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Polynomial-Time Reductions from 3SAT to Kurotto and Juosan Puzzles*

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SUMMARY Kurotto and Juosan are Nikoli's pencil puzzles. We study the computational complexity of Kurotto and Juosan puzzles. It is shown that deciding whether a given instance of each puzzle has a solution is NP-complete.

key words: *Kurotto, Juosan, pencil puzzle, NP-complete*

1. Introduction

The Kurotto puzzle is played on a rectangular grid of cells (see Fig. 1 (a)). Initially, some of the cells contain circles, where each circle contains a number or no number. The purpose of the puzzle is to fill in cells in black in the following rules [1]: (1) The number in a circle indicates the sum of the number of continuous black cells extending from it, vertically and horizontally (see three cells a, c and d extending from ③ in Fig. 1 (c)). (2) Empty circles may have any number of black cells around them. (3) Cells with circles cannot be colored black.

Figure 1 (a) is an initial configuration of a Kurotto puzzle. In this figure, there are six circles, five of which contain numbers. From Figs. 1 (b)–(f), the reader can understand the basic technique for finding a solution. (b) Consider ② in the red cell. If two grey cells a and b are colored black, then ① in the yellow cell is connected to two continuous black cells. Thus, cells a and c are colored black (see (c)), and cell b must not be colored black; such a cell is indicated by • in Fig. 1. (c) Circled number ③ in the blue cell is connected to black cells c, a and d. (d) Consider ④ in the red cell. If four grey cells containing cell e are colored black, then ① in the yellow cell is connected to four continuous black cells. Thus, cells f and g, h, i, j are colored black (see (e)). (f) is one of the multiple solutions.

The Juosan puzzle is played on a rectangular grid of cells (see Fig. 2 (a)). Initially, the grid of cells are divided into rectangular *territories*. Each territory contains a number or no number. The purpose of the puzzle is to fill in cells, each with a mark "**–**" or a mark "**|**," in the following rules [2]: (1) The number in a territory shows the number of **–**-marks if **–**-marks have majority, or the number of **|**-marks if **|**-marks have majority. However, there are also cases where the numbers of **–**-marks and **|**-marks are the same. Territories with no numbers may have any number of **–**-marks and **|**-marks. (2) **–**-marks can extend across more than three cells horizontally, but not more than two cells vertically. (3) **|**-marks can extend across more than three cells vertically, but not more than two cells horizontally.

Figure 2 (a) is an initial configuration of a Juosan puzzle. In this figure, cells are partitioned into ten territories, eight of which contain numbers. (b) The blue 3 × 1 territory

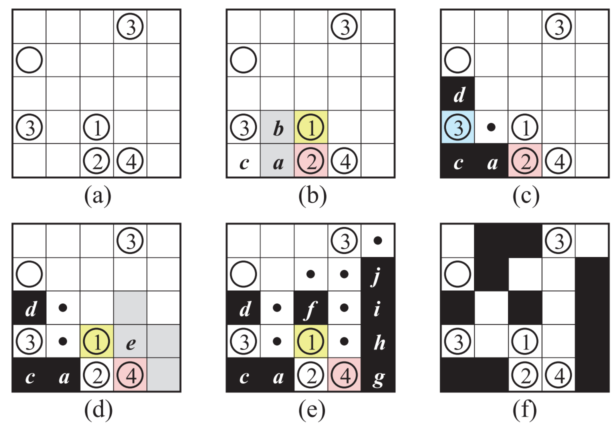

Fig. 1 (a) Initial configuration of a Kurotto puzzle. (b)–(f) are the progress from the initial configuration to a solution.

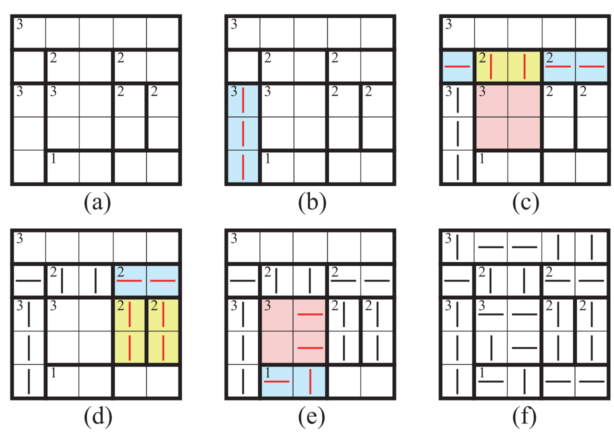

Fig. 2 (a) Initial configuration of a Juosan puzzle. (b)–(f) are the progress from the initial configuration to a solution.

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has number 3, so three |-marks are filled in it. (c) If there are three |-marks in the red 2×2 territory, then |-marks extend across three cells horizontally. Thus, the red 2×2 territory will contain three −-marks (see (f)), which implies that the yellow 1×2 territory must contain two |-marks. (d) Since the blue 1×2 territory contains two −-marks, two yellow 2×1 territories contain four |-marks. (e) Since two −-marks are placed in the red 2×2 territory, a −-mark and a |-mark are filled in the blue 1×2 territory. (f) is one of the multiple solutions.

In this paper, we study the computational complexity of the decision version of Kurotto and Juosan puzzles. The instance of the *Kurotto puzzle problem* is defined as a rectangular grid of cells, where some of the cells contain circles, and each circle contains a number or no number. The instance of the *Juosan puzzle problem* is a rectangular grid of cells, which are divided into rectangular territories. Each territory contains a number or no number. Each problem is to decide whether there is a solution to the instance.

Theorem 1: The Kurotto and Juosan puzzle problems are NP-complete.

It is clear that the Kurotto puzzle problem belongs to NP, since the game ends when all empty cells are colored black or white. The Juosan puzzle problem also belongs to NP, since the game ends when all cells are filled by a −-mark or a |-mark.

There has been a huge amount of literature on the

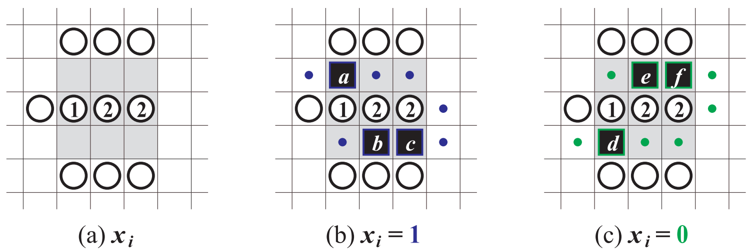

Fig. 3 (a) Variable gadget of Kurotto. (b) and (c) are solutions corresponding to $x_i = 1$ and $x_i = 0$, respectively.

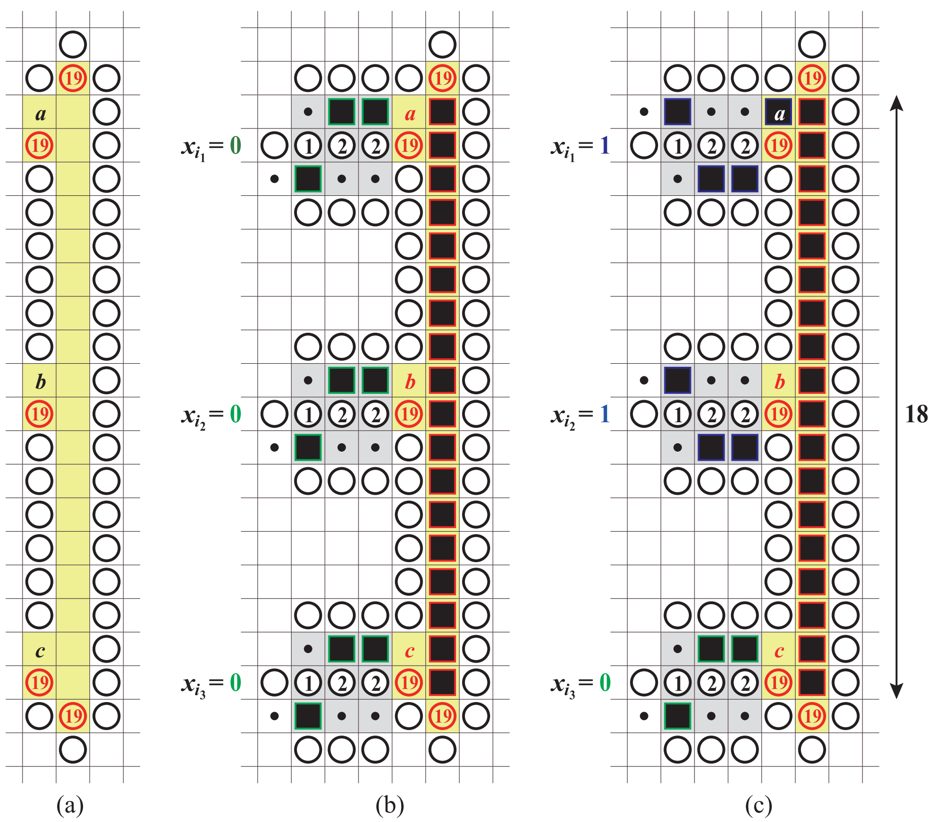

Fig. 4 (a) Clause gadget of Kurotto. Suppose $c_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. (b) is an invalid placement of black cells, since the number of continuous black cells extending from each ⑲ is 18. (c) If either cell a or b is colored black, then continuous 19 black cells are extended from each ⑲. One can see that at least one of variables x_{i_1}, x_{i_2}, and x_{i_3} is 1 if and only if there is a solution to the clause gadget.

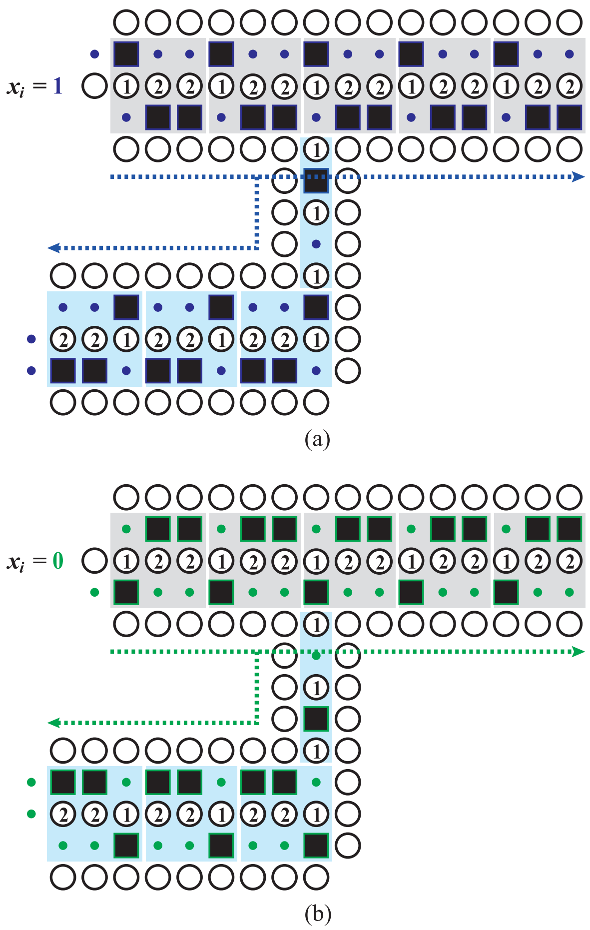

Fig. 5 Right branch gadget. (a) and (b) are solutions when $x_i = 1$ and $x_i = 0$, respectively. The set of connected cells in the blue areas is a right turn gadget.

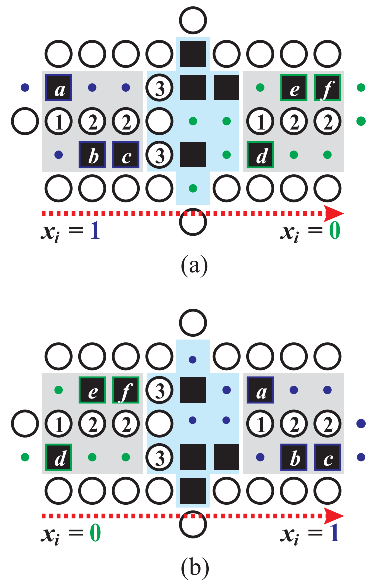

Fig. 6 Not gadget. If the left grey area is $x_i = 1$ (resp. $x_i = 0$), then the right grey area is $x_i = 0$ (resp. $x_i = 1$).

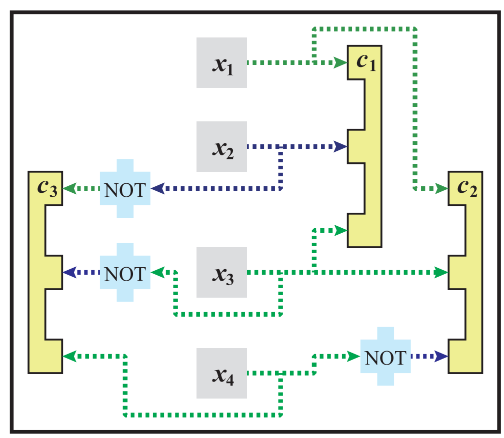

Fig. 7 Top-level description of the puzzle transformed from $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{x_1, x_3, \overline{x_4}\}$, and $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$. From this figure, one can see that $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfies all clauses.

computational complexities of games and puzzles. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [8]. After the publication of this book, the following Nikoli's pencil puzzles were shown to be NP-complete: Dosun-Fuwari [13], Fillmat [17], Hashiwokakero [5], Herugolf and Makaro [12], Kurodoko [14], Numberlink [3], Pencils [15], Pipe Link [18], Shakashaka [6], Shikaku and Ripple Effect [16], Sto-Stone [4], Usowan [11], Yajilin and Country Road [9], and Yosenabe [10]. The reductions given in [6], [17], [18] are parsimonious, i.e., they preserve the number of solutions. Counting the number of solutions to a Shakashaka puzzle is #P-complete [6], and Fillmat [17] and Pipe Link [18] are ASP-complete. Proving the #P-completeness and ASP-completeness of other pencil puzzles is an interesting future research.

2. NP-Completeness of Kurotto

We present a polynomial-time transformation from an arbitrary instance C of PLANAR 3SAT to a Kurotto puzzle such that C is satisfiable if and only if the puzzle has a solution.

2.1 3SAT Problem

The definition of 3SAT is mostly from [7]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. A clause is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over U such that (i) $|c_j| = 3$ for each $c_j \in C$ and (ii) the bipartite graph $G = (V, E)$, where $V = U \cup C$ and E contains exactly those pairs $\{x, c\}$ such that either literal x or $\overline{x}$ belongs to the clause c, is pla-

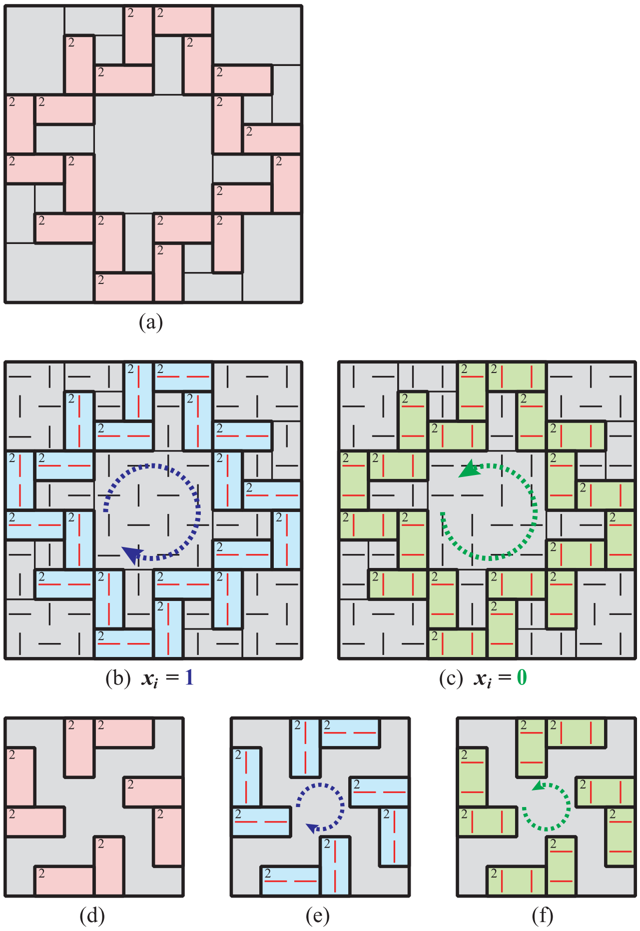

Fig. 8 (a) Variable gadget of Juosan. (b) and (c) are solutions corresponding to $x_i = 1$ and $x_i = 0$, respectively. (d)–(f) are simplified illustration of (a)–(c), respectively.

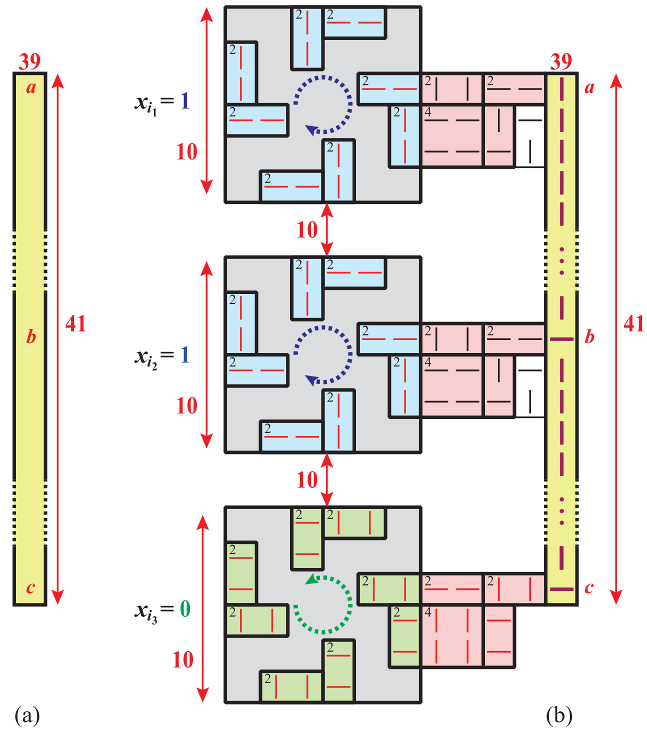

Fig. 9 (a) Clause gadget of Juosan. (b) Suppose $c_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$, where $x_{i_1} = x_{i_2} = 1$ and $x_{i_3} = 0$. Cell c must contain a −-mark. If either cell a or b contains a |-mark, then the yellow territory can contain 39 |-marks. One can see that at least one of variables x_{i_1}, x_{i_2}, and x_{i_3} is 1 if and only if there is a solution to the clause gadget.

nar. The PLANAR 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. The PLANAR 3SAT is known to be NP-complete [7].

For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3\}$, and $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{x_1, x_3, \overline{x_4}\}$, $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$ provide an instance of PLANAR 3SAT. For this instance, the answer is "yes," since there is a truth assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfying all clauses.

2.2 Transformation from an Instance of PLANAR 3SAT to a Kurotto Puzzle

Variable x_i is transformed into a variable gadget as shown in Fig. 3 (a), which is composed of three circles ①, ②, ② and seven empty circles. Figures 3 (b) and 3 (c) are solutions corresponding to $x_i = 1$ and $x_i = 0$, respectively.

Clause $c_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ is transformed into a clause gadget as shown in Fig. 4 (a), which is composed of 36 empty circles and 26 yellow cells; five of the yellow cells contain circled numbers ⑲. This gadget is connected to three variable gadgets of x_{i_1}, x_{i_2}, and x_{i_3} (see Figs. 4 (b) and 4 (c)). Three cells a, b, and c play a key role in this gadget.

If $x_{i_1} = x_{i_2} = x_{i_3} = 0$ (see Fig. 4 (b)), then cells a, b, and c must not be colored black. Thus, Fig. 4 (b) is an invalid placement of black cells, since the number of continuous black cells extending from each ⑲ is 18. Suppose $x_{i_1} = x_{i_2} = 1$ and $x_{i_3} = 0$ (see Fig. 4 (c)). If either cell a or b is colored black, then continuous 19 black cells are extended from each ⑲. Now one can see that at least one of variables x_{i_1}, x_{i_2}, and x_{i_3} is 1 if and only if there is a solution to the clause gadget.

Figure 5 is a right branch gadget. The values $x_i = 1$ and $x_i = 0$ are transmitted in the directions of two arrows indicated in Figs. 5 (a) and 5 (b), respectively. The set of cells in the blue areas of Fig. 5 (a) is a right turn gadget. A left branch gadget and a left turn gadget can be constructed similarly.

Figure 6 is a NOT gadget. In Fig. 6 (a) (resp. Fig. 6 (b)), if cells a, b, and c (resp. d, e, and f) in the left grey area are colored black, then cells d, e, and f (resp. a, b, and c) in the right grey area are colored black.

Figure 7 is the top-level description of a Kurotto puzzle transformed from $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{x_1, x_3, \overline{x_4}\}$, and $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$. From this construction, the Kurotto puzzle has a solution if and only if the instance of PLANAR 3SAT is satisfiable.

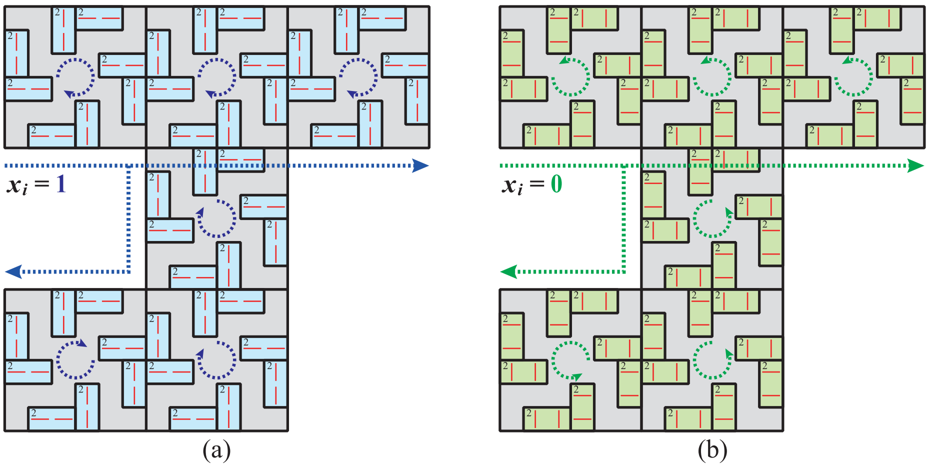

Fig. 10 A right branch gadget with a right turn gadget when (a) $x_i = 1$ and (b) $x_i = 0$.

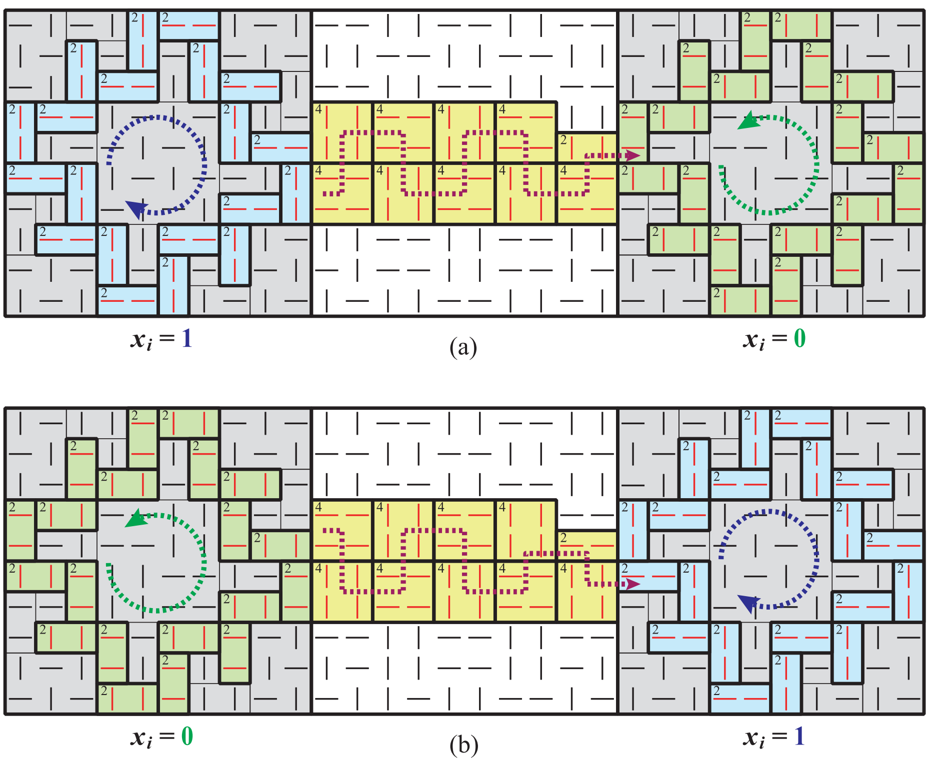

Fig. 11 Not gadget.

3. NP-Completeness of Juosan

We present a polynomial-time transformation from an arbitrary instance C of PLANAR 3SAT to a Juosan puzzle such that C is satisfiable if and only if the puzzle has a solution.

Variable x_i is transformed into a variable gadget as shown in Fig. 8 (a), which is composed of 10×10 cells. Those cells are divided into red and grey cells. The red cells are composed of ten 2×1 territories and ten 1×2 territories, which play a key role of this gadget. In Fig. 8 (b) (resp. Fig. 8 (c)), 20 |-marks and 20 −-marks in the blue (resp. green) area are a solution to the red area of Fig. 8 (a). The grey areas of Fig. 8 (a) are partitioned into 17 territories. Two of the multiple solutions to those grey areas are given in Figs. 8 (b) and 8 (c). Figures 8 (d)–(f) are simplified illustration of Figs. 8 (a)–(c), respectively.

The yellow 41×1 territory of Fig. 9 (a) is a clause gadget. This territory contains number 39, and is connected to three variable gadgets of x_{i_1}, x_{i_2}, and x_{i_3} through 12 red territories (see Fig. 9 (b)). Suppose $x_{i_1} = x_{i_2} = 1$ and $x_{i_3} = 0$. Cell c must contain a −-mark. If either cell a or b contains a |-mark, then the yellow territory can contain 39 |-marks. Now one can see that at least one of variables x_{i_1}, x_{i_2}, and

x_{i_3} is 1 if and only if there is a solution to the clause gadget.

Figure 10 is a right branch gadget with a right turn gadget when (a) $x_i = 1$ and (b) $x_i = 0$. Figure 11 is a NOT gadget. In Fig. 11 (a), the left blue area contains the solution for $x_i = 1$ if and only if the right green area contains the solution for $x_i = 0$.

By using variable, clause, branch, turn, and NOT gadgets of Figs. 8–11, we can construct a Juosan puzzle of Fig. 7 transformed from $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{x_1, x_3, \overline{x_4}\}$, and $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$. From this construction, the Juosan puzzle has a solution if and only if the instance of PLANAR 3SAT is satisfiable.

Acknowledgments

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