PAPER

Cyclic Vertex Connectivity of Trivalent Cayley Graphs

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SUMMARY A vertex subset $F \subseteq V(G)$ is called a cyclic vertex-cut set of a connected graph G if $G - F$ is disconnected such that at least two components in $G - F$ contain cycles. The cyclic vertex connectivity is the cardinality of a minimum cyclic vertex-cut set. In this paper, we show that the cyclic vertex connectivity of the trivalent Cayley graphs TG_n is equal to eight for $n \geq 4$.

key words: *interconnection network, trivalent Cayley graphs, fault-tolerance, conditional connectivity, cyclic vertex connectivity*

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph, where $V(G)$ and $E(G)$ are the vertex set and edge set, respectively. A vertex subset $F \subseteq V(G)$ (edge subset $F \subseteq E(G)$) is called a cyclic vertex-cut set (cyclic edge-cut set) if $G - F$ has at least two connected components containing cycles. The cyclic vertex-cut set may not exist. The cyclic vertex connectivity $\kappa_c(G)$ (cyclic edge connectivity $\lambda_c(G)$) is defined as the minimum cardinality over all cyclic vertex-cut sets (cyclic edge-cut sets) of G if G has a cyclic vertex-cut set (cyclic edge-cut set). The cyclic vertex (cyclic edge) connectivity has been studied in [1]–[5], [7], [8], [10]–[14].

Connectivity is a measurement for the fault-tolerance capability of interconnection network. Call the vertices in cyclic vertex-cut set F as faulty and vertices in $G - F$ as good. The cyclic vertex connectivity is an important measure for supporting the execution of parallel algorithms on cycles in a faulty and disconnected interconnection network. The cyclic vertex connectivity is determined for the following interconnection networks: star graphs [1], [11], bubble sort graphs [1], hierarchical cubic networks [2], complete cubic networks [3], and balanced hypercubes [14].

Cyclic vertex connectivity is related to a kind of conditional connectivity. In [6], the authors define the conditional connectivity as follows: for a connected graph G, a vertex subset $F \subseteq V(G)$ is called a R^k-vertex-cut set if $G - F$ is disconnected and each vertex in $G - F$ has at least k neighbors in $G - F$. The R^k-vertex-connectivity $\kappa^k(G)$ is the cardinality of a minimum R^k-vertex-cut set of G. Since every graph with minimum degree at least two has a cycle, we have $\kappa_c(G) \leq \kappa^2(G)$ if both $\kappa_c(G)$ and $\kappa^2(G)$ exist. In [11], the authors gave an example which shows that the strict inequality may hold and the gap between $\kappa_c(G)$ and $\kappa^2(G)$ can be arbitrarily large.

Trivalent Cayley graphs proposed in [9] are fixed vertex degree interconnection network which are suitable for VLSI implementation. In [9], the graphs are shown to have nice network properties such as regular, logarithmic diameter, and maximal fault tolerant. As fixed vertex degree graphs, trivalent Cayley graphs have the advantage to form a parallel architecture with large number of computing nodes. In this paper, we determine the cyclic vertex connectivity of the trivalent Cayley graphs.

The rest of the paper is organized as follows. Section 2 recalls and derives some structure properties of trivalent Cayley graphs. Section 3 derives some properties about shortest cycles in trivalent Cayley graphs. Section 4 determines the value of cyclic vertex connectivity of trivalent Cayley graph TG_n for $n \geq 4$. Section 5 provides concluding remarks.

2. Preliminaries

We use the following notations in this paper. Let v be a vertex of a graph G. $N_G(v)$ is the set of vertices adjacent to v in G. For a vertex subset V of the graph G, $N_G(V) = (\bigcup_{v \in V} N_G(v)) - V$. Sometimes, we use a graph to represent its vertex set. For example, $N_G(G_1)$ represents $N_G(V(G_1))$ where G_1 is a subgraph of G.

The structure of the trivalent Cayley graphs TG_n is stated in this section. Each vertex in the graph TG_n corresponds to a circular permutation of n symbols in lexicographic order where each symbol may be in either uncomplement or complement form. Let t_k, $1 \leq k \leq n$, denote the k-th symbol in the set of those n symbols. Denote t_i^* as either t_i or $\bar{t}_i$. Denote $u = a_1 a_2 \cdots a_n$ to represent vertex u with label $a_1 a_2 \cdots a_n$. If $a_1 = t_k^*$, then, for $2 \leq i \leq n$, $a_i = t^*_{(k+i-1)}$ if $k + i - 1 \leq n$, otherwise, $a_i = t^*_{(k+i-1) \ mod \ n}$. The edges of the graph TG_n are defined by three generators as follows:

$$f(a_1 a_2 \cdots a_n) = a_2 a_3 \cdots \bar{a}_1,$$
$$f^{-1}(a_1 a_2 \cdots a_n) = \bar{a}_n a_1 \cdots a_{n-1},$$
$$g(a_1 a_2 \cdots a_n) = a_1 a_2 \cdots \bar{a}_n.$$

Notice that $f^{-1} f = f f^{-1} = e$ and $gg = e$, where e is the identity mapping. The edges between u and $f(u)$ or $f^{-1}(u)$ are called a f-edge or f^{-1}-edge, respectively. Observe that a f-edge is also a f^{-1}-edge. An edge between u and $g(u)$

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is called a g-edge. We classify the g-edges as follows. Let $u = a_1a_2 \cdots a_n$. The g-edge incident on u is called a g_i-edge if a_n is symbol t_i^*. Clearly, each vertex has two f-edges and one g-edge incident on it. Every vertex in TG_n for $n \geq 2$ has vertex degree 3, see [9]. Any cycle in TG_n consists of f-edges is called a f-cycle. The following results can be found in [9].

Lemma 2.1: *(See [9])* All of the vertices of TG_n are partitioned into vertex disjoint f-cycles of length $2n$; number of f-cycles in TG_n is 2^{n-1}.

The complement of any vertex $u = u_1u_2 \cdots u_n$ in TG_n is the vertex $\overline{u}$ obtained by complementing the symbols in u, i.e., $\bar{u} = \bar{u}_1\bar{u}_2 \cdots \bar{u}_n$.

Lemma 2.2: *(See [9])* For an arbitrary pair of vertices u and v in TG_n such that $g(u) = v$, the complement vertices satisfy the same relation, i.e., $g(\bar{u}) = \bar{v}$.

Lemma 2.3: *(See [9])* For any vertex u in TG_n, both the u and $\overline{u}$ belong to the same f-cycles.

For each f-cycle in TG_n, the unique vertex with label starting with t_1 is called the *leader* vertex. Let $t_1t_2^* \cdots t_n^*$ be the leader vertex of some f-cycle. The leader vertex is mapped into a $(n-1)$-bit binary number $b_2b_3 \cdots b_n$ by assigning bit b_i to 1 if $t_i^* = t_i$ and 0 if $t_i^* = \bar{t}_i$. The f-cycle is assigned with the binary number associated with it's leader vertex. We denote f-cycle with $(n-1)$-bit binary number $b_2b_3 \cdots b_n$ as $f_{b_2b_3 \cdots b_n}$. For brevity, we also denote f-cycle as f_i for integer number i. Cycles f_i and f_j are said to be adjacent if there exists a vertex $u \in f_i$ and a vertex $v \in f_j$ such that $u = g(v)$ or $v = g(u)$. We reformulate the following result given in [9] for our need in this paper.

Lemma 2.4: *(See [9])* Each $f_{b_2b_3 \cdots b_n}$-cycle in TG_n is adjacent to the following n different f-cycles:

$f_{\bar{b}_2\bar{b}_3 \cdots \bar{b}_n}$ by g_1-edge, $f_{\bar{b}_2b_3 \cdots b_n}$ by g_2-edge,
$f_{b_2\bar{b}_3 \cdots b_n}$ by g_3-edge, $\cdots$, $f_{b_2b_3 \cdots \bar{b}_n}$ by g_n-edge.

From this lemma, we see that the binary numbers associated with any two adjacent f-cycles must have one bit different or all different.

By Lemmas 2.1, 2.2, 2.3, and 2.4, we see that adjacent f-cycles are connected by exactly two g_i-edges for some i, $1 \leq i \leq n$. Furthermore, there are exactly two g_i-edges, $1 \leq i \leq n$, incident on the vertices of the f-cycle.

Lemma 2.5: Let f_i and f_j be two distinct f-cycles in the trivalent Cayley graphs TG_n. Then, $N_{TG_n}(f_i) \cap N_{TG_n}(f_j) = \emptyset$.

Proof: Let vertex $w \in N_{TG_n}(f_i) \cap N_{TG_n}(f_j)$. Then, there exist two distinct vertices $u \in f_i$ and $v \in f_j$ such that $g(u) = w$ and $g(v) = w$. By the definition of generator g, we have $u = v$. Since $u \neq v$, this is a contradiction. □

A TG_n can be represented as a reduced graph RG_{n-1} as follows: condense each f-cycle into a single vertex and label that vertex with the $(n-1)$-bit binary number of the f-cycle; connect two vertices of RG_{n-1} if and only if the corresponding f-cycles are adjacent in TG_n. We have the following lemma.

Lemma 2.6: No cycle of length three in RG_{n-1} for $n \geq 4$.

Proof: Assume that RG_{n-1} has a cycle of length three. This implies that there are three f-cycles such that one of f-cycle f_i is adjacent to the other two f-cycles f_j and f_k. And, f_j and f_k must be adjacent f-cycles. Since f_i is adjacent to both of f_j and f_k, by Lemma 2.4, the labels of f_j and f_k have two or $n-2$ different bits. Since f_j is adjacent to f_k, by Lemma 2.4, the labels of f_j and f_k have one or $n-1$ different bits. Since $n-1 \neq 2$ and $n-2 \neq 1$ for $n \geq 4$, we see that f_j and f_k are not adjacent in TG_n. Therefore, there is no cycle of length three in RG_{n-1}. □

The following result in [9] is useful in deriving our result.

Lemma 2.7: *(See [9])* The vertex connectivity of RG_{n-1} corresponding to TG_n is n, i.e., for any two given source and destination vertices in RG_{n-1}, there are n vertex disjoint paths connecting the source and the destination.

Define the composition of generators as $(h_1 \cdot h_2)(u) = h_1(h_2(u))$. Let u be the leader vertex of a f-cycle. The $2n$ vertices in a f-cycle can be denoted by $f^k(u)$ for $1 \leq k \leq 2n$. Let $a = k \bmod n$. Then, the g-edge incident on the vertex $f^k(u)$ is g_a-edge if $a \neq 0$ and g_n-edge, otherwise. We have the following lemma.

Lemma 2.8: Let g_a-edge and g_b-edge be the g-edges incident on distinct vertices $f^{k_1}(u)$ and $f^{k_2}(u)$ of a f-cycle with leader vertex u, respectively. Let l be the number of f-edges in the f-cycle connecting the g_a and g_b. Then, $l = |k_1 - k_2|$ or $2n - |k_1 - k_2|$. Furthermore, $l = n$ if $a = b$.

Proof: Since f-cycle is a cycle consisting of f-edges, there are two paths connecting the g_a and g_b. Without loss of generality, let $k_1 < k_2$. Then, one of the paths must consist of $|k_1 - k_2|$ f-edges. By Lemma 2.1, the length of a f-cycle is $2n$. Thus, the other path consists of $2n - |k_1 - k_2|$ f-edges. Therefore, $l = |k_1 - k_2|$ or $2n - |k_1 - k_2|$.

If $a = b$, we must have $k_2 = k_1 + n$. Then, both of the paths consists of n f-edges. Thus, $l = n$. □

Let cycle C contains a sequence of g-edges. Let a and b be integers in the range from 1 to n. We call g_a-edge and g_b-edge adjacent in C if edges in C between g_a-edge and g_b-edge are all f-edges. Since f-cycles are vertex disjoint cycles by Lemmas 2.1, the edges between any two adjacent g_a-edge and g_b-edge are in the same f-cycle.

Lemma 2.9: Let C be a cycle with at least two g-edges in TG_n. Let l be the number of f-edges in C between any two adjacent g_a-edge and g_b-edge. Then

$$l = \begin{cases} |b-a| \text{ or } 2n-|b-a| \text{ or } n-|b-a| \text{ or } n+|b-a| & \text{if } b \neq a \\ n & \text{if } b = a \end{cases}$$

Proof: Since the edges in C between any two adjacent g_a-edge and g_b-edge are in the same f-cycle, there are two possible g_a-edges and two possible g_b-edges incident on the vertices of the f-cycle.

Let u be the leader vertex of the f-cycle. Let g_a-edge and g_b-edge be incident on vertices $f^{k1}(u)$ and $f^{k2}(u)$, respectively. Now, we have that $k_1 = a$ or $a + n$, $k_2 = b$ or $b + n$. Case 1: $k_1 = a$ and $k_2 = b$. By Lemma 2.8, $l = |b - a|$ or $2n - |b - a|$ if $b \neq a$. Otherwise, $l = n$. Case 2: $k_1 = a$ and $k_2 = b + n$. By Lemma 2.8, $l = n + |b - a|$ or $n - |b - a|$ if $b \neq a$. Otherwise, $l = n$. Case 3: $k_1 = a + n$ and $k_2 = b$. By Lemma 2.8, $l = n + |b - a|$ or $n - |b - a|$ if $b \neq a$. Otherwise, $l = n$. Case 4: $k_1 = a + n$ and $k_2 = b + n$. By Lemma 2.8, $l = |b - a|$ or $2n - |b - a|$ if $b \neq a$. Otherwise, $l = n$. This completes the proof. □

Notice that $1 \leq |b - a| \leq n - 1$ since $a, b \in \{1, \cdots, n\}$.

Lemma 2.10: Let $P = u_1u_2 \cdots u_{n+1}$ be a path of length n in a f-cycle of TG_n. Then,

1. u_1 and u_{n+1} are incident on g_a-edges where $1 \leq a \leq n$
2. vertices other than u_1 and u_{n+1} are incident on $n - 1$ different g_i-edges where $i \neq a$.

Proof: Let u be the leader vertex of the f-cycle. Let vertex $u_1 = f^k(u)$ where $1 \leq k \leq 2n$. Notice that $u_i = f^{k+i-1}(u)$ for $1 \leq i \leq n + 1$. By the definition of generator f, the last symbols of the vertex labels on u_1 and u_{n+1} are the same symbol, say t^*_a, where $1 \leq a \leq n$. Thus, u_1 and u_{n+1} are incident on g_a-edges. By the definition of generator f, the last symbols of the vertex labels on u_i, $1 \leq i \leq n$, are all different. Thus, the g-edges incident on those vertices u_i are all different. Therefore, vertices other than u_1 and u_{n+1} are incident on $n - 1$ different g_i-edges where $i \neq a$. □

For any f-cycle f and any vertex $u \in V(f)$, u has exactly one neighbor outside of f, namely the vertex $g(u)$. Denote u' as the outside neighbor of u.

Lemma 2.11: Let path $P = u_1u_2 \cdots u_{n-1}$ of length $n - 2$ in a f-cycle of TG_n, the outside neighbors $u'_1, u'_2, \cdots, u'_{n-1}$ are in $n - 1$ different f-cycles.

Proof: By Lemma 2.10, the g_i-edges incident on vertices u_i of path P are all different. By Lemma 2.4, the outside neighbors $u'_1, u'_2, \cdots, u'_{n-1}$ are in $n - 1$ different f-cycles. □

We call $Y = \{u'_k | u'_k$ is the outside neighbor of vertex u_k in path $P\}$, the outside neighbor set of path P. Clearly, $Y \subseteq N_{TG_n}(f)$ if path P is in f-cycle f. The following two lemmas are useful in deriving the value of cyclic vertex connectivity $\kappa_c(TG_n)$, $n \geq 4$.

Lemma 2.12: Let f_i and f_j be two different f-cycles in TG_n, $n \geq 4$. Let G be the subgraph induced by $V(f_i) \cup V(f_j)$. Let C be a cycle in G such that $V(C) \cap V(f_i) \neq \emptyset$ and $V(C) \cap V(f_j) \neq \emptyset$. Then, there are two paths $P1$ and $P2$ in C of length $n - 2$ such that $P1$ is in f_i and $P2$ is in f_j. Furthermore, the outside neighbor sets $Y1$ of $P1$ and $Y2$ of $P2$ satisfy $Y1 \cap Y2 = \emptyset$.

Proof: Since C is a cycle in G such that $V(C) \cap V(f_i) \neq \emptyset$ and $V(C) \cap V(f_j) \neq \emptyset$, f_i and f_j must be adjacent f-cycles connected by two g_a-edges for some $a \in \{1, 2, \cdots, n\}$. By Lemma 2.9, there is a path in C of length n in each f-cycle.

The cycle can be constructed as follows. Let (u_1, v_1) and (u_2, v_2) be the two g_a-edges. Without loss of generality, let u_1 and u_2 be in f_i, while v_1 and v_2 in f_j. Then, cycle C contains a path in f_i from u_1 to u_2, a g_a-edge (u_2, v_2), a path in f_j from v_2 to v_1, and another g_a-edge (v_1, u_1).

Since there are two possible paths of length n connecting the two g_a-edges in each f-cycle, we have four possible constructions of cycle C. Clearly, there is a path in C of length $n - 2$ in each f-cycle, say $P1$ in f_i and $P2$ in f_j. Clearly, $Y1 \subseteq N_{TG_n}(f_i)$ and $Y2 \subseteq N_{TG_n}(f_j)$. By Lemma 2.5, $Y1 \cap Y2 = \emptyset$. □

Lemma 2.13: Let f_i, f_j and f_k be three different f-cycles in TG_n, $n \geq 4$. Let G be the subgraph induced by $V(f_i) \cup V(f_j) \cup V(f_k)$. Let C be a cycle in G such that $V(C) \cap V(f_i) \neq \emptyset$, $V(C) \cap V(f_j) \neq \emptyset$, and $V(C) \cap V(f_k) \neq \emptyset$. Then, there are two nonadjacent f-cycles, say f_i and f_j, among those three f-cycles such that two paths $P1$ and $P2$ in C of length $n-2$ are in f_i and f_j, respectively. Furthermore, the outside neighbor sets $Y1$ of $P1$ and $Y2$ of $P2$ satisfy $Y1 \cap Y2 = \emptyset$.

Proof: Since C is a cycle in G such that $V(C) \cap V(f_i) \neq \emptyset$, $V(C) \cap V(f_j) \neq \emptyset$ and $V(C) \cap V(f_k) \neq \emptyset$, RG_{n-1} contains a corresponding cycle of length 3 or a corresponding path of length 2 that connects those three f-cycles. By Lemma 2.6, no cycle of length 3 in RG_{n-1} for $n \geq 4$. Thus, RG_{n-1} has a corresponding path of length 2. Without loss of generality, we assume that f_i and f_j are not adjacent, and f_k is adjacent to both of f_i and f_j. Let f_k and f_i be connected by two distinct g_a-edges and f_k and f_j be connected by two distinct g_b-edges, where $a \neq b$ and $a, b \in \{1, 2, \cdots, n\}$. By Lemma 2.9, each of f_i and f_j has a path in C of length n.

The cycle C can be constructed as follows. Let (u_1, v_1) and (u_2, v_2) be the two g_a-edges where u_1 and u_2 are in f_i, while v_1 and v_2 are in f_k. Let (x_1, y_1) and (x_2, y_2) be the two g_b-edges where x_1 and x_2 are in f_k, while y_1 and y_2 are in f_j. Then, C contains a path of length n in f_i from u_1 to u_2, a g_a-edge (u_2, v_2), a path in f_k connecting a g_a-edge and a g_b-edge, say from v_2 to x_2, a g_b-edge (x_2, y_2), a path of length n in f_j from y_2 to y_1, another g_b-edge (y_1, x_1), a path in f_k connecting a g_b-edge and a g_a-edge, say from x_1 to v_1, and another g_a-edge (v_1, u_1).

In cycle C, vertices v_1 and v_2 are incident on g_a-edges in f_k, while vertices x_1 and x_2 on g_b-edges. By Lemma 2.9, there is a path P_1 of length n in f_k from v_1 to v_2, or P_2 from v_2 to v_1. By Lemma 2.10, there exists exactly one vertex, say x_1 in P_1, and another vertex, say x_2 in P_2, incident on g_b-edge. Thus, f_k is divided into four paths where each path is connecting between a g_a-edge and g_b-edge. Thus, there exist two vertex disjoint paths in f_k connecting between a g_a-edge and a g_b-edge in C.

Since there are two possible paths in each f-cycle for each path of C in f_i and f_j and two possible choices of paths connecting between a g_a-edge and a g_b-edge in f_k, we have eight possible constructions of cycle C. Clearly, there is a path in C of length $n - 2$ in each f-cycle, say $P1$ in f_i and $P2$ in f_j. Clearly, $Y1 \subseteq N_{TG_n}(f_i)$ and $Y2 \subseteq N_{TG_n}(f_j)$. By Lemma 2.5, $Y1 \cap Y2 = \emptyset$. □

3. Shortest Cycles of Trivalent Cayley Graphs

In this section, we characterize the shortest cycles in trivalent Cayley graphs.

Lemma 3.1: Let C be a cycle with $m \leq 3$ g-edges in TG_n for $n \geq 4$. Then, m is equal to 0 or 2. Furthermore,

1. C is a f-cycle of length $2n$ if $m = 0$.
2. C is a cycle of length $2n + 2$ and is contained in two adjacent f-cycles if $m = 2$.

Proof: Observe that each f-cycle connected in C must have even number of g-edges in C incident on its vertices. This implies that the cycle C in TG_n will correspond to an isolated vertex if $m = 0$, a path of length one if $m = 2$, or a cycle of length three if $m = 3$ in RG_{n-1}.

An isolated vertex in RG_{n-1} is a f-cycle in TG_n. By Lemma 2.1, the length of C is $2n$. A path of length one in RG_{n-1} corresponds to a cycle connecting two adjacent f-cycles in TG_n. Since adjacent f-cycles are connected by exactly two g_i-edges for some i, $1 \leq i \leq n$. By Lemma 2.9, the number of f-edges between the two adjacent g_a-edges is n. Therefore, C is of length $2n + 2$. Since cycle of length three does not exist in RG_{n-1} for $n \geq 4$ by Lemma 2.6, cycle with three g-edges does not exist. □

The following lemma determines the structure of shortest cycles.

Lemma 3.2: Let cycle C be a shortest cycle in TG_n for $n \geq 4$. Cycle C is of length 8. Furthermore,

1. Cycle C is a f-cycle for $n = 4$ if C does not contain g-edges.
2. Cycle C contains four distinct g-edges and four distinct f-edges. Edges in C are f-edge and g-edge alternating.

Proof: The shortest cycle can be a cycle with or without g-edges. First, we consider the cycles without g-edges. Let cycle C' be such a cycle. Clearly, C' is a f-cycle. By Lemma 2.1, C' is of length $2n$. Since $n \geq 4$, the shortest cycles without g-edges have length 8 for $n = 4$.

Now, we consider the cycles with g-edges. Let cycle C' has m g-edges. Let l be the length of cycle C'. By Lemma 2.9, we see that the minimum number of f-edges between any two adjacent g-edges is equal to one. Therefore, $l \geq 2m$. Clearly, $l \geq 8$ for $m \geq 4$. To show that the shortest cycle with g-edges has length $l = 8$, we need the following two claims.

Claim 1: The cycles with m g-edges where $0 < m < 4$ have length $l > 8$.

Claim 2: There exists a cycle with four g-edges of length $l = 8$.

Proof of Claim 1: By Lemma 3.1, we only need to consider the case that $m = 2$. In this case, the cycle has two g-edges and has length $l = 2n + 2$. Since $n \geq 4$, we have $l > 8$.

Proof of Claim 2: Since cycle C' with m g-edges has length $l \geq 2m$, the shortest cycle has length $l = 8$ if $m = 4$. This implies that the shortest cycle C with g-edges must have four distinct g-edges and the number of f-edges between adjacent g-edges is equal to one. Thus, edges in C are f-edge and g-edge alternating. Now, we show that such cycle C exists in TG_n for $n \geq 4$. Consider the following cases.

Case 1: The cycle C uses g_a-edge, g_b-edge, g_c-edge, and g_d-edge for four distinct integers $a, b, c, d \in \{1, \cdots, n\}$. Let C first go through g_a-edge, then g_b-edge, then g_c-edge, finally g_d-edge. By Lemma 2.9, the values of a, b, c, and d must satisfy the following equations.

$$
\begin{aligned}
|a-b| &= 1 \quad \text{or} \quad n-1\\
|b-c| &= 1 \quad \text{or} \quad n-1\\
|c-d| &= 1 \quad \text{or} \quad n-1\\
|d-a| &= 1 \quad \text{or} \quad n-1
\end{aligned}
$$

Now, we show that if the equations have solutions then $n = 4$.

Case 1.1: The solution contains both of 1 and n. Without loss of generality, let $a = 1$ and $b = n$. Since values of a, b, c, and d are four distinct integers, we have two possible choices of values of c and d. For the choice that $d = 2$ and $c = 3$. From the equations, we must have $|b-c| = |n-3| = 1$ or $n - 1$. Another choice is that $c = n - 1$ and $d = n - 2$. From the equations, we must have $|d - a| = |(n-2) - 1| = 1$ or $n - 1$. For both choices of values c and d, we must have $n = 4$.

Case 1.2: The solution does not contain both of 1 and n. This implies that the solution must contain four consecutive integers. Without loss of generality, let a be the value of the smallest of the four consecutive integers. From the equations, we have $b = a + 1$, $c = b + 1$, $d = c + 1$, and $|d - a| = |(a + 3) - a| = 1$ or $n - 1$. This implies that $n = 4$.

Since $n = 4$, we may assume that $a = 1$, $b = 2$, $c = 3$, and $d = 4$. Now, C first go through g_1-edge, then g_2-edge, then g_3-edge, finally g_4-edge. Let C start at f-cycle $f_{b_2b_3b_4}$ where $b_2b_3b_4$ is the binary number associated with the f-cycle. By Lemma 2.4 and going through the g_1-edge, C reaches f-cycle $f_{\bar{b}_2\bar{b}_3\bar{b}_4}$. Then, C reaches f-cycle $f_{b_2\bar{b}_3\bar{b}_4}$ after passing through the g_2-edge. After passing through the g_3-edge and g_4-edge, C reaches the starting f-cycle.

Now, we construct cycle C as follows. Let $t_1^*t_2^*t_3^*t_4^*$ be the label of vertex u. The cycle C starting with vertex u is given below:

$$
\begin{aligned}
& u = t_1^*t_2^*t_3^*t_4^* && \xrightarrow{f} t_2^*t_3^*t_4^*\bar{t}_1^*\\
& \xrightarrow{g_1} t_2^*t_3^*t_4^*t_1^* && \xrightarrow{f} t_3^*t_4^*t_1^*\bar{t}_2^*\\
& \xrightarrow{g_2} t_3^*t_4^*t_1^*t_2^* && \xrightarrow{f} t_4^*t_1^*t_2^*\bar{t}_3^*\\
& \xrightarrow{g_3} t_4^*t_1^*t_2^*t_3^* && \xrightarrow{f} t_1^*t_2^*t_3^*\bar{t}_4^*\\
& \xrightarrow{g_4} t_1^*t_2^*t_3^*t_4^* = u
\end{aligned}
$$

It is easy to check that the vertices in the cycle are all distinct.

Case 2: The cycle C uses g_a-edge, g_b-edge, and g_c-edge for three distinct integers $a, b, c \in \{1, \cdots, n\}$. Assume that the cycle C has two g_b-edges. By Lemma 2.9, the number of f-edges between two distinct and adjacent g_b-edges is equal to n, where $n \geq 4$. Since the number of f-edges between adjacent g-edges must be equal to one in cycle C, the two g_b-edges can not be adjacent in cycle C. This implies that g_a-edge and g_b-edge are adjacent g-edges. And, g_b-edge and g_c-edge are adjacent g-edges. Without loss of generality, let C first go through g_a-edge, then g_b-edge, then g_c-edge, finally another g_b-edge. By Lemma 2.9, the values of a, b, and c must satisfy the following equations.

$$|a-b| = 1 \quad \text{or} \quad n-1$$
$$|b-c| = 1 \quad \text{or} \quad n-1$$

Clearly, the solution of the equations contains any three cyclic consecutive integers of the set $\{1, \cdots, n\}$.

Now, we show that any solution of the equations can not have a corresponding cycle in TG_n. Let C start at f-cycle $f_{b_2b_3\cdots b_n}$ where $b_2b_3\cdots b_n$ is the binary number associated with the f-cycle. By Lemma 2.4 and passing through the four g-edges, C reaches f-cycle $f_{b_2\cdots\bar{b}_a b_b \bar{b}_c\cdots b_n}$ if $a \neq 1$, $b \neq 1$, and $c \neq 1$. Or, C reaches f-cycle such that its associated binary number has the bit b_a and bit b_c complemented if $b = 1$. Or, C reaches f-cycle such that its associated binary number has each bit $b_i, 2 \leq i \leq n$, complemented except the bit b_c or bit b_a if $a = 1$ or $c = 1$, respectively. Clearly, the reached f-cycle is not the same as the starting f-cycle with each corresponding solution. Thus, this choice of g-edges has no corresponding cycle in TG_n.

Case 3: The cycle C uses g_a-edge and g_b-edge for two distinct integers $a, b \in \{1, \cdots, n\}$. Clearly, adjacent g-edges must be g_a-edge and g_b-edge and $|b-a| = 1$ or $n-1$ for $n \geq 4$ by Lemma 2.9. Without loss of generality, the values of a and b can be chosen as follows. For $|b-a| = n-1$, let $a = n$ and $b = 1$. For $|b-a| = 1$, let $b = a+1$ for $1 \leq a < n$. By Lemma 2.4 and passing through even number of g_a-edges and even number of g_b-edges, C reaches the starting f-cycle.

Now, we construct cycle C as follows. For $|b-a| = 1$, let t_a be the symbol corresponding to g_a-edge. Then, the symbol corresponding to g_b-edge is t_{a+1}. Let $t^*_{a+1}\cdots t^*_n t^*_1\cdots t^*_a$ be the label of vertex u. The cycle C starting with vertex u is given below:

$$\begin{aligned}
& u = t^*_{a+1}\cdots t^*_n t^*_1\cdots t^*_a && \xrightarrow{f} t^*_{a+2}\cdots t^*_n t^*_1\cdots t^*_a \bar{t}^*_{a+1} \\
& \xrightarrow{g_b} t^*_{a+2}\cdots t^*_n t^*_1\cdots t^*_a t^*_{a+1} && \xrightarrow{f^{-1}} \bar{t}^*_{a+1}\cdots t^*_n t^*_1\cdots t^*_a \\
& \xrightarrow{g_a} \bar{t}^*_{a+1}\cdots t^*_n t^*_1\cdots \bar{t}^*_a && \xrightarrow{f} t^*_{a+2}\cdots t^*_n t^*_1\cdots \bar{t}^*_a t^*_{a+1} \\
& \xrightarrow{g_b} t^*_{a+2}\cdots t^*_n t^*_1\cdots \bar{t}^*_a \bar{t}^*_{a+1} && \xrightarrow{f^{-1}} t^*_{a+1}\cdots t^*_n t^*_1\cdots \bar{t}^*_a \\
& \xrightarrow{g_a} t^*_{a+1}\cdots t^*_n t^*_1\cdots t^*_a = u
\end{aligned}$$

It is easy to check that the vertices in the cycle are all distinct. Similarly, the cycle C can be constructed for the case that $|b-a| = n-1$.

Case 4: The cycle C uses four g_a-edges for $a \in \{1, \cdots, n\}$. Since adjacent f-cycles are connected by exactly two g_i-edges, cycle C does not exist in TG_n.

This completes the proof. □

4. Cyclic Vertex Connectivity of Trivalent Cayley Graphs

This section determines the value of $\kappa_c(TG_n)$ where $n \geq 4$.

Lemma 4.1: Let C be a shortest cycle in TG_n, $n \geq 4$. Then $N_{TG_n}(C)$ is a cyclic vertex-cut set of TG_n.

Proof: Clearly, $TG_n - N_{TG_n}(C)$ is disconnected with C as a component. To prove the lemma, we must show that subgraph $G' = TG_n - N_{TG_n}(C) - C$ has a cycle. Since graph with every vertex degree at least 2 has a cycle, we prove that the degree of each vertex in G' is at least 2.

Assume that there exists a vertex $u \in G'$ such that $deg(u) \leq 1$. This implies that u has at least two neighbors in $N_{TG_n}(C)$. Let $v_1 \in N_{TG_n}(C)$ and $v_2 \in N_{TG_n}(C)$ be two distinct neighbors of u. Also, let x_1 and x_2 be the vertices in C adjacent to vertices v_1 and v_2, respectively. Notice that x_1 and x_2 must be two distinct vertices. Otherwise, we have a cycle $C_4 = x_1v_1uv_2$ of length 4 which is impossible by Lemma 3.2. Then, there is a cycle C' consisting of path $x_1v_1uv_2x_2$ of length 4 and a path in C between x_1 and x_2. Let l be the length of the path in C between x_1 and x_2. Since C is a cycle, there are two paths between x_1 and x_2 in C. If one of the paths has length l, then the other path has length $8-l$. Since the shortest cycle is of length 8 by Lemma 3.2, we must have $l = 4$. Thus, there are two possible C' cycles such that each cycle is of length 8 and contains a path in C of length 4.

Now, we show that cycle C' does not exist. Since cycle C' is of length 8, it must have the structure of shortest cycles in TG_n. By Lemma 3.2, we have two cases. Case 1: C is a f-cycle and $n = 4$. This implies that edges (x_1, v_1) and (x_2, v_2), which are the edges between C and $N_{TG_n}(C)$, must be g-edges. Case 2: C contains four distinct g-edges and four distinct f-edges. By Lemma 3.2, the edges in C are f-edge and g-edge alternating. This implies that edges (x_1, v_1) and (x_2, v_2), which are the edges between C and $N_{TG_n}(C)$, must be f-edges. Then, C' has two consecutive f-edges incident on either x_1 or x_2. In both cases, cycle C' is not a cycle with edges that are all f-edges or that are with f-edge and g-edge alternating. By Lemma 3.2, cycle C' does not exist. From this, we conclude that the degree of each vertex in G' is at least 2. □

Theorem 4.2: For any integer $n \geq 4$, $\kappa_c(TG_n) = 8$.

Proof: By Lemma 4.1, $N_{TG_n}(C)$ is a cyclic vertex-cut set for a shortest cycle C of length 8. So, we have $\kappa_c(TG_n) \leq |N_{TG_n}(C)|$. Since C is the shortest cycle of trivalent Cayley graph TG_n for $n \geq 4$, no two vertices on C have a common neighbor in $N_{TG_n}(C)$. Thus, $|N_{TG_n}(C)| = 8$ since every vertex in TG_n for $n \geq 2$ has fixed vertex degree 3. This implies that $\kappa_c(TG_n) \leq 8$.

Now, we show that no minimum cyclic vertex-cut set F

such that $|F| \leq 7$. Suppose that we have a minimum cyclic vertex-cut set F such that $|F| \leq 7$. Let $F_i = F \cap f_i$ where f_i is a f-cycle associated with number i, $i \in \{0, 1, \cdots, 2^{n-1} - 1\}$. Define $I = \{i \mid |F_i| \geq 2\}$.

Claim 1: Let $G1$ be the subgraph of TG_n induced by $\bigcup_{i \notin I} V(f_i - F_i)$. Then $G1$ is connected.

Proof of Claim 1: Let $u \in V(f_i - F_i)$ and $v \in V(f_j - F_j)$ with $|F_i| \leq 1$ and $|F_j| \leq 1$ for some i and j, where i and j may be the same. To prove the claim, we have to show that there is a path between vertices u and v.

Case 1: $i = j$. Since $|F_i| \leq 1$, subgraph induced by $f_i - F_i$ is connected. Thus, there is a path connecting u and v.

Case 2: $i \neq j$. Observed that the two g-edges between adjacent f-cycles corresponding to an edge in RG_{n-1}. To delete an edge in RG_{n-1}, each g-edge must have one vertex deleted. Thus, deleting an edge in RG_{n-1} requires deleting at least two vertices in TG_n. Since every vertex in TG_n is incident on exactly one g-edge, vertices in TG_n incident on different g-edge are different vertices. Thus, at most one incident edge of vertex f_i in RG_{n-1} is deleted. Similarly, at most one incident edge of vertex f_j in RG_{n-1} is deleted. Since $|F| \leq 7$, at most three edges in RG_{n-1} are deleted. Since $|F| \leq 7$, RG_{n-1} has at most three vertices (f-cycles) f_k such that $k \in I$. Since $k \in I$, $f_k \notin G1$. By Lemma 2.7 and $n \geq 4$, there is a path P in RG_{n-1} connecting f_i and f_j such that at most one vertex f_s where $s \neq i$ and $s \neq j$ in P has $|F_s| = 1$ and other vertices f_l where $l \neq i$ and $l \neq j$ in P have $|F_l| = 0$. Since $|F_s| \leq 1$, subgraph induced by $f_s - F_s$ is connected. Since $|F_i| \leq 1$ and $|F_j| \leq 1$, induced subgraphs $f_i - F_i$ and $f_j - F_j$ are connected. Thus, there exists a path connecting u and v in $G1$. This completes the proof of the Claim 1.

By Claim 1, if $|I| = 0$, then subgraph induced by $V(TG_n - F)$ is $G1$ which is a connected graph, contradicting that F is a vertex-cut set. Therefore, we have $|I| > 0$. Since $|F| \leq 7$, RG_{n-1} has at most three vertices (f-cycle) f_k such that $k \in I$. Thus, $1 \leq |I| \leq 3$. Let $G2$ be the subgraph of TG_n induced by $\bigcup_{i \in I} V(f_i - F_i)$.

Claim 2: Let T be a connected component of $G2$ containing at least a cycle. Then, T is connected to $G1$.

Proof of Claim 2: We prove the claim by contradiction. Assume that T is not connected to $G1$. First, we show that $N_{TG_n}(T) \subseteq F$. Clearly, $N_{TG_n}(T) \bigcap V(f_i) \subseteq F_i$ for $i \in I$ since T is a component of $G2$. Let $v \in N_{TG_n}(T) \bigcap V(f_i)$ for some $i \notin I$. Since $|F_i| \leq 1$ for $i \notin I$, $f_i - F_i$ is connected. If $v \notin F_i$, then $v \in G1$. This implies that T is connected to $G1$. So, we must have $v \in F_i$. Thus, $N_{TG_n}(T) \bigcap V(f_i) \subseteq F_i$ for $i \notin I$. Therefore, $N_{TG_n}(T) \subseteq F$. Since $|F| \leq 7$, we have the following condition:

$$|N_{TG_n}(T)| \leq 7.$$

In the following, we derive a contradiction that this condition can not be hold.

Let C be a cycle in T. Since $1 \leq |I| \leq 3$, we have the following three cases.

Case 1: C is a cycle in subgraph G induced by $V(f_i)$ for some $i \in I$. Since $|F_i| \geq 2$, the subgraph induced by $V(f_i - F_i)$ consists of paths or isolated vertices. Thus, T contains isolated vertex or paths in G. This implies that C can not be a cycle in G.

Case 2: Let G be the subgraph induced by $V(f_i) \cup V(f_j)$ for some $i \in I$, $j \in I$, and $i \neq j$. C is a cycle in G such that $V(C) \cap V(f_i) \neq \emptyset$ and $V(C) \cap V(f_j) \neq \emptyset$.

This implies that C is a cycle connecting two adjacent f-cycles. Since adjacent f-cycles are connected by exactly two g_a-edges for $1 \leq a \leq n$, the number of f-edges between the two adjacent g_a-edges is n by Lemma 2.9. By Lemma 2.12, C contains two paths of length $n-2$ where one of the paths is in f_i and the other one is in f_j. By Lemma 2.10, we can choose a path $P1 = u_1 u_2 \cdots u_{n-1}$ in f_i such that the g-edge incident on u_s, $1 \leq s \leq n-1$, is not a g_a-edge. By Lemma 2.11, the outside neighbors $u_1^{'}$, $u_2^{'}$, $\cdots$, $u_{n-1}^{'}$ are in $n-1$ different f-cycles. Let $Y1 = \{u_1^{'}, u_2^{'}, \cdots, u_{n-1}^{'}\}$ be the outside neighbor set of $P1$. Let $u_s^{'} \in Y1$. Since the g-edge incident on u_s is not a g_a-edge, we have $u_s^{'} \notin V(f_j)$. Clearly, outside neighbor $u_s^{'} \notin V(f_i)$. We have $Y1 \cap (f_i \cup f_j) = \emptyset$.

Now, we count the number of vertices in $Y1$ that belong to $N_{TG_n}(T)$. Let outside neighbor $u_s^{'} \in Y1$. Let outside neighbor $u_s^{'} \in V(f_k)$ for some $k \notin I$. Clearly, $u_s^{'} \in N_{TG_n}(T)$. Let $u_s^{'} \in V(f_k)$ for some $k \in I$ where $k \neq i$ and $k \neq j$. By Lemma 2.11 and $|I| \leq 3$, at most one of the outside neighbors, say $u_s^{'}$, belongs to $V(G2)$. Thus, at least $|Y1| - 1 = n-2$ outside neighbors are in $N_{TG_n}(T)$.

Similarly, let $P2 = v_1 v_2 \cdots v_{n-1}$ be the path in f_j such that the g-edge incident on v_s, $1 \leq s \leq n-1$, is not a g_a-edge. Let $Y2 = \{v_1^{'}, v_2^{'}, \cdots, v_{n-1}^{'}\}$ be the outside neighbor set of $P2$. By the same argument on path $P1$, we have $Y2 \cap (f_i \cup f_j) = \emptyset$ and at least $|Y2| - 1 = n-2$ outside neighbors are in $N_{TG_n}(T)$. By Lemma 2.12, $Y1 \cap Y2 = \emptyset$.

By Lemma 2.6, no f-cycle can be adjacent to both f_i and f_j since f_i and f_j are adjacent f-cycles. Thus, $Y1 \cup Y2$ contains at most one vertex in f_k for $k \in I$ where $k \neq i$ and $k \neq j$. Thus, we have at least $|Y1| + |Y2| - 1 = 2n-3$ outside neighbors in $N_{TG_n}(T)$.

Observed that subgraph GF induced by $V(f_i - F_i)$ with $|F_i| \geq 2$ consists of paths or isolated vertices. Since T contains a path or an isolated vertex of GF, $N_{f_i}(T) \subseteq F_i$ and $N_{f_i}(T) \geq 2$. By the same argument, $N_{f_j}(T) \subseteq F_j$ and $N_{f_j}(T) \geq 2$. Notice that $F_i \cap F_j = \emptyset$. Now, we have

$$\begin{aligned} &|N_{TG_n}(T)| \\ &\geq |Y1| + |Y2| - 1 + |N_{f_i}(T)| + |N_{f_j}(T)| \\ &\geq 2n - 3 + 2 + 2 = 2n + 1. \end{aligned}$$

From this, we have $|N_{TG_n}(T)| \geq 2n+1 \geq 9$ for $n \geq 4$. Since $|N_{TG_n}(T)| \leq 7$ by the assumption, this is a contradiction.

Case 3: Let G be the subgraph induced by $V(f_i) \cup V(f_j) \cup V(f_k)$ where f_i, f_j, and f_k are three distinct f-cycles and i, j, $k \in I$. Let C be a cycle in G such that $V(C) \cap V(f_i) \neq \emptyset$, $V(C) \cap V(f_j) \neq \emptyset$, and $V(C) \cap V(f_k) \neq \emptyset$. By Lemma 2.6, no cycle of length 3 in RG_{n-1} for $n \geq 4$. So, using the construction of C in the proof of Lemma 2.13, f_i

and f_j are not adjacent f-cycles, f_i and f_k are connected by two g_a-edges, and f_j and f_k are connected by two g_b-edges, where $a \neq b$ and $a, b \in \{1, 2, \cdots, n\}$. By Lemma 2.10, we can choose a path $P1 = u_1 u_2 \cdots u_{n-1}$ in f_i such that the g-edge incident on u_s, $1 \leq s \leq n-1$, is not a g_a-edge. By Lemma 2.11, the outside neighbors $u_1^{'}$, $u_2^{'}$, $\cdots$, $u_{n-1}^{'}$ are in $n-1$ different f-cycles. Let $Y1 = \{u_1^{'}, u_2^{'}, \cdots, u_{n-1}^{'}\}$ be the outside neighbor set of $P1$. Let $u_s^{'} \in Y1$. Since the g-edge incident on u_s is not a g_a-edge, we have $u_s^{'} \notin V(f_k)$. Clearly, outside neighbor $u_s^{'} \notin V(f_i)$. Since f_i and f_j are not adjacent f-cycles, $u_s^{'} \notin V(f_j)$. We have $Y1 \cap (f_i \cup f_j \cup f_k) = \emptyset$.

Now, we count the number of vertices in $Y1$ that belong to $N_{TG_n}(T)$. Let $u_s^{'} \in Y1$. Since $Y1 \cap (f_i \cup f_j \cup f_k) = \emptyset$, $u_s^{'} \notin G2$. By the same argument in Case 2, $u_s^{'} \in N_{TG_n}(T)$ if $u_s^{'} \in V(f_l)$ for $l \notin I$. Thus, all $|Y1| = n-1$ outside neighbors $u_s^{'}$ are in $N_{TG_n}(T)$.

Similarly, let $P2 = v_1 v_2 \cdots v_{n-1}$ be the path in f_j such that the g-edge incident on u_s, $1 \leq s \leq n-1$, is not a g_b-edge. Let $Y2 = \{v_1^{'}, v_2^{'}, \cdots, v_{n-1}^{'}\}$ be the outside neighbor set of $P2$. By the same argument on path $P1$ in this case, we have $Y2 \cap (f_i \cup f_j \cup f_k) = \emptyset$ and all $|Y2| = n-1$ outside neighbors $u_s^{'}$ are in $N_{TG_n}(T)$.

Notice that F_i, F_j, and F_k are pairwise disjoint and $|F_s| \geq 2$ for $s \in I$. By the same argument in Case 2, we have $N_{f_s}(T) \subseteq F_s$ and $N_{f_s}(T) \geq 2$ for $s \in I$. By Lemma 2.13, we have $Y1 \cap Y2 = \emptyset$. Now, we have

$$
\begin{aligned}
&|N_{TG_n}(T)| \\
&\geq |Y1| + |Y2| + |N_{f_i}(T)| + |N_{f_j}(T)| + |N_{f_k}(T)| \\
&\geq 2(n-1) + 6 = 2n + 4.
\end{aligned}
$$

From this, we have $|N_{TG_n}(T)| \geq 2n + 4 \geq 12$ for $n \geq 4$. Since $|N_{TG_n}(T)| \leq 7$ by the assumption, this is a contradiction.

This complete the proof of Claim 2.

Let $\widetilde{G1}$ be the connected component containing $G1$. From Claim 2, every connected component in $G2$ which has a cycle is connected to $G1$. Thus, every such connected component in $G2$ is in $\widetilde{G1}$. Therefore, $TG_n - F - \widetilde{G1}$ consists of acyclic connected components, contradicting to F is a cyclic vertex-cut set. This completes the proof. □

By considering all possible vertex subsets F and distributing of the vertices in F to the f-cycles, we can determine the values of the cyclic vertex connectivity $\kappa_c(TG_2)$ and $\kappa_c(TG_3)$ of TG_2 and TG_3, respectively.

The TG_2 has two f-cycles and eight vertices. It is easy to check that the cyclic vertex connectivity $\kappa_c(TG_2)$ does not exist.

The TG_3 has four f-cycles. The shortest cycle C of TG_3 is of length 6. The $N_{TG_3}(C)$ can be checked as a cyclic vertex-cut set of TG_3. Since $|N_{TG_3}(C)| = 6$ and any vertex subset F such that $1 \leq |F| \leq 5$ can not be checked as a cyclic vertex-cut set, the cyclic vertex connectivity $\kappa_c(TG_3) = 6$

5. Concluding Remarks

In this paper, the value of the cyclic vertex connectivity $\kappa_c(TG_n)$ for the trivalent Cayley graphs TG_n for $n \geq 4$ is determined and is a constant.

Without a proof, we also determine the values of the cyclic vertex connectivity $\kappa_c(TG_2)$ and $\kappa_c(TG_3)$ for TG_2 and TG_3, respectively.

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