t|\} + 1 = |C_0| + 1$, because C_t can be obtained from C'_t by only deleting the vertices in $C'_t \setminus C_t$; and hence $f^*_G(C_0, C_t) \leq f(\mathcal{C})$.

It is well known that, if a vertex subset C is a vertex cover of a graph G, then $V(G) \setminus C$ forms an independent set of G, and *vice versa*. Let $I_0 = V(G) \setminus C_0$ and $I'_t = V(G) \setminus C'_t$, then they are independent sets of G having the same cardinality. Therefore, by Lemma 1 there is a reconfiguration sequence $\mathcal{I} = \langle I_0, I_1, \ldots, I_\ell \rangle$ between I_0 and $I_\ell = I'_t$ such that $|I_i| \geq |I_0| - 1$ for all independent sets $I_i \in \mathcal{I}$.

Consider the sequence $\mathcal{C} = \langle C_0, C_1, \ldots, C_\ell \rangle$ of vertex covers of G corresponding to $\mathcal{I}$, that is, $C_i = V(G) \setminus I_i$ for each i, $0 \leq i \leq \ell$. Since $|I_{i-1} \triangle I_i| = 1$ holds for every i, $1 \leq i \leq \ell$, two vertex covers C_{i-1} and C_i are adjacent. Therefore, $\mathcal{C}$ is a reconfiguration sequence between C_0 and $C_\ell = C'_t$. Furthermore, since $|I_i| \geq |I_0| - 1$ hold for all $I_i \in \mathcal{I}$, any vertex cover $C_i \in \mathcal{C}$ satisfies

$$|C_i| = |V(G)| - |I_i| \leq |V(G)| - |I_0| + 1 = |C_0| + 1.$$

Thus, $f(\mathcal{C}) = \max\{|C_i| : C_i \in \mathcal{C}\} \leq |C_0| + 1$ holds. □

3.2 Linear-Time Algorithm

We now give our linear-time algorithm. Let C_0 and C_t be two given vertex covers of an even-hole-free graph G. We may assume without loss of generality that $|C_0| \geq |C_t|$. Then, $\max\{|C_0|, |C_t|\} = |C_0|$. Note that Lemma 2 and Eq. (1) imply that $f^*_G(C_0, C_t) \in \{|C_0|, |C_0| + 1\}$.

A vertex cover C of a graph G is said to be *minimal* if there is no vertex $u \in C$ such that $C \setminus \{u\}$ is a vertex cover of G. We can easily check whether a vertex cover of G is minimal or not in linear time. Then, our algorithm is described as follows:

1. **if** C_0 is minimal, then return $f^*_G(C_0, C_t) = |C_0| + 1$;
2. **else if** C_t is minimal, then return
$$f^*_G(C_0, C_t) = \max\{|C_0|, |C_t| + 1\};$$
3. **otherwise** return $f^*_G(C_0, C_t) = |C_0|$.

The algorithm above clearly runs in linear time, because it just checks the minimality of vertex covers C_0 and C_t. Therefore, to complete the proof of Theorem 1, we now prove the correctness of our algorithm.

Case (1): C_0 is a minimal vertex cover of G.

In this case, our algorithm returns $f^*_G(C_0, C_t) = |C_0|+1$. Lemma 2 says that $f^*_G(C_0, C_t) \leq |C_0| + 1$, and hence we will prove that $f^*_G(C_0, C_t) \geq |C_0| + 1$.

Let $C^* = \langle C_0, C^*_1, C^*_2, \ldots, C^*_{\ell-1}, C^*_\ell \rangle$ be an arbitrary reconfiguration sequence between C_0 and $C^*_\ell = C_t$ such that $f(C^*) = f^*_G(C_0, C_t)$. Since C_0 is a minimal vertex cover of G, the vertex cover C^*_1 must be obtained by adding some vertex to C_0. Therefore, we have $|C^*_1| = |C_0| + 1$, and hence $f^*_G(C_0, C_t) = f(C^*) \geq |C^*_1| = |C_0| + 1$, as claimed.

Case (2): C_0 is not minimal, but C_t is minimal.

In this case, our algorithm returns $f^*_G(C_0, C_t) = \max\{|C_0|, |C_t| + 1\}$. We prove this case according to the following two sub-cases.

First, consider the sub-case where $|C_0| = |C_t|$. Then, our algorithm returns $f^*_G(C_0, C_t) = \max\{|C_0|, |C_t|+1\} = |C_t|+1$. Since C_t is minimal, the symmetric argument of Case (1) above proves the correctness of this sub-case.

Second, consider the sub-case where $|C_0| > |C_t|$. Then, $|C_0| \geq |C_t| + 1$, and hence our algorithm returns $f^*_G(C_0, C_t) = |C_0|$. By Eq. (1) it suffices to prove that $f^*_G(C_0, C_t) \leq |C_0|$, that is, there exists a reconfiguration sequence C between C_0 and C_t such that $f(C) = |C_0|$. Since C_0 is not minimal, there exists a vertex $u \in C_0$ such that $C_0 \setminus \{u\}$ forms a vertex cover of G. Let $C_1 = C_0 \setminus \{u\}$, then C_0 and C_1 are adjacent and $|C_1| = |C_0| - 1$. Then, Lemma 2 implies that $f^*_G(C_1, C_t) \leq \max\{|C_1|, |C_t|\} + 1$, and hence there is a reconfiguration sequence $C' = \langle C_1, C_2, \ldots, C_\ell \rangle$ between C_1 and $C_\ell = C_t$ such that $f(C') \leq \max\{|C_1|, |C_t|\} + 1$. Since $|C_1| = |C_0| - 1$ and $|C_0| - 1 \geq |C_t|$, we have

$$f(C') \leq \max\{|C_1|, |C_t|\} + 1 = (|C_0| - 1) + 1 = |C_0|.$$

As our reconfiguration sequence C between C_0 and C_t, we put C_0 in front of C', that is, let $C = \langle C_0, C_1, C_2, \ldots, C_\ell \rangle$. Then, $f(C) = \max\{|C_0|, f(C')\} = |C_0|$, as claimed.

Case (3): neither C_0 nor C_t are minimal.

In this case, our algorithm returns $f^*_G(C_0, C_t) = |C_0|$. By Eq. (1) it suffices to prove that $f^*_G(C_0, C_t) \leq |C_0|$, that is, there exists a reconfiguration sequence C between C_0 and C_t such that $f(C) = |C_0|$.

Since C_0 is not minimal, there exists a vertex cover C_1 of G such that $|C_1| = |C_0| - 1$ and C_0 and C_1 are adjacent. Similarly, since C_t is not minimal, there exists a vertex cover C_{t-1} of G such that $|C_{t-1}| = |C_t| - 1$ and C_{t-1} and C_t are adjacent. Then, Lemma 2 implies that

$$\begin{aligned} f^*_G(C_1, C_{t-1}) &\leq \max\{|C_1|, |C_{t-1}|\} + 1 \\ &= \max\{|C_0|, |C_t|\} \\ &= |C_0|. \end{aligned}$$

Therefore, there is a reconfiguration sequence $C' = \langle C_1, C_2, \ldots, C_{\ell-1} \rangle$ between C_1 and $C_{\ell-1} = C_{t-1}$ such that $f(C') \leq |C_0|$. As our reconfiguration sequence C between C_0 and C_t, let $C = \langle C_0, C_1, C_2, \ldots, C_{\ell-1}, C_t \rangle$. We then have $f(C) = \max\{|C_0|, f(C'), |C_t|\} = |C_0|$, as claimed.

This completes the proof of Theorem 1. □

4. Upper Bound on the Reconfiguration Index

In this section, we give an upper bound on the reconfiguration index $f^*_G(C_0, C_t)$.

4.1 Definitions

For a vertex cover C_i of a graph G and a subgraph G' of G, we denote by $C_{i,G'} = C_i \cap V(G')$ the *restriction* of C_i to G'. Observe that $C_{i,G'}$ is a vertex cover of G', because C_i is a vertex cover of G and G' is a subgraph of G.

Let C_i and C_j be two vertex covers of a graph G, and let D be any vertex subset of $C_i \cap C_j$. Then, we introduce the *reconfiguration index $f^*_G(C_i, C_j; D)$ under the constraint of D*, as follows:

$$f^*_G(C_i, C_j; D) = \min\{f(C_D) : C_D \text{ is a reconfiguration sequence between } C_i \text{ and } C_j \text{ such that } D \subseteq C_k \text{ for all } C_k \in C_D\}. \quad (2)$$

Note that $f^*_G(C_i, C_j; D)$ is well defined for any vertex subset $D \subseteq C_i \cap C_j$; recall the trivial reconfiguration sequence C_D between C_i and C_j via the vertex cover $C_i \cup C_j$ (in Sect. 2), then $D \subseteq C_k$ holds for every $C_k \in C_D$. We clearly have

$$f^*_G(C_i, C_j) = f^*_G(C_i, C_j; \emptyset). \quad (3)$$

Furthermore, we have the following lemma.

Lemma 3: Let C_i and C_j be any pair of vertex covers of a graph G, and let D and D' be any vertex subsets such that $D \subseteq D' \subseteq C_i \cap C_j$. Then, $f^*_G(C_i, C_j; D) \leq f^*_G(C_i, C_j; D')$.

Proof. Let $\mathcal{C}_{D'}$ be any reconfiguration sequence between C_i and C_j such that $f(\mathcal{C}_{D'}) = f^*_G(C_i, C_j; D')$ and $D' \subseteq C_k$ for all vertex covers $C_k \in \mathcal{C}_{D'}$. Since $D \subseteq D'$, we have $D \subseteq C_k$ for all $C_k \in \mathcal{C}_{D'}$. Therefore, $\mathcal{C}_{D'}$ is a reconfiguration sequence such that $D \subseteq C_k$ for all vertex covers $C_k \in \mathcal{C}_{D'}$. By Eq. (2) we thus have $f^*_G(C_i, C_j; D) \leq f(\mathcal{C}_{D'}) = f^*_G(C_i, C_j; D')$, as claimed. □

Lemma 3 and Eq. (3) imply that, for any vertex subset $D \subseteq C_0 \cap C_t$,

$$f^*_G(C_0, C_t) \leq f^*_G(C_0, C_t; D). \tag{4}$$

4.2 Our Upper Bound

We now give our upper bound, whose proof will be given in Sect. 4.3.

Theorem 2: Let α be a fixed integer, and let G be any graph. Suppose that

$$f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) \leq \max\{|C_{0,G''}|, |C_{t,G''}|\} + \alpha$$

for every 2-connected subgraph G'' of G. Then,

$$f^*_G(C_0, C_t; C_0 \cap C_t) \leq \max\{|C_0|, |C_t|\} + \alpha.$$

A graph G is a *cactus* if every edge is part of at most one cycle in G [5]. Using Theorem 2, we give the following corollary.

Corollary 1: Let C_0 and C_t be any pair of vertex covers of a graph G. Then,
(a) $f^*_G(C_0, C_t) \leq \max\{|C_0|, |C_t|\} + 1$ if G is a tree; and
(b) $f^*_G(C_0, C_t) \leq \max\{|C_0|, |C_t|\} + 2$ if G is a cactus.

Proof. (a) Every 2-connected subgraph G'' of a tree G consists of a single edge vw. Then, any vertex cover of G contains at least one of v and w, and hence $\max\{|C_{0,G''}|, |C_{t,G''}|\} \geq 1$. Note that $V(G'') = \{v, w\}$ forms a vertex cover of G'', and hence there always exists a reconfiguration sequence between $C_{0,G''}$ and $C_{t,G''}$ via the vertex cover $V(G'')$. Furthermore, $C_{0,G''} \cap C_{t,G''} \subseteq V(G'')$ holds. Therefore,

$$\begin{aligned} f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) &\leq |V(G'')| \\ &\leq \max\{|C_{0,G''}|, |C_{t,G''}|\} + 1. \end{aligned}$$

Then, Theorem 2 implies that $f^*_G(C_0, C_t; C_0 \cap C_t) \leq \max\{|C_0|, |C_t|\} + 1$. By Eq. (4) we thus have

$$f^*_G(C_0, C_t) \leq f^*_G(C_0, C_t; C_0 \cap C_t) \leq \max\{|C_0|, |C_t|\} + 1$$

if G is a tree.

(b) Every 2-connected subgraph G'' of a cactus G consists of either a single edge or a cycle. Similarly as in (a) above, we have $f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) \leq \max\{|C_{0,G''}|, |C_{t,G''}|\} + 1$ if G'' is a single edge.

We thus consider the case where G'' is a cycle. We choose an arbitrary vertex u in $C_{t,G''}$, and let $P = G[V(G'') \setminus \{u\}]$. Since P is a path (and hence a tree), by the proof of Corollary 1(a) we have

$$f^*_P(C_{0,P}, C_{t,P}; C_{0,P} \cap C_{t,P}) \leq \max\{|C_{0,P}|, |C_{t,P}|\} + 1.$$

Therefore, there is a reconfiguration sequence $\mathcal{C}$ between $C_{0,P}$ and $C_{t,P}$ such that $f(\mathcal{C}) \leq \max\{|C_{0,P}|, |C_{t,P}|\} + 1$ and $C_{0,P} \cap C_{t,P} \subseteq C_i$ for all vertex covers $C_i \in \mathcal{C}$ of P. For each vertex cover $C_i \in \mathcal{C}$ of P, notice that $C_i \cup \{u\}$ forms a vertex cover of G''. Since $C_{0,G''} \cap C_{t,G''} \subseteq (C_{0,P} \cap C_{t,P}) \cup \{u\} \subseteq C_i \cup \{u\}$, we have

$$\begin{aligned} f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) &\leq f(\mathcal{C}) + |\{u\}| \\ &\leq \max\{|C_{0,P}|, |C_{t,P}|\} + 2. \end{aligned}$$

Since $C_{0,P} \subseteq C_{0,G''}$ and $C_{t,P} \subseteq C_{t,G''}$, we have $|C_{0,P}| \leq |C_{0,G''}|$ and $|C_{t,P}| \leq |C_{t,G''}|$. Therefore,

$$f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) \leq \max\{|C_{0,G''}|, |C_{t,G''}|\} + 2.$$

In this way, $f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) \leq \max\{|C_{0,G''}|, |C_{t,G''}|\} + 2$ holds for any 2-connected subgraph G'' of a cactus G. Then, Theorem 2 implies that $f^*_G(C_0, C_t; C_0 \cap C_t) \leq \max\{|C_0|, |C_t|\} + 2$. By Eq. (4) we thus have

$$f^*_G(C_0, C_t) \leq f^*_G(C_0, C_t; C_0 \cap C_t) \leq \max\{|C_0|, |C_t|\} + 2$$

if G is a cactus. □

We note that Corollary 1(a) gives another proof of Lemma 2 for trees. Conversely, Corollary 1(b) cannot be obtained from Lemma 2, because a cactus is not always even-hole free.

Furthermore, we note that our upper bound on $f^*_G(C_0, C_t)$ is best possible in some sense. For example, consider an even-length cycle G and its two vertex covers C_0 and C_t, each of which forms an independent set of G. (See Fig. 2.) Since G is a cycle (and hence a cactus), by Corollary 1(b) we have $f^*_G(C_0, C_t) \leq \max\{|C_0|, |C_t|\} + 2$. Indeed, we have to add at least two vertices to C_0 in order to delete any vertex in C_0. Therefore, $f^*_G(C_0, C_t) = \max\{|C_0|, |C_t|\} + 2$.

4.3 Proof of Theorem 2

In this subsection, as a proof of Theorem 2, we construct a reconfiguration sequence $\mathcal{C}$ between C_0 and C_t such that $f(\mathcal{C}) \leq \max\{|C_0|, |C_t|\} + \alpha$ and $C_0 \cap C_t \subseteq C_i$ for all vertex covers $C_i \in \mathcal{C}$. Then, the theorem follows, because $f^*_G(C_0, C_t; C_0 \cap C_t) \leq f(\mathcal{C})$ holds for such a reconfiguration sequence $\mathcal{C}$.

Roughly speaking, our idea is as follows. We first decompose a graph G into its 2-connected subgraphs, and then

(a) C_0 (b) C_t

Fig. 2 Instance for a cycle G such that $f^*_G(C_0, C_t) = \max\{|C_0|, |C_t|\} + 2$.

separately construct a reconfiguration sequence for each 2-connected subgraph G'' which transforms the vertex cover $C_0 \cap V(G'')$ into the target one $C_t \cap V(G'')$. Of course, we need to extend the reconfiguration sequence for G'' to one for the whole graph G. Furthermore, we need to find a clever ordering of 2-connected subgraphs of G to be transformed so that all intermediate vertex covers C_i of G satisfy $|C_i| \le \max\{|C_0|, |C_t|\} + \alpha$.

Therefore, we first introduce some notions and properties of vertex covers in subgraphs, and then give our reconfiguration sequence.

4.3.1 Notions and Properties

Let C_i and C_j be two vertex covers of a graph G. Then, for a subgraph G' of G, we define the *difference* $\delta(G', C_i, C_j)$ from C_i to C_j, as follows:

$$\delta(G', C_i, C_j) = |C_{j,G'}| - |C_{i,G'}|, \tag{5}$$

that is, the cardinality of the vertex cover of G' is increased by $\delta(G', C_i, C_j)$ if we transform the vertex cover $C_i \cap V(G')$ into $C_j \cap V(G')$. Clearly,

$$\delta(G', C_i, C_j) = -\delta(G', C_j, C_i).$$

Let $G' = (V', E')$ be any induced subgraph of a graph $G = (V, E)$. Let u be an arbitrary cut vertex of G', and suppose that the induced subgraph $G[V' \setminus \{u\}]$ consists of p connected components $G'_1, G'_2, \ldots, G'_p$. Note that $p \ge 2$ since u is a cut vertex of G'. Let $G'_a = (V'_a, E'_a)$ be the connected component in $G[V' \setminus \{u\}]$ such that

$$\delta(G'_a, C_0, C_t) = \min\{\delta(G'_i, C_0, C_t) : 1 \le i \le p\}. \tag{6}$$

Then, the *bipartition* (G'_L, G'_R) *of* G' *with a cut vertex* u is to decompose G into two subgraphs G'_L and G'_R, defined as follows (see Fig. 3):

$$G'_L = \begin{cases} G'_a & \text{if } u \in C_0; \\ G[V'_a \cup \{u\}] & \text{if } u \notin C_0, \end{cases} \tag{7}$$

and $G'_R = G[V' \setminus V(G'_L)]$. Therefore, $V(G'_L)$ and $V(G'_R)$ form a partition of $V(G')$, that is, $V(G'_L) \cup V(G'_R) = V(G')$ and $V(G'_L) \cap V(G'_R) = \emptyset$.

Based on the bipartition (G'_L, G'_R) of a subgraph G' with a cut vertex u, we construct a reconfiguration sequence from $C_{0,G'}$ to $C_{t,G'}$, as follows:

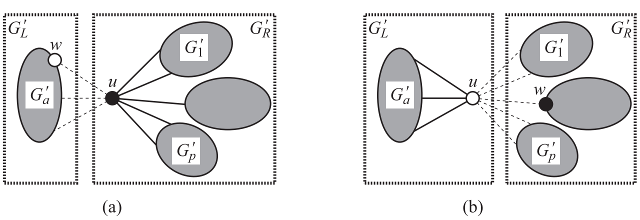

Fig. 3 The bipartition (G'_L, G'_R) of a subgraph G' with a cut vertex u for the cases where (a) $u \in C_0$ and (b) $u \notin C_0$.

(1) transform C_{0,G'_L} into C_{t,G'_L} without adding/deleting any vertex in G'_R; and
(2) transform C_{0,G'_R} into C_{t,G'_R} without adding/deleting any vertex in G'_L.

The following lemma will be used in Lemma 6 for proving that any vertex subset appearing in the reconfiguration sequence above is a vertex cover of G'. For notational convenience, let $C_{i,L} = C_{i,G'_L}$ and $C_{i,R} = C_{i,G'_R}$.

Lemma 4: Let (G'_L, G'_R) be the bipartition of G' with a cut vertex u. Then,
(a) for any vertex cover C'_L of G'_L, the vertex subset $C'_L \cup C_{0,R}$ forms a vertex cover of G'; and
(b) for any vertex cover C'_R of G'_R such that $C_{0,R} \cap C_{t,R} \subseteq C'_R$, the vertex subset $C_{t,L} \cup C'_R$ forms a vertex cover of G'.

Proof. (a) Since $C_{0,R}$ is a vertex cover of G'_R, every edge in $E(G'_R)$ is covered by some vertex in $C_{0,R}$. Similarly, every edge in $E(G'_L)$ is covered by some vertex in C'_L. Therefore, it suffices to show that every edge e in $E(G') \setminus (E(G'_L) \cup E(G'_R))$ is covered by some vertex in $C'_L \cup C_{0,R}$. Note that such an edge e is incident to the cut vertex u; let $e = uw$. (See the thin dotted edges in Fig. 3.)

We first consider the case where $u \in C_0$. Then, by Eq. (7) we have $u \in V(G'_R)$, and hence $u \in C_{0,R} \subseteq C'_L \cup C_{0,R}$. (See Fig. 3(a).) Therefore, e is covered by $u \in C'_L \cup C_{0,R}$.

We then consider the case where $u \notin C_0$. By Eq. (7) we have $u \in V(G'_L)$, and hence $w \in V(G'_R)$. Since $u \notin C_0$ and $C_{0,G'}$ is a vertex cover of G', the other endpoint w must be in C_0. Therefore, $w \in C_{0,R}$, and hence e is covered by $w \in C'_L \cup C_{0,R}$.

In this way, $C'_L \cup C_{0,R}$ forms a vertex cover of G'.

(b) Every edge in $E(G'_L) \cup E(G'_R)$ is covered by some vertex in $C_{t,L} \cup C'_R$ because $C_{t,L}$ and C'_R are vertex covers of G'_L and G'_R, respectively. We thus show that every edge $e = uw$ in $E(G') \setminus (E(G'_L) \cup E(G'_R))$ is covered by some vertex in $C_{t,L} \cup C'_R$.

We first consider the case where $u \in C_0$. (See Fig. 3(a).) Then, $u \in V(G'_R)$ and $w \in V(G'_L)$. If $u \in C_t$, then u is contained in both C_0 and C_t and hence $u \in C_{0,R} \cap C_{t,R}$. By the assumption that $C_{0,R} \cap C_{t,R} \subseteq C'_R$ holds, the edge e is covered by $u \in C_{t,L} \cup C'_R$ if $u \in C_t$. If $u \notin C_t$, then the other endpoint w must be in C_t. Since $w \in V(G'_L)$, we have $w \in C_t \cap V(G'_L) = C_{t,L}$ and hence the edge e is covered by $w \in C_{t,L} \cup C'_R$.

We then consider the case where $u \notin C_0$. (See Fig. 3(b).) Then, $u \in V(G'_L)$ and $w \in V(G'_R)$. If $u \in C_t$, then u is contained in $C_{t,L}$ and hence e is covered by $u \in C_{t,L} \cup C'_R$. If $u \notin C_t$, then w must be in both C_0 and C_t. Therefore, $w \in C_{0,R} \cap C_{t,R}$. By the assumption that $C_{0,R} \cap C_{t,R} \subseteq C'_R$ holds, the edge e is covered by $w \in C_{t,L} \cup C'_R$.

In this way, $C_{t,L} \cup C'_R$ forms a vertex cover of G'. $\square$

Let $C'_q = C_{t,L} \cup C_{0,R}$. Lemma 4(a) implies that C'_q is a vertex cover of G'. Furthermore, $|C'_q| = |C_{t,L}| + |C_{0,R}|$ since $V(G'_L) \cap V(G'_R) = \emptyset$. Then, we give the following lemma.

Lemma 5: The following (a) and (b) hold:

(a) $\delta(G'_R, C_0, C_t) \geq 0$ if $\delta(G'_L, C_0, C_t) \geq 1$; and

(b) $|C'_q| \leq \max\{|C_{0,G'}|, |C_{t,G'}|\}$.

Proof. (a) We first note that

$$-1 \leq \delta(G[\{u\}], C_0, C_t) \leq 1 \tag{8}$$

clearly holds for the cut vertex u, because $G[\{u\}]$ consists only of a single vertex u.

We now consider the case where $u \in C_0$. Then, by Eq. (7) we have $G'_L = G'_a$. By the assumption that $\delta(G'_L, C_0, C_t) \geq 1$ holds, we have

$$\delta(G'_a, C_0, C_t) = \delta(G'_L, C_0, C_t) \geq 1,$$

and hence by Eq. (6)

$$\delta(G'_i, C_0, C_t) \geq \delta(G'_a, C_0, C_t) \geq 1 \tag{9}$$

for any connected component G'_i, $1 \leq i \leq p$, in $G[V' \setminus \{u\}]$. Since $G'_L = G'_a$,

$$V(G'_R) = V' \setminus V(G'_L) = \{u\} \cup \bigcup_{1 \leq i \leq p, i \neq a} V(G'_i),$$

as illustrated in Fig. 3(a). Then, Eqs. (8) and (9) imply that

$$\begin{aligned}\delta(G'_R, C_0, C_t) &= \delta(G[\{u\}], C_0, C_t) + \sum_{1 \leq i \leq p, i \neq a} \delta(G'_i, C_0, C_t) \\ &\geq (-1) + (p-1) \cdot 1 \\ &= p - 2.\end{aligned}$$

Recall that $p \geq 2$ since u is a cut vertex of G'. We thus have $\delta(G'_R, C_0, C_t) \geq 0$.

We then consider the case where $u \notin C_0$. By Eq. (7) we have $G'_L = G[V'_a \cup \{u\}]$, and hence

$$\delta(G'_L, C_0, C_t) = \delta(G'_a, C_0, C_t) + \delta(G[\{u\}], C_0, C_t).$$

Then, by the assumption of the lemma and Eq. (8) we have

$$\begin{aligned}\delta(G'_a, C_0, C_t) &= \delta(G'_L, C_0, C_t) - \delta(G[\{u\}], C_0, C_t) \\ &\geq 1 - 1 \\ &= 0.\end{aligned}$$

Therefore, Eq. (6) implies that $\delta(G'_i, C_0, C_t) \geq 0$ for any connected component G'_i, $1 \leq i \leq p$, in $G[V' \setminus \{u\}]$. Since

$$V(G'_R) = V' \setminus V(G'_L) = \bigcup_{1 \leq i \leq p, i \neq a} V(G'_i)$$

as illustrated in Fig. 3(b), we have

$$\delta(G'_R, C_0, C_t) = \sum_{1 \leq i \leq p, i \neq a} \delta(G'_i, C_0, C_t) \geq 0.$$

(b) Suppose for a contradiction that $|C'_q| > \max\{|C_{0,G'}|, |C_{t,G'}|\}$. Then, $|C'_q| > \max\{|C_{0,G'}|, |C_{t,G'}|\} \geq |C_{0,G'}|$, and hence we have $|C'_q| - |C_{0,G'}| \geq 1$. Since $V(G'_L) \cap V(G'_R) = \emptyset$, by Eq. (5) we thus have

$$\begin{aligned}\delta(G'_L, C_0, C_t) &= |C_{t,L}| - |C_{0,L}| \\ &= (|C_{t,L}| + |C_{0,R}|) - (|C_{0,L}| + |C_{0,R}|) \\ &= |C'_q| - |C_{0,G'}| \\ &\geq 1.\end{aligned} \tag{10}$$

Since $|C'_q| > \max\{|C_{0,G'}|, |C_{t,G'}|\}$, we also have $|C'_q| > |C_{t,G'}|$. Therefore, we have

$$\begin{aligned}\delta(G'_R, C_0, C_t) &= |C_{t,R}| - |C_{0,R}| \\ &= (|C_{t,L}| + |C_{t,R}|) - (|C_{t,L}| + |C_{0,R}|) \\ &= |C_{t,G'}| - |C'_q| \\ &< 0.\end{aligned} \tag{11}$$

In this way, both Eqs. (10) and (11) hold; this contradicts Lemma 5(a). □

4.3.2 Reconfiguration Sequence

We now give our reconfiguration sequence between C_0 and C_t of a graph G, based on a *decomposition tree* T of G which is recursively defined as follows:

(A) the root r of T corresponds to the whole graph G; and

(B) if there is a cut vertex u in the subgraph G' corresponding to a node v of T, then v has two children v_L and v_R in T which correspond to the subgraphs G'_L and G'_R, respectively, where (G'_L, G'_R) is the bipartition of G' with u.

Then, each leaf of T corresponds to a 2-connected subgraph of G.

We now prove the key lemma.

Lemma 6: Let α be a fixed integer, and T be a decomposition tree of a graph G. For every 2-connected subgraph G'' of G, suppose that

$$f^*_{G''}(C_{0,G''}, C_{t,G''}; C_{0,G''} \cap C_{t,G''}) \leq \max\{|C_{0,G''}|, |C_{t,G''}|\} + \alpha. \tag{12}$$

Then, for the subgraph G' corresponding to each node v of T, there is a reconfiguration sequence $\mathcal{C}' = \langle C'_0, C'_1, \ldots, C'_\ell \rangle$ such that

(a) $C'_0 = C_{0,G'}$ and $C'_\ell = C_{t,G'}$;

(b) $C_{0,G'} \cap C_{t,G'} \subseteq C'_i$ for all vertex covers $C'_i \in \mathcal{C}'$; and

(c) $f(\mathcal{C}') \leq \max\{|C_{0,G'}|, |C_{t,G'}|\} + \alpha$.

Proof. We prove the lemma by induction based on the decomposition tree T.

Base step.

Suppose that v is a leaf of T, and let G' be the subgraph corresponding to v. Then, G' is 2-connected, and hence Eq. (12) holds for G'. Therefore, Eqs. (2) and (12) imply that there exists a reconfiguration sequence between $C_{0,G'}$ and $C_{t,G'}$ satisfying all the three conditions (a)–(c).

Inductive step.

Let v be an internal node of T having two children v_L

and v_R. Let G', G'_L and G'_R be the subgraphs corresponding to v, v_L and v_R, respectively, and hence (G'_L, G'_R) is a bipartition of G'. Suppose that the lemma holds for G'_L and G'_R. Then, G'_L has a reconfiguration sequence $\mathcal{C}^L = \langle C^L_0, C^L_1, \ldots, C^L_{\ell_L}\rangle$ such that

(a-L) $C^L_0 = C_{0,L}$ and $C^L_{\ell_L} = C_{t,L}$;
(b-L) $C_{0,L} \cap C_{t,L} \subseteq C^L_i$ for all vertex covers $C^L_i \in \mathcal{C}^L$; and
(c-L) $f(\mathcal{C}^L) \le \max\{|C_{0,L}|, |C_{t,L}|\} + \alpha$.

Similarly, G'_R has a reconfiguration sequence $\mathcal{C}^R = \langle C^R_0, C^R_1, \ldots, C^R_{\ell_R}\rangle$ such that

(a-R) $C^R_0 = C_{0,R}$ and $C^R_{\ell_R} = C_{t,R}$;
(b-R) $C_{0,R} \cap C_{t,R} \subseteq C^R_j$ for all vertex covers $C^R_j \in \mathcal{C}^R$; and
(c-R) $f(\mathcal{C}^R) \le \max\{|C_{0,R}|, |C_{t,R}|\} + \alpha$.

From the induction hypothesis above, we now construct a sequence $\mathcal{C}' = \langle C'_0, C'_1, \ldots, C'_\ell\rangle$ of vertex subsets of G', where $\ell = \ell_L + \ell_R$, as follows:

(i) $C'_i = C^L_i \cup C_{0,R}$ for all i, $0 \le i \le \ell_L$; and
(ii) $C'_i = C_{t,L} \cup C^R_{i-\ell_L}$ for all i, $\ell_L < i \le \ell_L + \ell_R = \ell$.

Then, $C'_{\ell_L} = C_{t,L} \cup C_{0,R}$. In the following, we will show that $\mathcal{C}'$ is a reconfiguration sequence for G' satisfying all the three conditions (a)–(c). Let $\mathcal{C}'_{0,\ell_L} = \langle C'_0, C'_1, \ldots, C'_{\ell_L}\rangle$ and $\mathcal{C}'_{\ell_L,\ell} = \langle C'_{\ell_L}, C'_{\ell_L+1}, \ldots, C'_{\ell_L+\ell_R}\rangle$. Note that, for notational convenience, C'_{ℓ_L} appears in both $\mathcal{C}'_{0,\ell_L}$ and $\mathcal{C}'_{\ell_L,\ell}$.

We first show that $\mathcal{C}'$ satisfies the condition (a). By the construction (i) above and the condition (a-L), we have $C'_0 = C^L_0 \cup C_{0,R} = C_{0,L} \cup C_{0,R} = C_{0,G'}$, as required. Similarly, by the construction (ii) above and the condition (a-R), we have $C'_\ell = C_{t,L} \cup C^R_{\ell_R} = C_{t,L} \cup C_{t,R} = C_{t,G'}$. Thus, $\mathcal{C}'$ satisfies the condition (a).

Before showing the conditions (b) and (c), we now prove that $\mathcal{C}'$ is a reconfiguration sequence between $C_{0,G'}$ and $C_{t,G'}$. It suffices to show that $\mathcal{C}'_{0,\ell_L}$ is a reconfiguration sequence from $C_{0,G'} = C_{0,L} \cup C_{0,R}$ to $C'_{\ell_L} = C_{t,L} \cup C_{0,R}$, and that $\mathcal{C}'_{\ell_L,\ell}$ is a reconfiguration sequence from $C'_{\ell_L} = C_{t,L} \cup C_{0,R}$ to $C_{t,G'} = C_{t,L} \cup C_{t,R}$.

Recall that $\mathcal{C}^L = \langle C^L_0, C^L_1, \ldots, C^L_{\ell_L}\rangle$ is a reconfiguration sequence between $C^L_0 = C_{0,L}$ and $C^L_{\ell_L} = C_{t,L}$, and hence each $C^L_i \in \mathcal{C}^L$ is a vertex cover of G'_L. Since $C'_i = C^L_i \cup C_{0,R}$ for each vertex subset $C'_i \in \mathcal{C}'_{0,\ell_L}$, Lemma 4(a) implies that C'_i is a vertex cover of G'. Therefore, the sequence $\mathcal{C}'_{0,\ell_L}$ is a reconfiguration sequence from $C_{0,G'} = C_{0,L} \cup C_{0,R}$ to $C'_{\ell_L} = C_{t,L} \cup C_{0,R}$.

Recall also that $\mathcal{C}^R = \langle C^R_0, C^R_1, \ldots, C^R_{\ell_R}\rangle$ is a reconfiguration sequence between $C^R_0 = C_{0,R}$ and $C^R_{\ell_R} = C_{t,R}$, and hence each $C^R_j \in \mathcal{C}^R$ is a vertex cover of G'_R. Furthermore, by the condition (b-R) we have $C_{0,R} \cap C_{t,R} \subseteq C^R_j$ for all $C^R_j \in \mathcal{C}^R$. Since $C'_i = C_{t,L} \cup C^R_{i-\ell_L}$ for each vertex subset $C'_i \in \mathcal{C}'_{\ell_L,\ell}$, Lemma 4(b) implies that C'_i is a vertex cover of G'. Therefore, the sequence $\mathcal{C}'_{\ell_L,\ell}$ is a reconfiguration sequence from $C'_{\ell_L} = C_{t,L} \cup C_{0,R}$ to $C_{t,L} \cup C_{t,R} = C_{t,G'}$.

In this way, $\mathcal{C}'$ is a reconfiguration sequence between $C_{0,G'}$ and $C_{t,G'}$.

We then show that $\mathcal{C}'$ satisfies the condition (b). Since $V(G'_L) \cap V(G'_R) = \emptyset$, we have

$$C_{0,G'} \cap C_{t,G'} = (C_{0,L} \cap C_{t,L}) \cup (C_{0,R} \cap C_{t,R}).$$

By the condition (b-L), we have $C_{0,L} \cap C_{t,L} \subseteq C^L_i$ for all i, $0 \le i \le \ell_L$. Therefore, for all vertex covers C'_i in $\mathcal{C}'_{0,\ell_L}$, by the construction (i) above we have

$$\begin{aligned} C_{0,G'} \cap C_{t,G'} &= (C_{0,L} \cap C_{t,L}) \cup (C_{0,R} \cap C_{t,R}) \\ &\subseteq C^L_i \cup C_{0,R} \\ &= C'_i, \end{aligned}$$

as required. Similarly, by the condition (b-R), we have $C_{0,R} \cap C_{t,R} \subseteq C^R_{i-\ell_L}$ for all i, $\ell_L \le i \le \ell_L + \ell_R$. Therefore, for all vertex covers C'_i in $\mathcal{C}'_{\ell_L,\ell}$, by the construction (ii) above we have

$$\begin{aligned} C_{0,G'} \cap C_{t,G'} &= (C_{0,L} \cap C_{t,L}) \cup (C_{0,R} \cap C_{t,R}) \\ &\subseteq C_{t,L} \cup C^R_{i-\ell_L} \\ &= C'_i, \end{aligned}$$

as required. In this way, $C_{0,G'} \cap C_{t,G'} \subseteq C'_i$ for all vertex covers $C'_i \in \mathcal{C}'$, and hence $\mathcal{C}'$ satisfies the condition (b).

We finally prove that $\mathcal{C}'$ satisfies the condition (c). Notice that

$$f(\mathcal{C}') = \max\{f(\mathcal{C}'_{0,\ell_L}), f(\mathcal{C}'_{\ell_L,\ell})\}. \tag{13}$$

Recall that $V(G'_L) \cap V(G'_R) = \emptyset$. Then, by the construction (i) above and the condition (c-L), we have

$$\begin{aligned} f(\mathcal{C}'_{0,\ell_L}) &= f(\mathcal{C}^L) + |C_{0,R}| \\ &\le \max\{|C_{0,L}|, |C_{t,L}|\} + \alpha + |C_{0,R}| \\ &= \max\{|C_{0,L}| + |C_{0,R}|, |C_{t,L}| + |C_{0,R}|\} + \alpha \\ &= \max\{|C_{0,G'}|, |C'_{\ell_L}|\} + \alpha. \end{aligned}$$

Since $C'_{\ell_L} = C_{t,L} \cup C_{0,R}$, by Lemma 5(b) we thus have

$$f(\mathcal{C}'_{0,\ell_L}) \le \max\{|C_{0,G'}|, |C_{t,G'}|\} + \alpha. \tag{14}$$

Similarly, by the construction (ii) above and the condition (c-R), we have

$$\begin{aligned} f(\mathcal{C}'_{\ell_L,\ell}) &= |C_{t,L}| + f(\mathcal{C}^R) \\ &\le |C_{t,L}| + \max\{|C_{0,R}|, |C_{t,R}|\} + \alpha \\ &= \max\{|C_{t,L}| + |C_{0,R}|, |C_{t,L}| + |C_{t,R}|\} + \alpha \\ &= \max\{|C'_{\ell_L}|, |C_{t,G'}|\} + \alpha. \end{aligned}$$

Therefore, by Lemma 5(b) we have

$$f(\mathcal{C}'_{\ell_L,\ell}) \le \max\{|C_{0,G'}|, |C_{t,G'}|\} + \alpha. \tag{15}$$

Equations (13), (14) and (15) prove that $\mathcal{C}'$ satisfies the condition (c). □

4.3.3 Proof of Theorem 2

Recall that the root r of a decomposition tree T of a graph G corresponds to the whole graph G. Therefore, by Lemma 6 there exists a reconfiguration sequence $\mathcal{C}$ between $C_{0,G} = C_0$

and $C_{t,G} = C_t$ such that $C_{0,G} \cap C_{t,G} \subseteq C_i$ for all vertex covers $C_i \in \mathcal{C}$ and $f(\mathcal{C}) \leq \max\{|C_{0,G}|, |C_{t,G}|\} + \alpha$. By Eq. (2) we thus have

$$f^*_G(C_0, C_t; C_0 \cap C_t) \leq f(\mathcal{C}) \leq \max\{|C_0|, |C_t|\} + \alpha,$$

as required. This completes the proof of Theorem 2. □

5. Concluding Remarks

In this paper, we gave algorithmic results for the two reconfiguration problems on VERTEX COVER. We note again that our upper bound on the reconfiguration index gives an approximation algorithm with absolute performance guarantee.

Recently, Mouawad et al. [13] proposed a linear-time algorithm to solve VERTEX COVER RECONFIGURATION for even-hole-free graphs and cacti. Their proof method is different from ours, and hence both results can coexist. As one of the interesting points of our paper, we proved that the reconfiguration index of a whole graph can be bounded only by the local computation, that is, the reconfiguration index of each 2-connected subgraph; this fact suggests that 2-connected subgraphs are essential for the problem.

In addition, it has been proved recently that the reconfiguration problem on INDEPENDENT SET under the TAR-model can be solved in polynomial time for cographs [1], [3]. Thus, VERTEX COVER RECONFIGURATION is solvable in polynomial time for cographs. Note that the classes of even-hole-free graphs and cographs are non-comparable with each other.

It remains open to obtain an upper bound on the reconfiguration index in terms of the cardinality of separators in a graph. (Our upper bound can be seen as the case where the separator is of cardinality one.) This would help a better understanding of the reconfiguration index.

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