LETTER

Finding the Minimum Number of Open-Edge Guards in an Orthogonal Polygon is NP-Hard

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SUMMARY We study the problem of determining the minimum number of open-edge guards which guard the interior of a given orthogonal polygon with holes. Here, an open-edge guard is a guard which is allowed to be placed along open edges of a polygon, that is, the endpoints of the edge are not taken into account for visibility purpose. It is shown that finding the minimum number of open-edge guards for a given orthogonal polygon with holes is NP-hard.
key words: *open-edge guards, orthogonal polygons, art gallery problem*

1. Introduction

The art gallery problem is to determine the minimum number of guards who can observe the interior of a gallery. Chvátal [2] proved that $\lfloor n/3 \rfloor$ guards are lower and upper bounds for this problem; namely, $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary for observing the interior of an n-vertex simple polygon. This $\lfloor n/3 \rfloor$-bound is replaced by $\lfloor n/4 \rfloor$ if the instance is restricted to a simple orthogonal polygon [4].

Another approach to the art gallery problem is to study the complexity of locating the minimum number of guards in a polygon. The NP-hardness of this problem was shown by Lee and Lin [6]. Furthermore, Schuchardt and Hecker [9] proved that this problem remains NP-hard if we restrict our attention to simple orthogonal polygons. Even guarding the vertices of a simple orthogonal polygon is NP-hard [5].

An *edge guard* is a guard that is only allowed to be placed on the edges of a polygon, and the edge guard can move between the endpoints of the edge (see Fig. 1 (b)). For the edge guarding problem for n-vertex polygons, it is known that $\lfloor n/4 \rfloor$ and $\lfloor 3n/10 \rfloor$ are lower and upper bounds, respectively [10], and the minimum edge-guarding problem is NP-hard [6].

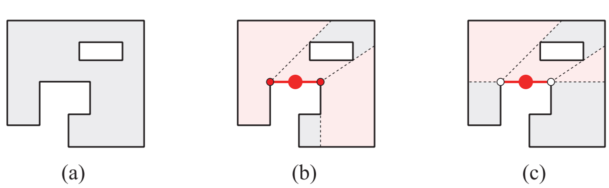

Fig. 1 (a) An orthogonal polygon with holes. (b) A (closed) edge guard. (c) An open-edge guard.

An *open-edge guard* is an edge guard such that the endpoints of the edge are not taken into account for visibility purpose (see Fig. 1 (c)). The motivation of the open-edge guarding problem is given in [12], where the problem is informally stated as "How many windows should we place on the walls of a dark building to illuminate its interior?"

Tóth, Toussaint, and Winslow proved that $\lfloor n/3 \rfloor$ and $\lfloor n/2 \rfloor$ are lower and upper bounds of the minimum number of open-edge guards, respectively [11]. They also conjectured that $\lfloor n/3 \rfloor$ is both a lower and upper bound.

In this paper, we investigate the complexity of the open-edge guarding problem. It is shown that finding the minimum number of open-edge guards for a given orthogonal polygon with holes is NP-hard.

2. Definitions and Results

The definitions of a polygon and a polygon with holes are mostly from [7], [8]. A *polygon* is defined by a finite set of segments such that every segment extreme is shared by exactly two edges and no subset of edges has the same property. The segments are the *edges* and their endpoints are the *vertices* of the polygon. If each edge of a polygon is perpendicular to one of the coordinate axes, then the polygon is called *orthogonal*. Thus, an orthogonal polygon is a polygon all of whose edge intersections are at right angles. An orthogonal polygon is sometimes called a *rectilinear polygon*.

A *polygon with holes* is a polygon P enclosing several other polygons $H_1, H_2, \ldots, H_t$, the holes. None of the boundaries of $P, H_1, H_2, \ldots, H_t$ may intersect, and each of the holes is empty. P is said to bound a *multiply-connected region* with t holes: the region of the plane interior to or on the boundary of P, but exterior to or on the boundary of $H_1, H_2, \ldots, H_t$. (Such a region is called a "polygon P with holes" or simply "polygon P" in the rest of this paper.) Similarly, we define an *orthogonal polygon with holes* to be an orthogonal polygon with orthogonal holes, with all edges aligned with the same pair of orthogonal axes (see Fig. 1 (a)).

Given a polygon P, two points x and y are said to be *visible* if the line segment xy does not contain any points outside the polygon P. A point x in the polygon P is said to be *visible from an edge* if there exists an *interior point* y of the edge such that x and y are visible. Here, an interior point of an edge is a point which is on the edge but is not on the

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endpoints of the edge. A set of edges is said to *guard* the polygon P if every point in P is visible from at least one of these edges.

An instance of the *open-edge guarding problem* in this paper is an orthogonal polygon with holes and a positive integer g. The problem asks whether there exists a set of g edges which guards the region of the orthogonal polygon with holes.

Theorem 1: *The open-edge guarding problem for orthogonal polygons with holes is NP-hard.*

3. Proof of Theorem 1

In Sect. 3.2, we will show a polynomial-time transformation from an arbitrary instance C of the 3SAT problem to a polygon with holes and integer g such that C is satisfiable if and only if there exists a set of g edges which guards the region of the polygon with holes.

3.1 3SAT Problem

The definition of 3SAT is mostly from [LO1] of [3]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over U such that $|c_j| \leq 3$ for each $c_j \in C$. The 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. (An example of C is given in the caption of Fig. 8.) It is known that the 3SAT problem is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in C [1].

3.2 Transformation from 3SAT-Instance to Polygon

Figure 2 is an outline drawing of orthogonal polygon P of width $3w$. Each of the left and right parts of width w has a comb-like structure. The top side of the center part of width w has m projections labeled with $c_1, c_2, \ldots, c_m$. (See also the detailed drawing of c_i and c_j in Fig. 3.)

Each comb has $2m$ teeth, and thick and thin teeth appear alternately. One can see that the red area of Fig. 2 can be guarded by $2m$ edges, indicated by red nodes of thick teeth. The width w of the comb-like part is sufficiently large so that (i) at least $2m$ edges are required for guarding thin teeth, and (ii) those $2m$ edges cannot guard any projection c_i. Let d be the thickness of thin teeth (see Fig. 2). In the figure, the thickness of thick teeth is $4d$, the depth of each projection is d, and the width of the comb-like part is $w = 32d$. If the value of d is determined so that the width p of projections c_1 and c_m is less than the value of $4d$, then conditions (i) and (ii) are satisfied.

Figure 3 is a gadget for variable x_h, which appears in clause c_i positively and in c_j and c_k negatively in C. The detailed drawing of the variable gadget for x_h is shown in Figs. 4, 5, and 6.

In order to see the red area of Fig. 6, at least one guard must be placed on an edge which touches that red area (see the edge indicated by blue node s of Fig. 4). By the same reason, at least nine guards must be placed in the variable gadget x_h in Fig. 4 (see the nine blue edges indicated by blue nodes in that figure). Those nine edges see the left and right rectangular *halls* A and B excepting two small white triangular areas near the green nodes a and b.

The red area of Fig. 4 is a *connecting corridor* between two halls A and B. It is not difficult to show that at least two edges touching the red area are required to guard all points of the connecting corridor. If there is a guard on the edge indicated by green node a (resp. green circle b) of Fig. 4, then small white triangular area near b (resp. a) is guarded by one of the two green edges in the corridor indicated by

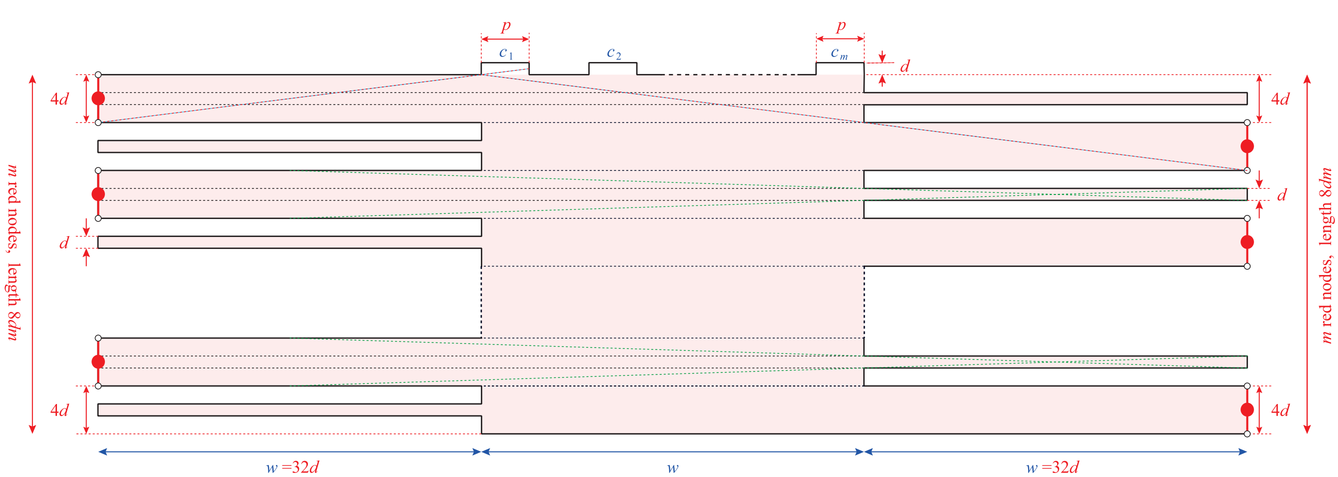

Fig. 2 An outline drawing of orthogonal polygon P of width $3w$. The center part of width w has m projections labeled with $c_1, c_2, \ldots, c_m$.

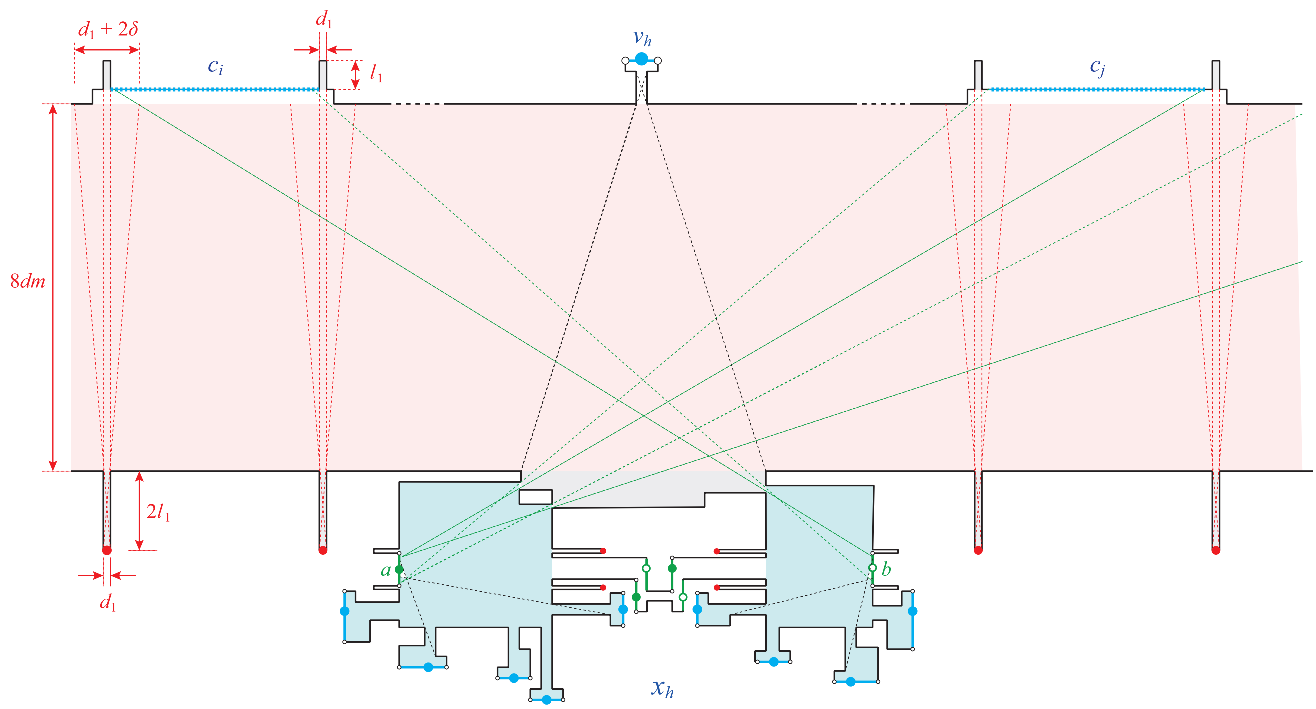

Fig. 3 Variable gadget for x_h. Here, variable x_h appears in clause c_i positively and in c_j and c_k negatively in C. (In the figure, c_k is omitted due to space limitations.)

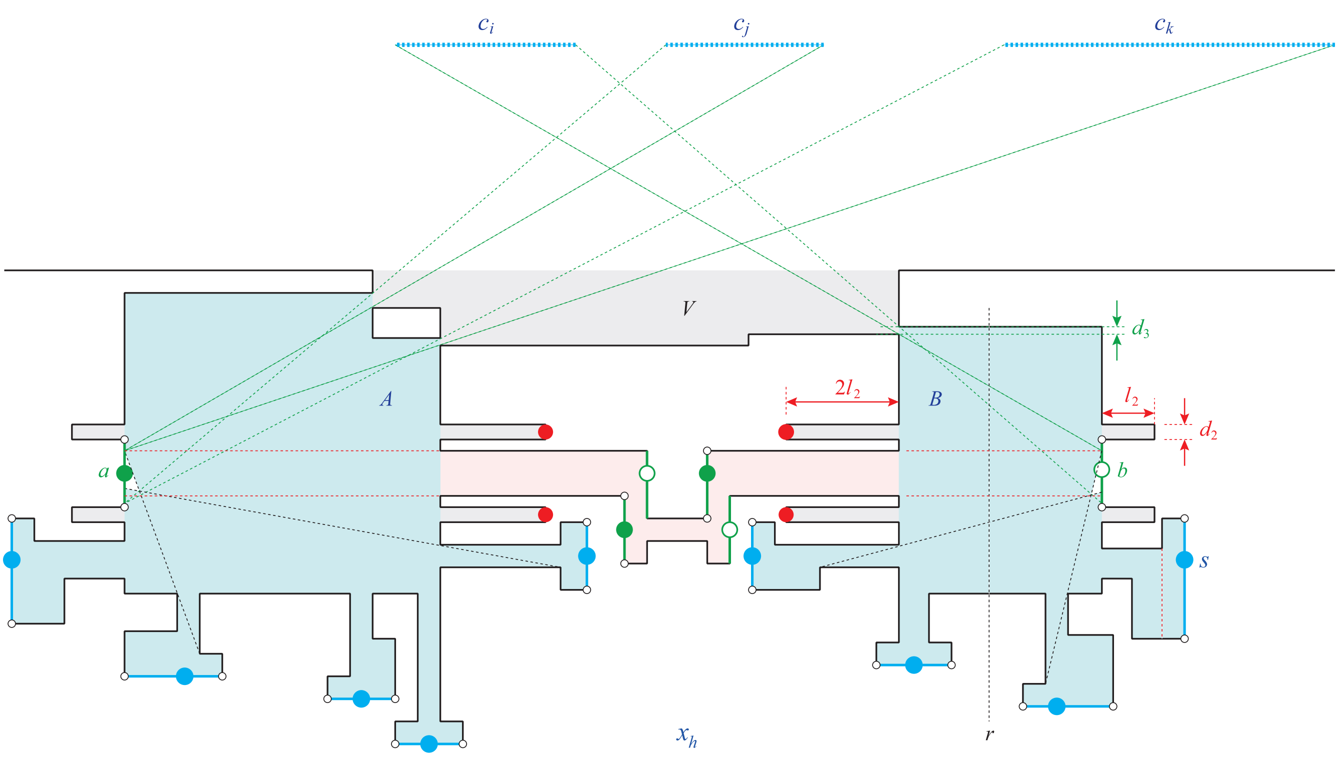

Fig. 4 Detailed drawing of the variable gadget for x_h. A simplified illustration is given in Fig. 9.

two green nodes (resp. two green circles). Three edges indicated by green nodes (resp. green circles) correspond to the assignment $\overline{x_h} = 1$ (resp. $x_h = 1$).

In Fig. 3, there are four pairs of narrow *rectangular portions* of size $l_1 \times d_1$ and $2l_1 \times d_1$. Those portions can be guarded by four edges (see the four red nodes of rectangular portions of size $2l_1 \times d_1$). Here, we assume that width d_1 is sufficiently small as compared with length l_1 so that the

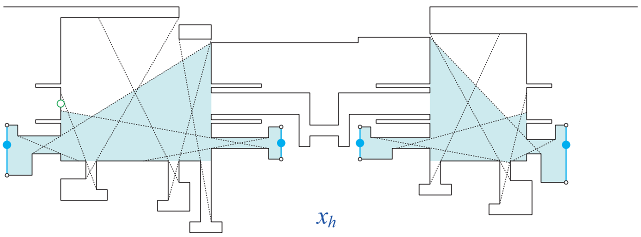

Fig. 5 Visible areas from four vertical edges indicated by blue nodes.

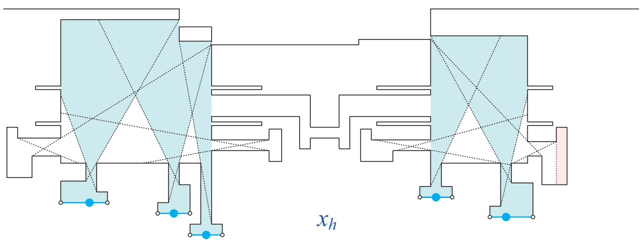

Fig. 6 Visible areas from five horizontal edges indicated by blue nodes.

value $d_1 + 2\delta$ in Fig. 3 is almost equivalent to d_1. Namely, for an arbitrary small constant $\delta > 0$, the value d_1 is determined so that $\frac{d_1}{2l_1} \leq \frac{\delta}{8dm}$ holds (i.e., $d_1 \leq \frac{\delta l_1}{4dm}$), where $8dm$ is the height of the red area of Figs. 2 and 3. One can see four edges are sufficient and necessary to guard the four pairs of rectangular portions.

Similar rectangular portions of size $l_2 \times d_2$ and $2l_2 \times d_2$ can be found in Fig. 4; four edges are sufficient and necessary for guarding such portions (see the four edges indicated by four red nodes of Fig. 4). Here, width d_2 is sufficiently small as compared with l_2. (The exact value of d_2 can be computed by a manner similar to the previous paragraph.)

Consider a "hammer-shaped" gadget labeled with v_h of Fig. 3. This gadget is for guarding the grey area V of Fig. 4. Obviously, one blue edge is sufficient and necessary for guarding both grey area V and the inside of the hammer.

Consider projections labeled with c_i, c_j, and c_k of Figs. 3 and 4. Edges a and b of Fig. 4 can see edges labeled with c_j, c_k and c_i, respectively. (Recall that variable x_h appears in clauses c_j and c_k negatively and in c_i positively.)

The position of blue dotted segment c_i of Fig. 4 can be controlled by moving all the 26 vertices placed on the right side of dotted line r, to the right or to the left. The length of the blue dotted segment c_i can be controlled by changing the gap d_3. The position and length of segments c_j and c_k can be controlled similarly. If the movement of those 26 vertices causes a collision with a $2l_1 \times d_1$ rectangular portion in Fig. 3, then we reconstruct a variable gadget so that halls A and B have a free space for the movement as shown in Fig. 7. Figure 7 (a) is a simplified illustration of hall B of Fig. 4. We reconstruct the variable gadget so that the size of halls is reduced as shown in Fig. 7 (b). Note that two green visibility lines in Fig. 7 (a) are coincident with those in Fig. 7 (b). Furthermore, in Fig. 7 (a), a hammer-shaped portion (illustrated by dotted lines) can be replaced with smaller one such that both of them have the same visibility line.

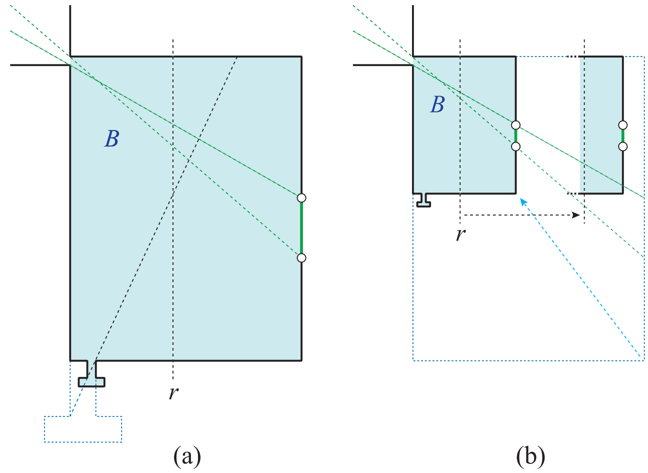

Fig. 7 (a) Simplified figure of hall B. Large and small hammer-shaped portions have the same visibility line. (b) Small-sized hall, which has the same pair of green visibility lines as (a).

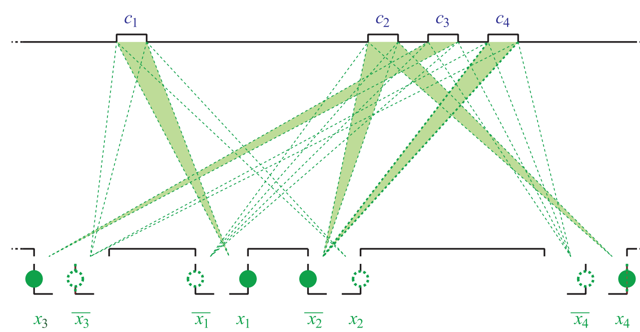

Fig. 8 Orthogonal polygon P transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_3, \overline{x_4}\}$, and $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. From this figure, one can see that there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfying all clauses. (In this figure, the detailed drawing of Fig. 4 is illustrated as a simplified figure of Fig. 9.)

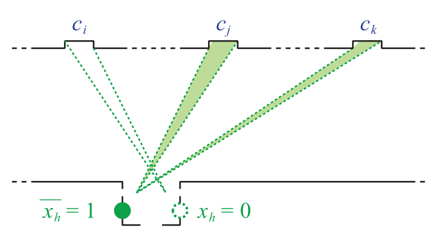

Fig. 9 Simplified illustration of Fig. 4.

Figure 8 is an orthogonal polygon P (with holes) transformed from a 3SAT-instance C, where C is given in the caption of Fig. 8. In this figure, the detailed drawing of Fig. 4 is illustrated as a simplified figure of Fig. 9. Figure 10 is the variable gadget for x_3. The variable gadget for x_4 is a mirror image of this figure. From Fig. 8, one can see that there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfying all clauses.

In Fig. 3, there are 2 red nodes for every clause $c_i \in \{c_1, c_2, \ldots, c_m\}$; 10 blue nodes, 4 red nodes, and 3 green nodes for every variable $x_h \in \{x_1, x_2, \ldots, x_n\}$. In Fig. 2, there

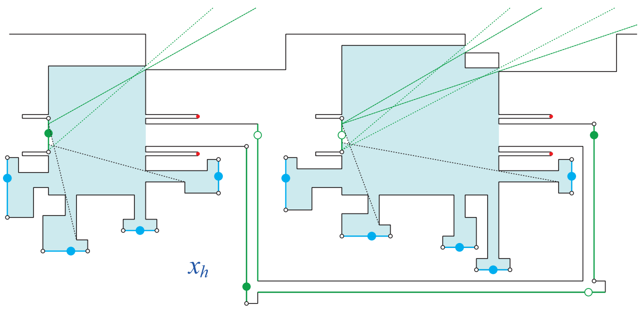

Fig. 10 Variable gadget for x_3 in Fig. 8. Variable gadget for x_4 is a mirror image of this figure.

are $2m$ red nodes. Hence, let $g = 2m + (10 + 4 + 3)n + 2m = 4m + 17n$. From this construction of polygon P with holes, one can see that C is satisfiable if and only if there exists a set of g edges which guards polygon P with holes.

Acknowledgments

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