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The Explicit Formula of the Presumed Optimal Recurrence Relation for the Star Tower of Hanoi

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SUMMARY In this paper, we show the explicit formula of the recurrence relation for the Tower of Hanoi on the star graph with four vertices, where the perfect tower of disks on a leaf vertex is transferred to the central vertex. This gives the solution to the problem posed at the 17th International Conference on Fibonacci Numbers and Their Applications [11]. Then, the recurrence relation are generalized to include the ones for the original 4-peg Tower of Hanoi and the Star Tower of Hanoi of transferring the tower from a leaf to another.

key words: *Tower of Hanoi, four pegs, star graph, Frame-Stewart algorithm, recurrence relation*

1. Introduction

The Tower of Hanoi puzzle was invented by French mathematician E. Lucas in 1883 [15]. The original puzzle has 3 pegs with a tower of n disks of different sizes initially piled on one peg in decreasing order from the bottom. The purpose of the puzzle is to transfer all the disks from the initial peg to the other in the minimum number of steps, under the condition that at each step one of the topmost disks is transferred to another peg not to put a disk on a smaller one. This simple mathematical puzzle was then extended to use 4 pegs by Dudeney in 1907 and was called the Reve's puzzle [7]. The original three-peg puzzle is easily solved, but somewhat surprisingly, any algorithm for the Reve's puzzle was not shown to be optimal for more than 100 years. For the general Tower of Hanoi using k ($\geq$ 4) pegs, Frame [8] and Stewart [18] independently presented recursive algorithms that produce the same number of steps for transferring n disks; thus, they are generically called the *Frame-Stewart algorithm*. When $k = 4$, it uses the following procedure:

1. transfer the topmost $n - m$ disks of the initial peg to an intermediate peg using 4 pegs;
2. transfer the larger m disks from the initial peg to the final (destination) peg using 3 pegs;
3. finally, transfer the $n - m$ disks from the intermediate peg to the final peg using 4 pegs.

The algorithm chooses the integer m ($0 < m \leq n$) such that the total number of steps of the above procedure is minimized. Let $H(n)$ be the number of steps for transferring the n disks. Then, $H(n)$ satisfies the following recurrence relation: $H(0) = 0$ and for $n \geq 1$,

$$H(n) = \min_{0<m\leq n} \{2 \cdot H(n - m) + 2^m - 1\}. \tag{1}$$

The explicit formula of $H(n)$ and its various properties have been obtained [12]–[14]. Since the values of $H(n)$'s (and those for general k pegs) are optimal at least by computer search, it had been believed to be optimal and was called the Frame-Stewart conjecture. As for the lower bounds, there had been little progress until Szegedy gave non-trivial lower bounds [20]. Then, there had been further improvement [6], [9]. Recently, Bousch [3] finally proved the Frame-Stewart conjecture for the case of 4 pegs in the affirmative way, that is, the solution of the Frame-Stewart algorithm is optimal for the case of 4 pegs.

Along with the original Tower of Hanoi, there is a Tower of Hanoi variant that limits the pairs of pegs to be used for transferring disks. The problem can be regarded as the Tower of Hanoi on graphs, by assigning each of the pegs to a different vertex and allowing the transfer of disks only when there is an edge between the pair of vertices. This variant has been studied for both undirected and directed graphs. For example, the original Tower of Hanoi with k ($\geq$ 3) pegs is regarded as the problem on the complete graphs with k vertices. The Tower of Hanoi on graphs have been studied on cyclic graphs, path graphs, star graphs, etc. [1], [2], [5], [12], [16], [17], [19].

The Tower of Hanoi on star graphs (Fig. 1), which is called the *Star Tower of Hanoi* [12], is the main topic of this paper. The Star Tower of Hanoi has the following two types of problems. The first one is to transfer all the disks on one leaf of the star graph to another leaf. The second one is to transfer the disks from a leaf to the center of the graph. The first type of problem with 4 pegs was studied by Stockmeyer [19], in which he gave a Frame-Stewart-type recursive algorithm. Then, Bousch [4] again proved its optimality. The algorithm was further generalized for the case of k ($\geq$ 4) pegs and its explicit solution was obtained in [5]. The second type of Star Tower of Hanoi problem is to transfer the n disks from one leaf to the center. A Frame-Stewart-type algorithm can be also designed in this case and its solution was checked to be optimal up to $n = 15$ by computer search [10], [11]; thus, called to be "presumed optimal". However, neither of the explicit solution nor its optimality are shown yet and are posed as open problems [11].

In this paper, we make exact analysis of this recurrence

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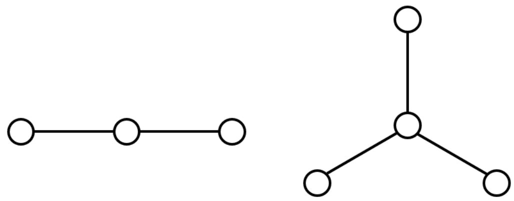

(a) (b)

Fig. 1 Star graphs with (a) 3 vertices; and (b) 4 verticees.

relation. The key step is to make a recursive definition of an integer sequence that is exactly the sequence of differences of the target Tower of Hanoi. Then, we prove its correctness by induction. We further generalize and analyze the recurrence relation to cover the problems of the original 4-peg Tower of Hanoi, i.e., the Reve's puzzle, and the leaf-to-leaf Star Tower of Hanoi.

This paper is organized as follows: In Sect. 2, we summarize the Star Tower of Hanoi problems with 3 pegs and also 4 pegs in the leaf-to-leaf case. In Sect. 3, we first summarize the Frame-Stewart-type algorithm for the Star Tower of Hanoi in the leaf-to-center case, define an integer sequence for the differences, and then state the main theorem. In Sect. 4, the proof of the main theorem is shown. In Sect. 5, a generalized recurrence relation is analyzed and finally, concluding remarks are given in Sect. 6.

2. Preliminaries

2.1 The Star Tower of Hanoi with 3 Pegs

The star graph with 3 vertices is shown in Fig. 1 (a), which is also regarded as the path graph. On this graph, the following two kinds of Tower of Hanoi problems are considered. Given n disks on a leaf peg, we transfer all the disks from the initial leaf to another, which is denoted as *leaf-to-leaf*, or to the center peg, which is denoted as *leaf-to-center*. Note that the leaf-to-center problem is equivalent to the problem of transferring the disks in the opposite direction, that is, from the center to a leaf. The optimal algorithms and their analysis are rather straightforward, but we summarize them as basis for later analysis.

For the leaf-to-leaf problem with 3 pegs, (i) we first move the smaller $n-1$ disks to another leaf (destination); (ii) then move the largest disk to the center; (iii) move the $n-1$ disks back to the initial leaf; (iv) move the largest disk to the destination leaf; and (v) finally, move the $n-1$ disks to the destination. Let $S_3(n)$ be the number of steps of this recursive algorithm. Then, $S_3(n)$ satisfies the following recurrence relation:

$$S_3(0) = 0,\ S_3(n) = 3S_3(n-1) + 2 \text{ for } n \geq 1.$$

Therefore, $S_3(n) = 3^n - 1$ and $\Delta S_3(n) = S_3(n) - S_3(n-1) = 2 \cdot 3^{n-1}$ for $n \geq 1$.

For the leaf-to-center problem, (i) we first move the smaller $n-1$ disks to another leaf with the algorithm for the leaf-to-leaf problem (with $S_3(n-1)$ moves); (ii) move the largest disk to the center; and (iii) finally, move the $n-1$ disks to the center. The recurrence relation of this algorithm is the following:

$$T_3(0) = 0,\ T_3(n) = S_3(n-1) + 1 + T_3(n-1) \text{ for } n \geq 1.$$

It is simplified as $T_3(n) = T_3(n-1) + 3^{n-1}$, so $T_3(n) = \frac{3^n-1}{2}$ and $\Delta T_3(n) = T_3(n) - T_3(n-1) = 3^{n-1}$ for $n \geq 1$.

We will use these results in the Star Tower of Hanoi problems with 4 pegs.

2.2 The 4-Peg Star Tower of Hanoi: Leaf-to-Leaf Case

In this section, we summarize the results on the Tower of Hanoi on the star graph in Fig. 1 (b) in the case that the n disks are transferred from one leaf to another [19].

Let $S_4(n)$ be the number of steps of the recursive algorithm to be stated. The task of transferring n disks is achieved by the following procedure:

1. transfer the topmost $n-m$ disks of one leaf to a non-final leaf using 4 pegs;
2. transfer the larger m disks from the initial leaf to the final leaf using 3 pegs;
3. finally, transfer the $n-m$ disks from the non-final leaf to the final leaf using 4pegs.

Similarly to the original Frame-Stewart algorithm, the algorithm chooses the integer m $(0 < m \leq n)$ such that the total number of steps of the above procedure is minimized. Then, the following recurrence relation holds: $S_4(0) = 0$ and for $n \geq 1$,

$$\begin{aligned} S_4(n) &= \min_{0<m\leq n} \{2 \cdot S_4(n-m) + S_3(m)\} \\ &= \min_{0<m\leq n} \{2 \cdot S_4(n-m) + (3^m - 1)\}. \end{aligned} \quad (2)$$

Stockmeyer [19] obtained the following explicit formulas:

$$m_{\min} = \lfloor \log_3 s_n \rfloor + 1,$$
$$\Delta S_4(n) = S_4(n) - S_4(n-1) = 2s_n,$$
$$S_4(n) = 2\sum_{i=1}^{n} s_i,$$

where $m_{\min}$ is the value of m that minimizes the recurrence relation for $S_4(n)$ and

$$s_n = 1, 2, 3, 4, 6, 8, 9, 12, \cdots$$

is the sequence of 3-smooth numbers $2^j \cdot 3^k$, $j, k \geq 0$ lined in increasing order. To summarize:

Lemma 1: For the 4-peg Star Tower of Hanoi of transferring n disks from one leaf peg to another, the Frame-Stewart-type recursive algorithm uses $S_4(n) = 2\sum_{i=1}^{n} s_i$ steps to achieve the task, where $\{s_n\}$ is the sequence of 3-smooth numbers.

3. The 4-Peg Star Tower of Hanoi: Leaf-to-Center Case

3.1 Frame-Stewart-Type Algorithm and Its Recurrence

We first state the Frame-Stewart-type algorithm and the corresponding recurrence relation for the Star Tower of Hanoi of the leaf-to-center case. Let $T_4(n)$ be the number of steps used by the algorithm. The procedure of the algorithm is the following:

1. transfer the topmost $n-m$ disks of one leaf to another leaf using 4 pegs;
2. transfer the larger m disks from the initial leaf to the center (destination) using 3 pegs;
3. finally, transfer the $n-m$ disks from the intermediate leaf to the center using 4 pegs.

As before, the algorithm chooses the integer m ($0 < m \leq n$) such that the total number of steps for the procedure is minimized. Then, the following recurrence relation holds: $T_4(0) = 0$ and for $n \geq 1$,

$$\begin{aligned} T_4(n) &= \min_{0<m\leq n} \{S_4(n-m) + T_3(m) + T_4(n-m)\} \\ &= \min_{0<m\leq n} \left\{ T_4(n-m) + 2\sum_{i=1}^{n-m} s_i + \frac{3^m-1}{2} \right\}, \qquad (3) \end{aligned}$$

where Eq. (3) holds due to the results in Sect. 2. The values of $T_4(n)$, the differences $\Delta T_4(n) = T_4(n) - T_4(n-1)$, and the argument $m_{\min}$ at which the equation is minimized are shown in Table 1 for $1 \leq n \leq 10$. Here we note that the recurrence takes the minimum at two values of $m_{\min}$ at $n = 2, 5$ (then, at $n = 15$). In such cases, if we choose the larger values for $m_{\min}$, that is, if we regard $m_{\min} = 2$ for $n = 2$ and $m_{\min} = 3$ for $n = 5$, then we observe that up to $n = 10$, the following equality holds:

$$m_{\min} = \lfloor \log_3 \Delta T_4(n) \rfloor + 1. \qquad (4)$$

Furthermore, the sequence $\{\Delta T_4(n)\}$ consists of the terms of $\{\Delta T_4(n) + \Delta S_4(n)\}$ and $\{\Delta T_3(n)\}$ arranged in increasing order in the weak sense that there can be identical values. However, we should be careful for these computations because they are self-referential and in order to compute $\Delta T_4(n)$, the value of $m_{\min}$ itself is needed. By computer experiments, values of $T_4(n)$'s are confirmed to be optimal up to $n = 15$ [10], [11], but the exact analysis of the recurrence relation has not been done yet. In the next section, we present an integer sequence that coincides with the sequence $\{\Delta T_4(n)\}$.

Table 1 Values of $m_{\min}$, $T_4(n)$ and $\Delta T_4(n)$ for $1 \leq n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$m_{\min}$	1	1, 2	2	2	2, 3	3	3	3	3	4
$T_4(n)$	1	4	7	14	23	32	47	68	93	120
$\Delta T_4(n)$	1	3	3	7	9	9	15	21	25	27

3.2 Explicit Solutions for $\Delta T_4(n)$ and $T_4(n)$

From now on, we regard integer sequences as a multiset which allows the existence of multiple identical integers as elements. Instead of $\{\cdot\}$, $[\cdot]$ is used to denote a multiset.

Now we make definitions of a family of multisets A_i for $i \geq 0$ and an integer sequence $\{t_i\}_{i\geq 1}$, which is the candidate for $\{\Delta T_4(n)\}$. Let $\{s_n\}_{n\geq 1}$ be the sequence of 3-smooth numbers $2^j \cdot 3^k$ for $j, k \geq 0$. A_i's and t_i's are defined recursively as follows:

1. $A_0 = [3^n]_{n\geq 0} = [1, 3, 3^2, 3^3, \cdots]$ and $t_1 = \min A_0 = 1$, where $\min A_0$ is the minimum integer in A_0.
2. For $i \geq 1$, the multiset A_i is defined as
$$A_i = A_{i-1} \setminus \min A_{i-1} \cup [t_i + 2s_i],$$
where $\min A_{i-1}$ is the minimum integer in A_{i-1}. Then, t_{i+1} is defined as $t_{i+1} = \min A_i$.

Here we note that subtraction of $\min A_{i-1}$ from the multiset A_{i-1} is done only once and the other identical integers, if any, remain in $\min A_{i-1}$.

Example 1: We show some small values of A_n's and t_n's.

$$\begin{aligned} A_1 &= A_0 \setminus \min A_0 \cup [t_1 + 2s_1] = [3^n]_{n\geq 0} \setminus [1] \cup [2+1] \\ &= [3, 3, 3^2, 3^3, 3^4, \cdots]. \end{aligned}$$

So, $t_2 = \min A_1 = 3$. It implies $t_i + 2s_i$ can be a power of 3.

$$\begin{aligned} A_2 &= A_1 \setminus \min A_1 \cup [t_2 + 2s_2] = [3^n]_{n\geq 1} \setminus [3] \cup [4+3] \\ &= [3, 7, 3^2, 3^3, 3^4, \cdots]. \end{aligned}$$

So, $t_3 = \min A_2 = 3$.

$$\begin{aligned} A_3 &= A_2 \setminus \min A_2 \cup [t_3 + 2s_3] = A_2 \setminus [3] \cup [6+3] \\ &= [7, 3^2, 3^2, 3^3, 3^4, \cdots]. \end{aligned}$$

So, $t_4 = \min A_3 = 7$.

The values of t_n's for $1 \leq n \leq 10$ are shown in Table 2. Compared with Table 1, it is observed that the values of $\{\Delta T_4(n)\}$ and $\{t_n\}$ are exactly the same in these cases. We justify this observation by the following theorem.

Theorem 1: Suppose that $\{t_n\}_{n\geq 1}$ is the aforementioned integer sequence. Then, the following equality holds.

$$\Delta T_4(n) = t_n \text{ for } n \geq 1.$$

Therefore, $T_4(n)$ is computed as

$$T_4(n) = \sum_{i=1}^{n} t_i \text{ for } n \geq 1.$$

Table 2 Values of t_n for $1 \leq n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
t_n	1	3	3	7	9	9	15	21	25	27

4. Proof of Theorem 1

We prove $\Delta T_4(n) = t_n$ for $n \geq 1$ by induction on n.

When $n = 1$, $T_4(1) = t_1 = 1$ holds.

Suppose that $\Delta T_4(n) = t_n$ holds for $1 \leq n \leq k-1$. Then we show that $\Delta T_4(k) = t_k$. For this purpose, we first define the logarithm of t_k similarly to Eq. (4):

$$m_n = \lfloor \log_3 t_n \rfloor + 1. \tag{5}$$

As opposed to Eq. (4), m_n can be computed without self-reference since t_n is explicitly defined. We show one lemma on the relation between sequences $\{m_n\}$ and $\{t_n\}$.

Lemma 2: Let $\{m_n\}$, $\{t_n\}$, $\{s_n\}$ be the aforementioned integer sequences. Then for $n \geq 2$, m_n is expressed either as $m_n = m_{n-1} + 1$ or $m_n = m_{n-1}$. Furthermore, t_n is explicitly written as follows:

$$\begin{cases} \text{(i) When } m_n = m_{n-1} + 1, \ t_n = 3^{m_n - 1}. \\ \text{(ii) When } m_n = m_{n-1}, \ t_n = t_{n-m_n} + 2s_{n-m_n}. \end{cases}$$

Proof: The value t_n is by definition the nth smallest value in the multiset

$$[3^i]_{i\geq 0} \cup [t_i + 2s_i]_{1\leq i\leq n-1}.$$

We first divide the type of values of t_n's into three cases.

Case 1. When $t_n = 3^l$ for some l and when 3^l appears as an element of the sequence $\{t_i\}$ for the first time, then $3^{l-1} \leq t_{n-1} < t_n = 3^l$. Thus, $l - 1 \leq \log_3 t_{n-1} < \log_3 t_n = l$. Recall that $m_n = \lfloor \log_3 t_n \rfloor + 1$, so $l - 1 \leq m_{n-1} - 1 < m_n - 1 = l$. Therefore,

$$m_n = m_{n-1} + 1 \text{ and } t_n = 3^l = 3^{m_n - 1}.$$

Case 2. When $t_n = 3^l$ for some l but when it is not the first time for 3^l to appear in the sequence $\{t_n\}$, then $t_{n-1} = t_n = 3^l$. Therefore,

$$m_n = m_{n-1} = l + 1 \text{ and } t_n = 3^l = 3^{m_n - 1}.$$

Case 3. When $t_n = t_j + 2s_j$ for some j such that $1 \leq j \leq n-1$ and when t_n is not a power of 3, then t_n is bounded as $3^l < t_n < 3^{l+1}$ for some l. Then, t_{n-1} must be also bounded in the same interval as $3^l \leq t_{n-1} \leq t_n < 3^{l+1}$ since at least one 3^l must exist in $\{t_i\}_{i\leq n-1}$. Therefore,

$$m_n = m_{n-1} = l + 1.$$

Next we find the index j such that $t_n = t_j + 2s_j$. Since n integers $t_1, \cdots, t_n$ are chosen from the multiset $[3^i]_{i\geq 0} \cup [t_i + 2s_i]_{1\leq i\leq n-1}$ and since $3^{m_n - 1} < t_n < 3^{m_n}$, exactly m_n smallest 3^i's in $[3^n]_{n\geq 0}$, i.e., $\{1, 3, 3^2, \cdots, 3^{m_n - 1}\}$, are chosen and used in $\{t_i\}_{1\leq i\leq n-1}$. Since the remaining elements for $\{t_1, \cdots, t_n\}$ are chosen from $\{t_i + 2s_i\}$, $t_i + 2s_i$'s should appear $n - m_n$ times at the time of determining t_n, yielding $j = n - m_n$. Thus,

$$t_n = t_j + 2s_j = t_{n-m_n} + 2s_{n-m_n}.$$

This completes Case 3.

Now, we prove (i) and (ii) of the lemma using the results of Cases 1, 2, and 3. As for (i), since Cases 2 and 3 both satisfy $m_n = m_{n-1}$, when $m_n = m_{n-1} + 1$, Case 1 of $t_n = 3^{m_n - 1}$ holds and (i) of Lemma 2 is proved.

When $m_n = m_{n-1}$ of (ii) holds, by Cases 2 and 3, there are two kinds of expressions for t_n, that is, $t_n = 3^l$ for some l and $t_n = t_{n-m_n} + 2s_{n-m_n}$. But in Case 2, since 3^l appears at least twice, $t_{n-1} = t_n = 3^l$. So, this 3^l can be also written as $t_{n-m_n} + 2s_{n-m_n}$. This shows that when $m_n = m_{n-1}$, t_n can be expressed as $t_n = t_{n-m_n} + 2s_{n-m_n}$ as claimed in (ii).

This completes the proof of Lemma 2. □

Next, we show at which m the inner function of $T_4(k)$ takes the minimum.

Lemma 3: Under the assumption of induction, the function

$$f(m) = T_4(k-m) + S_4(k-m) + \frac{3^m - 1}{2}$$

takes the minimum at $m = m_k$.

Proof: When $i < m_k$,

$$\begin{aligned} & f(i+1) - f(i) \\ &= \left(T_4(k-(i+1)) + S_4(k-(i+1)) + \frac{3^{i+1}-1}{2}\right) \\ &\quad - \left(T_4(k-i) + S_4(k-i) + \frac{3^i - 1}{2}\right) \\ &= 3^i - (\Delta T_4(k-i) + \Delta S_4(k-i)) \\ &= 3^i - (t_{k-i} + 2s_{k-i}) \text{ (by Assumption)} \end{aligned} \tag{6}$$

By the definition of t_n and by Lemma 2, the sequence $\{t_1, t_2, \cdots, t_k\}$ consists of the k smallest numbers

$$[1, 3, \cdots, 3^{m_k - 1}] \cup [t_1 + 2s_1, \cdots, t_{k-m_k} + 2s_{k-m_k}]$$

in the multiset $[3^j]_{j\geq 0} \cup [t_j + 2s_j]_{1\leq j\leq k-1}$. This with $k - i > k - m_k$ implies that $t_{k-i} + 2s_{k-i}$ is larger or equal to both $t_{k-m_k} + 2s_{k-m_k}$ and $3^{m_k - 1}$ ($\geq 3^i$). Therefore, Eq. (6) leads to $f(i+1) - f(i) = 3^i - (t_{k-i} + 2s_{k-i}) \leq 0$ and $f(m)$ is monotonically decreasing for $m \leq m_k$.

When $i \geq m_k$, since $k - i \leq k - m_k$,

$$\begin{aligned} & f(i+1) - f(i) \\ &= 3^i - (\Delta T_4(k-i) + \Delta S_4(k-i)) \\ &= 3^i - (t_{k-i} + 2s_{k-i}) \text{ (by Assumption)} \\ &\geq 3^i - (t_{k-m_k} + 2s_{k-m_k}) \\ &\geq 0. \end{aligned} \tag{7}$$

Inequality (7) holds because the multiset that consists of the n smallest values in the multiset $[3^i]_{i\geq 0} \cup [t_i + 2s_i]_{1\leq i\leq k-1}$ is

$$[3^i]_{0\leq i\leq m_k - 1} \cup [t_i + 2s_i]_{1\leq i\leq k-m_k}$$

by the definition of t_n and by Lemma 2. Therefore, $f(m)$ is monotonically increasing for $m \geq m_k$.

Consequently, $f(m)$ takes the minimum at $m = m_k$. □

Now, we are ready to show that $\Delta T_4(k) = t_k$ under the assumption of induction. First, by Lemma 3, $\Delta T_4(k)$ is computed as follows:

$$\begin{aligned}
\Delta T_4(k) &= T_4(k) - T_4(k-1) \\
&= \left(T_4(k-m_k) + S_4(k-m_k) + \frac{3^{m_k}-1}{2}\right) - \\
&\quad \left(T_4(k-1-m_{k-1}) + S_4(k-1-m_{k-1}) + \frac{3^{m_{k-1}}-1}{2}\right) \\
&= (T_4(k-m_k) - T_4(k-1-m_{k-1})) \\
&\quad + (S_4(k-m_k) - S_4(k-1-m_{k-1})) \\
&\quad + \left(\frac{3^{m_k}-3^{m_{k-1}}}{2}\right).
\end{aligned} \tag{8}$$

For further computation, we divide into two cases according to the relation of m_k and m_{k-1} in Lemma 2.

Case 1. For k such that $m_k = m_{k-1} + 1$, $t_k = 3^{m_k - 1}$ by Lemma 2(i). Then, the values in the brackets of Eq. (8) are computed respectively as follows:

$$\begin{aligned}
&T_4(k-m_k) - T_4(k-1-m_{k-1}) \\
&= T_4(k-m_k) - T_4(k-1-(m_k-1)) = 0, \\
&S_4(k-m_k) - S_4(k-1-m_{k-1}) = 0, \\
&\frac{3^{m_k}-3^{m_{k-1}}}{2} = \frac{3^{m_k}-3^{m_k-1}}{2} = 3^{m_k-1}.
\end{aligned}$$

Therefore, by Lemma 2(i), Eq. (8) is simplified as

$$\Delta T_4(k) = 3^{m_k-1} = t_k.$$

Case 2. For k such that $m_k = m_{k-1}$, $t_k = t_{k-m_k} + 2s_{k-m_k}$ by Lemma 2(ii). Then, the values in brackets of Eq. (8) are computed respectively as follows:

$$\begin{aligned}
&T_4(k-m_k) - T_4(k-1-m_{k-1}) \\
&= T_4(k-m_k) - T_4(k-1-m_k) \\
&= \Delta T_4(k-m_k) \\
&= t_{k-m_k} \quad \text{(by Assumption)}, \\
&S_4(k-m_k) - S_4(k-1-m_{k-1}) \\
&= \Delta S_4(k-m_k) \\
&= 2\, s_{k-m_k} \quad \text{(by Lemma 1)}, \\
&\frac{3^{m_k}-3^{m_{k-1}}}{2} = \frac{3^{m_k}-3^{m_k}}{2} = 0.
\end{aligned}$$

Therefore, by Lemma 2(ii), Eq. (8) is simplified as

$$\Delta T_4(k) = t_{k-m_k} + 2s_{k-m_k} = t_k.$$

This completes the proof of Theorem 1. □

5. Generalized Recurrence Relation and Its Analysis

The recurrence relation for the 4-peg Star Tower of Hanoi and its analysis in Sects. 3 and 4 are generalized to include the cases of the original 4-peg Tower of Hanoi, i.e., the Reve's Puzzle, and the leaf-to-leaf Star Tower of Hanoi. Namely, we define a generalized recurrence relation as follows: Let $\{H(n)\}_{n\geq 0}$ be an integer sequence such that $H(0) = 0$ and that $h_n = \Delta H(n)$ is monotonically increasing on n (in the weak sense). Then, the recurrence relation for a function $G(n)$ is defined as follows: $G(0) = 0$ and for $n \geq 1$,

$$G(n) = \min_{0<m\leq n}\left\{G(n-m) + H(n-m) + \frac{a(q^m-1)}{q-1}\right\}, \tag{9}$$

where a and q are constant integers such that $a \geq 1$ and $q \geq 2$. We note that

$$\frac{a(q^m-1)}{q-1} - \frac{a(q^{m-1}-1)}{q-1} = a \cdot q^{m-1}.$$

Equation (9) generalizes Eq. (1) for the original 4-peg Tower of Hanoi by setting $G(n) = H(n)$ for all n and $(a, q) = (1, 2)$, Eq. (2) for the leaf-to-leaf Star Tower of Hanoi by setting $G(n) = H(n)$ for all n and $(a, q) = (2, 3)$, and the leaf-to-center Star Tower of Hanoi, i.e., Eq. (3) for $T_4(n)$ by setting $H(n) = S_4(n)$ for all n and $(a, q) = (1, 3)$. For other values of (a, q), we have not found a case in which Eq. (9) has some concrete meaning such as being a recurrence relation for some Tower of Hanoi variant, but emphasis and benefit of the generalization should lie at this point in understanding the algorithms of the three types of 4-peg Tower of Hanoi in the unified manner.

Now, the analysis for $T_4(n)$ in Sects. 3 and 4 is generalized for $G(n)$. Similarly to the definition of $\{t_i\}_{i\geq 1}$ in Sect. 3.2, we define the integer sequence $\{g_i\}_{i\geq 1}$ and the multisets A_i's for $i \geq 0$ as follows:

1. $A_0 = [a \cdot q^n]_{n\geq 0}$ and $g_1 = \min A_0 = a$.
2. For $i \geq 1$, the multiset A_i is defined as

   $$A_i = A_{i-1} \setminus \min A_{i-1} \cup [g_i + h_i],$$

   where $\min A_{i-1}$ is the minimum integer in A_{i-1}. Then, g_{i+1} is defined as $g_{i+1} = \min A_i$.

   When we define

   $$m_n = \left\lfloor \log_q \frac{g_n}{a} \right\rfloor + 1,$$

the following lemma holds, similarly to Lemma 2:

Lemma 4: m_n is written either as $m_n = m_{n-1} + 1$ or $m_n = m_{n-1}$. Furthermore, g_n is explicitly written as follows:

$$\begin{cases} \text{(i) When } m_n = m_{n-1} + 1,\ g_n = a \cdot q^{m_n-1}. \\ \text{(ii) When } m_n = m_{n-1},\ g_n = g_{n-m_n} + h_{n-m_n}. \end{cases}$$

Proof: The proof for Lemma 2 works in this case by replacing 3^n and $2s_n$ with $a \cdot q^n$ and h_n, respectively and by taking the logarithm with respect to q. □

We finally show the following theorem on $G(n)$ and g_n:

Theorem 2: Let $\{g_n\}_{n\geq 1}$ be the the sequence defined above. Then, the following equalities hold.

$$\Delta G(n) = g_n \text{ for } n \geq 1,$$
$$G(n) = \sum_{i=1}^{n} g_i \text{ for } n \geq 1.$$

Proof: We again prove by induction. First, $\Delta G(1) = G(1) = g_1 = a$.

Next, assume that $\Delta G(n) = g_n$ for all n such that $1 \leq n \leq k-1$. Then, the following lemma generalized from Lemma 3 holds:

Lemma 5: Under the assumption of induction, the function

$$f(m) = G(k-m) + H(k-m) + \frac{a(q^m - 1)}{q-1}$$

takes the minimum at $m = m_k$.

Proof of Lemma 5: When $i < m_k$, similarly to Lemma 3,

$$\begin{aligned} & f(i+1) - f(i) \\ &= \left(G(k-(i+1)) + H(k-(i+1)) + \frac{a(q^{i+1}-1)}{q-1}\right) \\ &\quad - \left(G(k-i) + H(k-i) + \frac{a(q^i - 1)}{q-1}\right) \\ &= a \cdot q^i - (g_{k-i} + h_{k-i}) \text{ (by Assumption)} \\ &\leq 0 \text{ (by Lemma 4).} \end{aligned}$$

When $i \geq m_k$, since $k - i \leq k - m_k$,

$$\begin{aligned} & f(i+1) - f(i) \\ &= a \cdot q^i - (g_{k-i} + h_{k-i}) \text{ (by Assumption)} \\ &\geq a \cdot q^i - (g_{k-m_k} + h_{k-m_k}) \\ &\geq 0 \text{ (by Lemma 4).} \end{aligned}$$

Consequently, $f(m)$ takes the minimum at $m = m_k$. □

Now, we show that $\Delta G(k) = g_k$. By Lemma 5,

$$\begin{aligned} \Delta G(k) &= G(k) - G(k-1) \\ &= (G(k-m_k) - G(k-1-m_{k-1})) \\ &\quad + (H(k-m_k) - H(k-1-m_{k-1})) \\ &\quad + \frac{a(q^{m_k} - q^{m_{k-1}})}{q-1}. \end{aligned} \tag{10}$$

We further divide into two cases.

Case 1. For k such that $m_k = m_{k-1} + 1$, $g_k = a \cdot q^{m_k - 1}$ by Lemma 4(i). Then, similarly to Theorem 1,

$$G(k-m_k) - G(k-1-m_{k-1}) = 0,$$
$$H(k-m_k) - H(k-1-m_{k-1}) = 0,$$
$$\frac{a(q^{m_k} - q^{m_{k-1}})}{q-1} = \frac{a(q^{m_k} - q^{m_k - 1})}{q-1} = a \cdot q^{m_k - 1}.$$

Therefore, by Lemma 4(i), Eq. (10) is simplified as

$$\Delta G(k) = a \cdot q^{m_k - 1} = g_k.$$

Case 2. For k such that $m_k = m_{k-1}$, $g_k = g_{k-m_k} + h_{n-m_k}$ by Lemma 4(ii). Then,

$$G(k-m_k) - G(k-1-m_{k-1}) = g_{k-m_k} \text{ (by Assumption)},$$
$$H(k-m_k) - H(k-1-m_{k-1}) = h_{k-m_k} \text{ (by Definition)},$$
$$\frac{a(q^{m_k} - q^{m_{k-1}})}{q-1} = \frac{a(q^{m_k} - q^{m_k})}{q-1} = 0.$$

Therefore, by Lemma 4(ii), Eq. (10) is simplified as

$$\Delta G(k) = g_{k-m_k} + h_{k-m_k} = g_k.$$

This completes the proof of Theorem 2. □

6. Concluding Remarks

We made exact analysis of the Frame-Stewart-type recurrence relation for the leaf-to-center 4-peg Star Tower of Hanoi. Then we generalized the recurrence relation to include the ones for the original 4-peg Tower of Hanoi and the leaf-to-leaf Star Tower of Hanoi.

Future work includes consideration of applying the generalized recurrence relation in Sect. 5 possibly to other Tower of Hanoi variants, analysis of the k-peg leaf-to-center Star Tower of Hanoi for $k \geq 5$, and exploration for the optimality of the solution obtained in this paper.

References

[1] D. Berend and A. Sapir, "The cyclic multi-peg Tower of Hanoi," ACM Trans. Algorithms, vol.2, no.3, pp.297–317, 2006.
[2] D. Berend, A. Sapir, and S. Solomon, "The Tower of Hanoi problem on path$_h$ graphs," Discrete Applied Mathematics, vol.160, no.10-11, pp.1465–1483, 2012.
[3] T. Bousch, "La quatrième Tour de Hanoi," Bull. Belg. Math. Soc. Simon Stevin, vol.21, no.5, pp.895–912, 2014.
[4] T. Bousch, "La tour de Stockmeyer," Séminaire Lotharingien de Combinatoire 77, article #B77d, 2017.
[5] J. Chappelon and A. Matsuura, "On generalized Frame-Stewart numbers," Discrete Mathematics, vol.312, no.5, pp.830–836, 2012.
[6] X. Chen and J. Shen, "On the Frame-Stewart conjecture about the Towers of Hanoi," SIAM Journal on Computing, vol.33, no.3, pp.584–589, 2004.
[7] H.E. Dudeney, "The Reve's puzzle," The Canterbury Puzzles and Other Curious Problems, Thomas Nelson and Sons, Ltd, London, 1907.
[8] J.S. Frame, "Solution to advanced problem 3918," Amer. Math. Monthly, vol.48, pp.216–217, 1941.
[9] C. Grosu, "A new lower bound for the Towers of Hanoi problem," Electronic Journal of Combinatorics, vol.23, article #P1.22, 2016.
[10] C.H. Heide, "Distances and automatic sequences in distinguished variants of Hanoi graphs," Dissertation, LMU München: Faculty of Mathematics, Computer Science and Statistics, 2016.
[11] C. Kimberling, "Problem proposals," Proc. 17th International Conference on Fibonacci Numbers and Their Applications, The Fibonacci Quarterly, vol.55, no.5, pp.213–221, 2017.
[12] A.M. Hinz, S. Klavžar, U. Milutinović, and C. Petr, The Tower of Hanoi - Myths and Maths, Springer, Basel, 2013.
[13] S. Klavžar and U. Milutinović, "Simple explicit formulas for the Frame-Stewart numbers," Annals of Combinatorics, vol.6, no.2, pp.157–167, 2002.
[14] S. Klavžar, U. Milutinović, and C. Petr, "On the Frame-Stewart algorithm for the multi-peg Tower of Hanoi problem," Discrete Applied Mathematics, vol.120, no.1-3, pp.141–157, 2002.

[15] E. Lucas, Récréation Mathématiques, vol.III, Gauthier-Villars, Paris, 1893.
[16] A. Matsuura, “Analysis of recurrence relations generalized from the 4-peg Tower of Hanoi,” IEICE Trans. Information and Systems, vol.E94-D, no.2, pp.220–225, 2011.
[17] A. Sapir, “The Tower of Hanoi with forbidden moves,” Computer Journal, vol.47, no.1, pp.20–24, 2004.
[18] B.M. Stewart, “Solution to advanced problem 3918,” Amer. Math. Monthly, vol.48, pp.217–219, 1939.
[19] P.K. Stockmeyer, “Variations on the four-post Tower of Hanoi puzzle,” Congressus Numerantium, vol.102, pp.3–12, 1994.
[20] M. Szegedy, “In how many steps the *k* peg version of the Towers of Hanoi game can be solved?,” Proc. STACS’99, LNCS 1563, pp.356–361, 1999.

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