# LETTER Performance Evaluation of Finite Sparse Signals for Compressed Sensing Frameworks\*

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**SUMMARY** In this paper, we consider to develop a recovery algorithm of a sparse signal for a compressed sensing (CS) framework over finite fields. A basic framework of CS for discrete signals rather than continuous signals is established from the linear measurement step to the reconstruction. With predetermined priori distribution of a sparse signal, we reconstruct it by using a message passing algorithm, and evaluate the performance obtained from simulation. We compare our simulation results with the theoretic bounds obtained from probability analysis.

key words: compressed sensing, finite fields, signal recovery, probabilistic decoding

## 1. Introduction

Over the past few years compressed sensing (CS) theory has attracted much interest in signal processing and information theory. A sparse signal in a certain domain is recovered from a small number of linear measurements [1]. The reconstruction of a sparse signal is performed through an optimization. For sparse signals with discrete values, e.g., bit streams for storage and pixel images, the recovery algorithms developed for the real-valued system of CS are less sufficient as they cannot effectively exploit the digitized nature of the source. This motivates us to design a recovery algorithm of a sparse signal for CS over finite fields.

The use of CS frameworks from linear codes, e.g., Low-Density Parity-Check (LDPC) codes [2], is an emerging approach as promising since it allows us to provide several improvements over the conventional CS. Although meaningful results of CS over finite fields have been currently shown, almost CS works were studied on based realvalued system. In other words, while discretization of realvalued signals for sensing and measurement results in loss of accuracy, performing operations over finite fields overcomes this drawback. For CS framework over finite fields, recovery bounds on sparse signals have been presented in [3] and [4]. The authors in [4] where considers CS framework over finite fields has shown theoretical results using error exponent technique. The authors derived probability of error using random sensing matrices. In [3], the authors showed theoretic bounds on reconstruction of sparse signals over finite fields. The sufficient and necessary conditions on perfect reconstruction for CS frameworks over finite fields have been shown in [3], where the theoretical recovery bounds over dense and sparse sensing matrices using a  $L_0$  norm minimization are coincided each other. Rather than the studies of [3] and [4] have mostly theoretical results, the authors in [5] have presented to exploit parity-check matrices, and linear decoding based on discrete-valued images. In [6], the authors have developed F<sup>2</sup>OMP (Finite Field OMP) as a recovery algorithm for images of CS over finite fields which it was utilized by the classical OMP. In [7], the authors have proposed a sparse recovery framework with sparse random network transfer matrices over finite fields to solve the network coding problems. In addition, the work of [8] has proposed to solve a discrete reconstruction problem as a  $L_1$  optimization by minimizing the sum of weighted absolute values. And this study has been extended to a symbol detection problem [9] and a discrete-valued control design [10]. Since the recovery algorithm proposed in this paper is a variant of the sum-product algorithm for LDPC codes, its complexity is  $O(Nq \log q)$  [15] for the length N of a signal and the size q of finite field. The linear programming for  $L_1$  optimization used in [8] has the complexity with the order of  $O(N^3)$ [14]. This is a major advantage of our work compared to the work [8].

In this paper, one measured sample is obtained by the inner product of a row of the so-called sensing matrix and a sparse signal. From some measured samples, the sparse signal can be reconstructed by using a CS recovery algorithm. This has a strong analogy with the syndrome decoding in the context of linear codes. In other words, sparse error patterns are identified from the syndrome equation which is obtained by multiplying the parity-check matrix to the received signal vector. From this observation, we aim to utilize the paritychecking frames as the sensing matrix over finite fields. In particular, we use the Gallager's parity-check matrices [2], and extend its probabilistic decoding (PD) method for CS context. For instance, low density frames over finite fields are used for sensing matrices. Then, we can see the possibility of using this framework for CS of discrete valued signals. We develop a recovery routine of a sparse signal which is a PD algorithm utilized of sparseness of the signal. We also provide extensive verification of this recovery algorithm with comparison of recently presented theoretic bounds.

Manuscript received July 25, 2017.

Manuscript revised September 28, 2017.

Manuscript publicized November 6, 2017.

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<sup>\*</sup>This paper was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (NRF-2017R1C1B5075823).

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DOI: 10.1587/transinf.2017EDL8166

#### 2. Compressed Sensing Framework over Finite Fields

#### 2.1 System of Interest

In this section, we describe a compressed sensing framework in a finite field of size q as  $\mathbb{F}_q$ . Let  $\mathbf{x} \in \mathbb{F}_q^N$  be a signal vector of length N with sparsity K,  $\mathbf{A} \in \mathbb{F}_q^{M \times N}$  be a  $M \times N$ sensing matrix with N > M. The measured signal  $\mathbf{y}$  is

$$\mathbf{y} = \mathbf{A}\mathbf{x}.\tag{1}$$

The signal  $\mathbf{z}$  obtained after passing the *q*-ary symmetric channel is

$$\mathbf{z} = \mathbf{y} + \mathbf{e}.\tag{2}$$

where  $\mathbf{e} \in \mathbb{F}_q^M$  is the  $M \times 1$  noise vector whose element follows an independently identical distribution (i.i.d.). The distribution  $\mathbf{p}_{\epsilon}$  of the *q*-ary symmetric channel for a noise vector  $\mathbf{e}$  is defined by

$$\Pr\left\{\mathbf{z}_{i}|\mathbf{y}_{i}\right\} = \begin{cases} 1 - \epsilon & \text{for } \mathbf{y}_{i} = \mathbf{z}_{i},\\ \epsilon/(q - 1) & \text{otherwise,} \end{cases}$$
(3)

where  $\mathbf{y}_i$  and  $\mathbf{z}_i$  are the *i*th elements of  $\mathbf{y}$  and  $\mathbf{z}$ , and  $\boldsymbol{\epsilon} = 0$  is the noiseless case. In this paper, we assume that each element of  $\mathbf{x}$  follows a probability distribution  $\mathbf{p}_x$  with i.i.d.,

$$\Pr\left\{\mathbf{x}_{j}=\theta\right\} = \begin{cases} 1-\delta & \text{if } \theta=0,\\ \delta/(q-1) & \text{if } \theta\neq 0, \end{cases}$$
(4)

where  $\mathbf{x}_j$  are the *j*th element of  $\mathbf{x}$ ,  $\delta$  is the sparsity ratio (= K/N), and a dummy variable  $\theta \in \mathbb{F}_q$ .

In this paper, we use a sparse matrix for **A**. The generation of a sparse sensing matrix **A** follows the Gallager approach named as a parity-check matrix randomly chosen from the ensemble of a regular  $(d_c, d_v)$  LDPC codes, where  $d_c$  and  $d_v$  are the number of nonzero entries in the column and the row of the matrix. Through this work, all arithmetic operations for multiplication and addition are performed over a finite field.

#### 2.2 Connection to Syndrome Decoding

Error correction is required for reliable communications, and performed by adding redundant parities to original information. Suppose that a codeword  $\mathbf{c} \in \mathbb{F}^n$  with length *n* is chosen from the codebook *C* over finite fields  $\mathbb{F}$ . And then, it is transmitted through a noisy channel, and received as  $\hat{\mathbf{c}}$ , where  $\hat{\mathbf{c}} = \mathbf{c} + \mathbf{w}$  and  $\mathbf{w} \in \mathbb{F}^n$  is the additive noise. Using the received codeword  $\hat{\mathbf{c}}$  and the knowledge of the codebook *C*, the decoder performs to estimate the correct codeword  $\mathbf{c}$ . The codebook *C* is given by  $m \times n$  parity-check matrix  $\mathbf{H} \in \mathbb{F}^{m \times n}$  as  $C = {\mathbf{c} \in \mathbb{F}^n | \mathbf{H}\mathbf{c} = 0}.$ 

At the receiver, a syndrome decoder performs the computation of the syndrome in the enabling way:  $\mathbf{r} = \mathbf{H}\hat{\mathbf{c}} =$  $\mathbf{H}(\mathbf{c} + \mathbf{w}) = \mathbf{H}\mathbf{w}$  since  $\mathbf{H}\mathbf{c} = 0$ . From the syndrome  $\mathbf{r}$ , it is desired to find the exact error pattern  $\mathbf{w}$  by using the calculated syndrome  $\mathbf{r}$  and the parity-check matrix  $\mathbf{H}$ . The error correction capability of this code *C* mainly relies on its minimum distance, which is the minimum Hamming weight (the number of nonzero elements) of any codeword. The tight connection between CS and coding theory was reported in [11] and [12].

# 3. Probabilistic Decoding Algorithm for Recovery of Sparse Signals

In this section, we propose a probabilistic decoding (PD) algorithm for the CS framework over finite fields. The *maximum a posterior* (MAP) detection is used which utilizes the prior knowledge that the distribution of each element defined in (4) is known for reconstruction. Then, the reconstruction problem of  $\mathbf{x}_j$  is defined as

$$\hat{\mathbf{x}}_j = \arg \max \Pr{\{\mathbf{x}_j = \theta | \mathbf{y}, \mathbf{A}\}}.$$
 (5)

We consider the graphical representation of the reconstruction problem, which is drawn from a sensing matrix **A** by mapping the rows to the measured signal **y** and the columns to the sparse signal **x** with the entries forming the edges of the graph. The presence of an edge between a sensing node and a signal node represents the nonzero coefficient of the sensing matrix **A**. In order to implement the PD algorithm in the graphical representation, we define the two extrinsic probabilities as follows:  $\mathbf{f}_{ij}$  is the probabilistic message from the *i*th sensing element  $\mathbf{y}_i$  to the *j*th signal element  $\mathbf{x}_j$ ;  $\mathbf{f}_{ji}$  is the probabilistic message from the *i*th sensing element  $\mathbf{x}_i$ .

We now discuss several related works. Sarvotham *et al.* proposed a belief propagation algorithm for recovery of real valued sparse signals in [13]. Donoho *et al.* in [14] proposed an approximate message passing (AMP) algorithm for CS with dense Gaussian sensing matrices, where authors utilized a variant of density evolution that provides a precise characterization of its performance.

The idea behind the AMP algorithm is based on decoding of nonbinary LDPC codes in Davey and Mackay [15]. The main difference between our proposed PD algorithm and the work [15] is that we exploit the knowledge of the prior information of sparse signals for reconstruction. Then, the initial process of the recovery algorithm is different as well as exchanging the probabilistic messages between sensing nodes and signal nodes. In Sect. 2, we set up the compressed sensing framework over finite fields as the following ways: an unknown sparse signal **x** is compressed into a measured signal **y**, and it is transmitted through the noisy channel. With the received signal **z** and a prior distribution of **x**, based on this framework, we determine the unknown sparse signal **x** by using the proposed PD algorithm.

For recovery of sparse signals, there are four main steps: i) initialization, ii) update of message  $\mathbf{f}_{ij}$ , iii) update of message  $\mathbf{f}_{ji}$ , and iv) tentative decoding. For the initialization, we set the values of the probabilistic messages for

Algorithm 1: Proposed Probabilistic Decoding
(PD) Algorithm
<b>Input:</b> Prior probability $\mathbf{p}_x$
Channel error probability $\mathbf{p}_{\epsilon}$
Sensing matrix A
<b>Output:</b> Estimated $\hat{\mathbf{x}}_j$
1) Initialization:
Set $\mathbf{p}_x$ and $\mathbf{p}_{\epsilon}$
Initial message $\mathbf{f}_{ji} \leftarrow \mathbf{p}_x$
Set maximum iterations
while $A\hat{\mathbf{x}} = \mathbf{z}$ or Maximum Iterations do
2) Update the message $f_{ij}$ :
$\mathbf{F}_{ij} = \left(\prod_{\tilde{j} \in \mathcal{L}(i) \setminus \{j\}} \mathbf{H}_q \tilde{\mathbf{f}}_{ji} ight) \mathbf{H}_q \mathbf{p}_{\epsilon}$
$\mathbf{f}_{ij} = \mathbf{H}_q^{-1} \mathbf{F}_{v,ij}$
3) Update the message $f_{ji}$ :
$\mathbf{f}_{ji} = \gamma \mathbf{p}_x \prod_{i \in \mathcal{M}(j) \setminus \{i\}} \mathbf{f}_{ij}$
4) Tentative decoding:
$\mathbf{f}_j = \mathbf{p}_x \prod_{i \in \mathcal{M}(j)} \tilde{\mathbf{f}}_{ij}$
$\hat{\mathbf{x}}_j = \arg \max \left\{ \mathbf{f}_j \right\}$
return Estimated $\hat{\mathbf{x}}$

all the nodes. The prior probability distribution  $\mathbf{p}_x$  of the *j*th signal  $\mathbf{x}$  defined in (4) is used. Also the transition probability for each sensing node  $\mathbf{z}$  is as given in (3). This information is utilized to determine the message  $\mathbf{f}_{ij}$ . This makes it possible to the major difference from Davey's work. In the next step, we update all the messages  $\mathbf{f}_{ij}$  as follows. The transformed version  $\mathbf{F}_{ij}$  of the message  $\mathbf{f}_{ij}$  is calculated

$$\mathbf{F}_{ij} = \left(\prod_{\tilde{j} \in \mathcal{L}(i) \setminus \{j\}} \mathbf{H}_q \tilde{\mathbf{f}}_{\tilde{j}i}\right) \mathbf{H}_q \mathbf{p}_{\epsilon},\tag{6}$$

where  $\mathcal{L}(i) = \{j : \mathbf{A}_{ij} \neq 0\}$  denotes the set of indices of  $\mathbf{x}_j$  that participate in the *i*th row of the sensing matrix,  $\mathbf{H}_q$  is the  $q \times q$  transform matrix, e.g., the Hardamard transform matrix or the Fourier transform matrix. In this case, we set the rearranged message  $\mathbf{\tilde{f}}_{ji}$  corresponding to its coefficient of the sensing matrix  $\mathbf{A}$ , which is initially the same with the signal probability  $\mathbf{p}_x$ . And then, using the inverse transformation, the message  $\mathbf{f}_{ij}$  is obtained from

$$\mathbf{f}_{ij} = \mathbf{H}_q^{-1} \mathbf{F}_{ij}.\tag{7}$$

The *i*th sensing node follows the constraint, i.e.,  $\mathbf{y}_i = \sum_i \mathbf{A}_{ij} \mathbf{x}_j$ .

In the third step, the computation of the message  $\mathbf{f}_{ji}$ from messages  $\mathbf{f}_{ij}$  is performed by

$$\mathbf{f}_{ji} = \gamma \mathbf{p}_x \prod_{\tilde{i} \in \mathcal{M}(j) \setminus \{i\}} \tilde{\mathbf{f}}_{\tilde{i}j}, \tag{8}$$

where  $\mathcal{M}(j) = \{i : \mathbf{A}_{ij} \neq 0\}$  denotes the set of indices of  $\mathbf{y}_i$  that participate in the *j*th column of the sensing matrix, the message  $\mathbf{\tilde{f}}_{ij}$  is obtained from rearranging the message  $\mathbf{f}_{ij}$  according to the coefficient of the sensing matrix,  $\gamma$  is the normalization factor for the total probability. Then  $\mathbf{f}_j$  denotes the posterior probability of the *j*th signal node  $\mathbf{x}_j$ , which is conditioned on the information obtained via  $\mathbf{y}_i$  and  $\mathbf{p}_x$ . Then,

the posterior probability is obtained from as follows,

$$\mathbf{f}_j = \mathbf{p}_x \prod_{i \in \mathcal{M}(j)} \tilde{\mathbf{f}}_{ij}.$$
(9)

The *j*th signal node  $\mathbf{x}_j$  is then estimated:  $\hat{\mathbf{x}}_j = \arg \max{\{\mathbf{f}_j\}}$ . The decoder checks if  $\hat{\mathbf{x}}$  satisfies the constraint condition, i.e.,  $A\hat{\mathbf{x}} = \mathbf{y}$ .

### 4. Simulation Results and Performance Comparison

In Fig. 1 to 4, we evaluate the performance of our CS framework considered in finite fields. In all simulations, the maximum number of iterations is set to 50 for the PD algorithm. We use a regular ( $d_c = 3$ ) sensing matrix **A** which was introduced in [2]. In order to demonstrate the proposed PD algorithm, we generate a sparse signal of length 1200, N = 1200. For the sparsity ratio  $\delta(= K/N)$ , the compression ratio  $\rho(= M/N)$ , and the error probability  $\epsilon$  of the *q*-ary symmetric channel, we obtain the failure probabilities for reconstruction of CS. Figure 1 shows the performance of our CS scheme with M = 600, over a finite field of each size:



**Fig.1** Failure probability for recovery of sparse signals with fixed N = 1200, M = 600, and  $\epsilon = 0.1$  over q = 2, 4, 16, 256.



**Fig. 2** Failure probability for recovery of sparse signals with fixed N = 1200, M = 600, and K = 120 over q = 4, 8, 16, 256.

q = 2, 4, 16, and 256. In this simulation, we set the error probability  $\epsilon = 0.1$  for the q-ary symmetric channel. With different sparsity of sparse signals, we evaluate the failure probability as shown in Fig. 1. We show that as the size of finite fields increases, the larger number of the sparsity can be successfully decoded. In Fig. 2, we show the failure probability with different error probability  $\epsilon$ . We observe that a larger finite alphabet is not sensible with the channel noise. In order for the effect of the number of measurements, we evaluate the performance of the CS framework and show the results in Fig. 3.

Figure 4 shows comparison of theoretic bounds with the simulation results for the failure probability of  $10^{-4}$ where theoretic bounds obtained from in [3] and Eq. (11) in [4] which are considered on fixed source signal with the i.i.d. uniformly sensing matrix. Simulation results as well as the bounds in [3] are considered sparse sensing matrices, and are tighter rather than the bounds in [4]. As the distribution of the element of the sensing matrix approaches uniformly, the curves of the bounds of [3] move to those of bounds in [4].



**Fig. 3** Failure probability for recovery of sparse signals with fixed N = 1200,  $\epsilon = 0.1$ , and K = 120 over q = 4, 8, 16, 256.



**Fig. 4** Comparisons with theoretic bounds and simulation results, which are obtained from the cases with the noiseless measurement and N = 1200: dots(simulation results for the failure probability of  $10^{-4}$ ), dot lines (bounds in [4]), solid lines (bounds in [3]).

#### 5. Conclusions

In conclusion, we considered the CS framework over finite fields. In this framework, low-density frames were used as the sensing matrices. We proposed a PD algorithm for recovery of sparse signals based the message passing algorithm which shows very good performance closely achieving the theoretical bounds in coding theory. This work allows us to utilize the low-density sensing matrices to be good reconstruction performance into a CS framework. Also we evaluated the performance of the proposed PD algorithm in the finite version of the CS framework. The simulation results show that larger size of finite fields achieves good reconstruction of sparse signals with respect to different compression and sparsity ratios.

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