LETTER *Special Section on Foundations of Computer Science — Frontiers of Theoretical Computer Science —*

Polynomial-Space Exact Algorithms for the Bipartite Traveling Salesman Problem

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SUMMARY Given an edge-weighted bipartite digraph $G = (A, B; E)$, the Bipartite Traveling Salesman Problem (BTSP) asks to find the minimum cost of a Hamiltonian cycle of G, or determine that none exists. When $|A| = |B| = n$, the BTSP can be solved using polynomial space in $O^*(4^{2n} n^{\log n})$ time by using the divide-and-conquer algorithm of Gurevich and Shelah (SIAM Journal of Computation, 16(3), pp.486–502, 1987). We adapt their algorithm for the bipartite case, and show an improved time bound of $O^*(4^{2n})$, saving the $n^{\log n}$ factor.

key words: *bipartite traveling salesman problem, exact algorithms, polynomial space, divide-and-conquer, Stirling's formula*

1. Introduction

Given an edge-weighted bipartite digraph $G = (A, B; E)$, we are interested in finding the minimum cost of a Hamiltonian cycle in G, or determine that none exists. We call this problem Bipartite Traveling Salesman Problem (BTSP), akin to the well-known Traveling Salesman Problem (TSP).

A Hamiltonian cycle must visit the vertices in A and B alternately, and obviously cannot exist unless $|A| = |B|$. Henceforth, let $|A| = |B| = n$, which means that G is a graph on $2n$ vertices. A straightforward reduction from the TSP tells us that the BTSP is also NP-hard, and previous works in the literature mainly report approximation algorithms [5], or exact solutions for special cases of the BTSP [2].

In exponential algorithms, the O^* notation suppresses polynomial factors. Exponential-time algorithms which also need exponential space are highly impractical, and recently developing exact algorithms which run in polynomial space, as well as improving their time bounds have gathered attention [1]. Gurevich and Shelah [3] have shown that the TSP in a k-vertex digraph is solvable in $O^*(4^k k^{\log k})$ time and polynomial space, by giving a divide-and-conquer algorithm. Their algorithm actually solves the Hamiltonian path problem for fixed terminals, and the TSP can be solved by calling this algorithm a polynomial number of times. In the "divide" step, the algorithm investigates all possible balanced bipartitions of the graph's vertex set, by creating $O(k2^k)$ sub-instances of $\lceil k/2 \rceil$ vertices.

Applying this algorithm to the BTSP where $|A| = |B| = n$, gives an $O^*(4^{2n} n^{\log n})$ time bound. However, in the bipartite setting, it is evident that not all possible balanced bipartitions of the vertex set $A \cup B$ yield feasible sub-instances. Based on this insight, we propose that instead of investigating all balanced bipartitions of the vertex set $A \cup B$, we investigate balanced bipartitions on each of the sets A and B individually, and state the following claim.

Theorem 1: Given an edge-weighted bipartite digraph $G = (A, B; E)$ where $|A| = |B| = n$, a minimum cost Hamiltonian cycle in G, if one exists, can be computed in $O^*(4^{2n})$ time and polynomial space.

To achieve a refined analysis on the time bound, we show that for a set of n elements, not more than $2^n/\sqrt{n}$ subsets need to be taken to yield all balanced bipartitions, given as the following claim.

Lemma 2: For any positive integer n, it holds that $\max_{k\in\{0,1,\ldots,n\}} \binom{n}{k} \le \binom{n}{\lfloor n/2 \rfloor} \le 2^n/\sqrt{n}$.

Proof. Since $\sqrt{2\pi n}\cdot(n/e)^n \le n! \le e\cdot\sqrt{n}\cdot(n/e)^n$ by Stirling's formula [6], we see that for an even integer $n = 2\ell$,

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{2\ell}{\ell} = \frac{(2\ell)!}{\ell!\cdot\ell!} \le \frac{e\sqrt{2\ell}(2\ell/e)^{2\ell}}{\pi 2\ell(\ell/e)^{2\ell}} \le \frac{2^{2\ell}}{\sqrt{2\ell}}.$$

From this, we see that for an odd integer $n = 2\ell+1$,

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{2\ell+1}{\ell} = \frac{1}{2}\binom{2\ell+2}{\ell+1} \le \frac{2^{2\ell+2}}{2\sqrt{2\ell+2}} \le \frac{2^{2\ell+1}}{\sqrt{2\ell+1}}.$$

□

Different proofs of Lemma 2 for an even integer n can be found elsewhere in the literature, e.g., Matoušek and Nešetřil [4].

2. Algorithm and Analysis

Let $\mathbb{R}$ denote the set of real numbers. Henceforth let $G = (A, B; E)$ be a bipartite digraph such that $|A| = |B|$, and $w : E \to \mathbb{R}$ be an edge weight function, where an edge with a tail u and a head v is denoted by (u, v) and the weight $w(e)$ of an edge $e = (u, v)$ is also written as $w(u, v)$. A path, or a v_1, v_k-path is defined to be a graph with a vertex set $\{v_1, v_2, \ldots, v_k\}$ and an edge set $\{(v_i, v_{i+1}) \mid i = 1, 2, \ldots, k-1\}$,

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which we denote by $P = (v_1, v_2, \dots, v_k)$ and whose cost $w(P)$ is defined to be $\sum_{i=1}^{k-1} w(v_i, v_{i+1})$. Let $A' \subseteq A$ and $B' \subseteq B$ be subsets. We call a path in G that contiguously and alternately visits all vertices in $A' \cup B'$, *A', B'-alternating*. Define $\mathrm{OPT}(A', B', x, y)$ to be the minimum cost $w(P)$ of an A', B'-alternating x, y-path P in G, and let $\mathrm{OPT}(A', B', x, y) = \infty$ if such a path does not exist. We easily observe that for any two vertices $x \in A'$ and $y \in B'$, the following property holds

$$\begin{aligned}
&\mathrm{OPT}(A',\ B', x, y)\\
&\quad = \min\{\mathrm{OPT}(A_1, B_1, x, u) + w(u, v) + \mathrm{OPT}(A_2, B_2, v, y) \mid\\
&\qquad\quad x \in A_1 \subseteq A', |A_1| = \lceil |A'|/2 \rceil, A_2 = A' \setminus A_1,\\
&\qquad\quad y \notin B_1 \subseteq B', |B_1| = \lceil |B'|/2 \rceil, B_2 = B' \setminus B_1,\\
&\qquad\quad (u, v) \in E, u \in B_1, v \in A_2\}. \qquad (1)
\end{aligned}$$

Equation (1) gives an obvious way of computing the minimum cost of a Hamiltonian cycle in G; we only need to evaluate $\mathrm{OPT}(A, B, x, y)$ for an arbitrary $x \in A$ and each of its $O(n)$ neighbors $y \in B$. As base case, the value of $\mathrm{OPT}(A', B', x, y)$ can be evaluated in constant time for any sets A', B' of fixed size. Hence, we give a recursive procedure to compute $\mathrm{OPT}(A', B', x, y)$ for any subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = |B'|$, and vertices $x \in A'$ and $y \in B'$, as Recursive Procedure BTSP-P(A', B', x, y).

Recursive Procedure BTSP-P(A', B', x, y)

Input: Two vertex sets $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| = |B'|$, and two vertices $x \in A'$ and $y \in B'$.
Output: The minimum cost of an A', B'-alternating x, y-path, and ∞ if such a path does not exist.

1: **if** $|A'| = |B'| \le 2$ **then**
2: **return** $\mathrm{OPT}(A', B', x, y)$
3: **else** /* $|A'| = |B'| \ge 3$ */
4: cost := ∞; $n_1 := \lceil A'/2 \rceil$;
5: **for each** pair $(A_1 \subseteq A', B_1 \subseteq B')$ such that $|A_1| = |B_1| = n_1$, $x \in A_1$, and $y \notin B_1$ **do**
6: $A_2 := A' \setminus A_1$; $B_2 := B' \setminus B_1$;
7: $\mathrm{cost}_1[u] :=$ BTSP-P(A_1, B_1, x, u) for each $u \in B_1$;
8: $\mathrm{cost}_2[v] :=$ BTSP-P(A_2, B_2, v, y) for each $v \in A_2$;
9: **for each** edge $(u, v) \in E$ with $u \in B_1$ and $v \in A_2$ **do**
10: cost := $\min\{\mathrm{cost}, \mathrm{cost}_1[u] + w(u, v) + \mathrm{cost}_2[v]\}$
11: **end for**
12: **end for**;
13: **return** cost
14: **end if**.

Lemma 3: Given vertex subsets $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| = |B'| = n$, the time complexity of Recursive Procedure BTSP-P is $O^*(4^{2n})$.

Proof. Let $T(n)$ be the number of recursive sub-calls of Recursive Procedure BTSP-P, each of which takes time polynomial in n. For $n \le 4$ the procedure finishes in polynomial time, and we proceed under the assumption that $n \ge 5$. Since the two terminal vertices x and y are fixed, there are $\binom{n-1}{\lceil n/2 \rceil}$ choices for the pair of subsets $A_1 \subseteq A'$ and $B_1 \subseteq B'$ in Line 5, and for each choice, the procedure is recursively called for each $u \in B_1$ and $v \in A_2 = A' \setminus A_1$, for which there are $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ candidates, respectively, from which we get

$$T(n) \le \binom{n-1}{\lceil n/2 \rceil}^2 (\lceil n/2 \rceil \cdot T(\lceil n/2 \rceil) + \lfloor n/2 \rfloor \cdot T(\lfloor n/2 \rfloor)). \quad (2)$$

To show the claim, it suffices to show that

$$T(n) \le n \cdot 4^{2n} \quad (3)$$

satisfies Eq. (2) for all $n \ge 5$. Substituting Eq. (3) into Eq. (2), we obtain from Lemma 2 that

$$\begin{aligned}
T(n) &\le \left(\frac{2^{n-1}}{\sqrt{n-1}}\right)^2 \cdot 2 \cdot \left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}\right) \cdot 4^{n+1}\\
&= n4^{2n} \cdot \frac{(n+1)^2}{2n(n-1)} \le n4^{2n} \quad (\text{by } n \ge 5),
\end{aligned}$$

as required. □

Note that Recursive Procedure BTSP-P can be implemented to use polynomial space in the size of the given subsets A' and B' locally at each call, and the depth of the recursion is not more than $O(\log |A'|)$, and therefore the entire space needed for a given input is at most polynomial. By Lemma 3 and the fact that Recursive Procedure BTSP-P can be used as a sub-procedure to develop an algorithm for the BTSP by calling it a polynomial number of times, we conclude a proof of Theorem 1. Note that since the O^* notation suppresses polynomial factors, the claim holds both for random access and log-cost access models.

It is an interesting question whether an improved analysis on a similar divide-and-conquer approach, especially introducing a non-trivial measure [1], can yield an improved bound on the time complexity for some special class of BTSP instances.

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