PAPER

Computational Complexity of the Vertex-to-Point Conflict-Free Chromatic Art Gallery Problem*

Chuzo IWAMOTO[†a)], ***Member and*** **Tatsuaki IBUSUKI**[††], ***Nonmember***

SUMMARY The art gallery problem is to find a set of guards who together can observe every point of the interior of a polygon P. We study a *chromatic* variant of the problem, where each guard is assigned one of k distinct colors. A chromatic guarding is said to be *conflict-free* if at least one of the colors seen by every point in P is unique (i.e., each point in P is seen by some guard whose color appears exactly once among the guards visible to that point). In this paper, we consider *vertex-to-point* guarding, where the guards are placed on vertices of P, and they observe every point of the interior of P. The *vertex-to-point conflict-free chromatic art gallery problem* is to find a colored-guard set such that (i) guards are placed on P's vertices, and (ii) any point in P can see a guard of a unique color among all the visible guards. In this paper, it is shown that determining whether there exists a conflict-free chromatic vertex-guard set for a polygon with holes is NP-hard when the number of colors is $k = 2$.

key words: *chromatic art gallery problem, polygons, visibility, NP-hard*

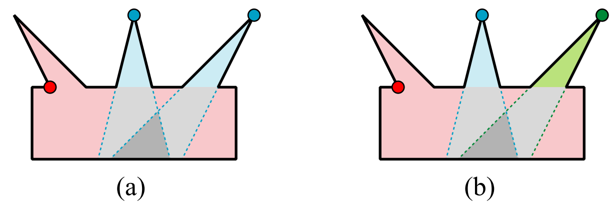


Fig. 1 (a) *Conflict-free* chromatic guarding. The dark gray area is seen by one red guard and two blue guards, where red is the unique color among the three guards. Each of the two light gray areas is seen by one red guard and one blue guard. (b) *Strong* chromatic guarding. The dark gray area is seen by three guards having three different colors.

1. Introduction

The art gallery problem is to determine the minimum number of guards who can observe the interior of a gallery. Chvátal [4] proved that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary for observing the interior of an n-vertex simple polygon. This $\lfloor n/3 \rfloor$-bound is replaced by $\lfloor n/4 \rfloor$ if the instance is restricted to a simple orthogonal polygon [9].

Another perspective to the art gallery problem is to study the complexity of locating the minimum number of guards in a polygon. The NP-hardness and APX-hardness of this problem were shown by Lee and Lin [13] and by Eidenbenz et al. [5], respectively. Furthermore, Schuchardt and Hecker [17] proved that this problem remains NP-hard even if we restrict our attention to simple orthogonal polygons. Even guarding the vertices of a simple orthogonal polygon was shown to be NP-hard [12].

In this paper, we consider *vertex-to-point* guarding, where the guards are placed on vertices of a polygon P, and they observe every point inside P. We study a *chromatic* version of the art gallery problem, where each guard is assigned one of k distinct colors. There are two chromatic variants, which are called *conflict-free* chromatic guarding and *strong* chromatic guarding [10] (see Fig. 1). A chromatic guarding is said to be *conflict-free* if at least one of the guards seen by every point in P has a unique color [1]. It is *strong* if no two guards with the same color have overlapping visibility regions [6].

The *vertex-to-point conflict-free chromatic art gallery problem* is to find a colored-guard set such that (i) guards are placed on P's vertices, and (ii) any point inside P can see a guard of a unique color among all the visible guards. In this paper, it is shown that determining whether there exists a conflict-free chromatic vertex-guard set which together observe every point in a given polygon with holes is NP-hard when the number of colors is $k = 2$.

The chromatic art gallery problem was motivated by the following application [2], [7]. Consider the problem of navigating a robot inside a polygon, where the robot communicates with radio beacons. The robot must be able to communicate with a radio beacon of a unique frequency in order to prevent interference. This motivates a *chromatic* version of the art gallery problem, where a guard corresponds to a radio beacon, and colors correspond to different frequencies.

The computational complexity of the chromatic art gallery problem was firstly investigated in [7]; the *point-to-point strong* chromatic art gallery problem was shown to be NP-hard for general polygons with holes. Recently, the current authors proved that the *point-to-point strong* chromatic art gallery problem with *r-visibility* is NP-hard for *orthogonal* polygons with holes [11]. Here, two points are said to be *r-visible* if the smallest axis-aligned rectangle containing them lies entirely within the polygon.

Çağırıcı et al. studied the *vertex-to-vertex conflict-*

Manuscript received December 16, 2022.
Manuscript revised March 28, 2023.
Manuscript publicized May 31, 2023.
[†]The author is with Graduate School of Advanced Science and Engineering, Hiroshima University, Higashihiroshima-shi, 739–8527 Japan.
[††]The author is with Graduate School of Engineering, Hiroshima University, Higashihiroshima-shi, 739–8527 Japan.
*Preliminary version of this paper appeared in Proc. WALCOM 2022 (LNCS 13174).
a) E-mail: chuzo@hiroshima-u.ac.jp
DOI: 10.1587/transinf.2022EDP7222

free chromatic guarding problem [2]; they proved the NP-hardness of the problem when the number of colors is $k = 2$. However, they mentioned that their proof does not imply the NP-hardness for the *vertex-to-point* case. Hence, the computational complexity of the vertex-to-point conflict-free chromatic guarding problem remained open. In the current paper, we solve this open problem.

Several results on the lower and upper bounds of the minimum number of colors can be found in [1], [6], [10] for general and orthogonal polygons under standard and orthogonal visibility conditions.

2. Definitions and Results

The definitions of a polygon and a polygon with holes are mostly from [14], [16]. A *polygon* is defined by a finite set of segments such that every segment endpoint is shared by exactly two segments and no subset of segments has the same property. The segments and their endpoints are called the *edges* and *vertices* of the polygon, respectively.

A *polygon with holes* is a polygonal domain defined by a polygon P enclosing several other polygons $H_1, H_2, \ldots, H_h$, the holes. None of the boundaries of P, $H_1, H_2, \ldots, H_h$ may intersect, and each of the holes is empty. P is said to bound a *multiply-connected* region with h holes: the region of the plane interior to or on the boundary of P, but exterior to or on the boundary of $H_1, H_2, \ldots, H_h$.

Two points v and u in a polygon P are said to be *visible* (or *v sees u*) if the line segment connecting them lies entirely within P. Here, the line segment may contain points on the boundary of P, but it must not across any hole of the polygon. An area is said to be *observed* by a point v if every point in the area is visible from v.

In this paper, we assume that each vertex of any polygon has integral coordinates. An instance of the *vertex-to-point conflict-free chromatic art gallery problem for polygons with holes* is $(P, H_1, H_2, \ldots, H_h; k)$, where P is a polygon with holes $H_1, H_2, \ldots, H_h$, and k is the number of colors. The problem asks whether there exists a conflict-free k-chromatic vertex-guard set which together observe every point in the polygonal domain defined by $(P, H_1, H_2, \ldots, H_h)$.

Theorem 1: *The vertex-to-point conflict-free chromatic art gallery problem for polygons with holes is NP-hard when the number of colors is two.*

3. NP-Completeness

3.1 3SAT Problem

The definition of 3SAT is mostly from [8], [15]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. A clause is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

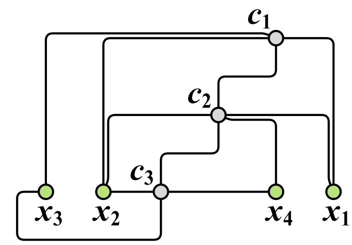


Fig. 2 Planar graph $G = (V, E_1)$ corresponding to CLAUSE-LINKED PLANAR 3SAT $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, \overline{x_2}, x_3\}$, $c_2 = \{x_1, x_2, \overline{x_4}\}$, and $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over U such that (i) $|c_j| = 3$ for each $c_j \in C$ and (ii) the graph $G = (V, E)$, defined by $V = U \cup C$ and $E = \{(x_i, c_j) \mid x_i \in c_j \in C \text{ or } \overline{x_i} \in c_j \in C\}$, is planar. PLANAR 3SAT asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C.

If E is replaced with

$$E_1 = E \cup \{(c_j, c_{j+1}) \mid 1 \le j \le m-1\},$$

then the problem is called CLAUSE-LINKED PLANAR 3SAT. This problem is NP-complete, since VARIABLE-CLAUSE-LINKED PLANAR 3SAT was shown to be NP-complete in [15], where the edge set E_2 is defined as

$$E_2 = E \cup \{(x_i, x_{i+1}) \mid 1 \le i \le n-1\} \cup \{(x_n, c_1)\} \\ \cup \{(c_j, c_{j+1}) \mid 1 \le j \le m-1\} \cup \{(c_m, x_1)\}.$$

Note that CLAUSE-LINKED PLANAR 3SAT in this paper is defined by a *chain* connecting $c_1, c_2, \ldots, c_m$ of length $m-1$, while VARIABLE-CLAUSE-LINKED PLANAR 3SAT in [15] is defined by a *cycle* of length $m+n$.

For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3\}$, and $c_1 = \{x_1, \overline{x_2}, x_3\}$, $c_2 = \{x_1, x_2, \overline{x_4}\}$, $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$ provide an instance of CLAUSE-LINKED PLANAR 3SAT. Figure 2 is a planar embedding of the graph corresponding to this instance. For this instance, the answer is "yes," since there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$ satisfying all clauses.

3.2 Bowl-Shaped Gadgets and Pocket Gadgets

In this section, we explain a *bowl*-shaped gadget (see Fig. 3 (a)) and a *pocket* gadget (see Fig. 3 (a)), introduced in [2].

A *bowl*-shaped gadget is a 20-vertex chain, which has the following two properties: (i) If a guard is placed on vertex p_1 or p_2 (see Fig. 3 (b)), then it can see all the vertices of $\{a_1, a_2, \ldots, a_9\} \cup \{c_1, c_2, \ldots, c_9\}$. (ii) Suppose that no guard is placed on p_1 or p_2 (see Fig. 3 (c)). Then, in order to observe vertices $a_1, a_2, \ldots, a_9$, both a red guard and a blue guard must be placed on two of the vertices in $\{a_1, a_2, \ldots, a_9\}$. Similarly, both a red guard and a blue guard must be placed on $\{c_1, c_2, \ldots, c_9\}$. Those four guards see vertices p_1 and

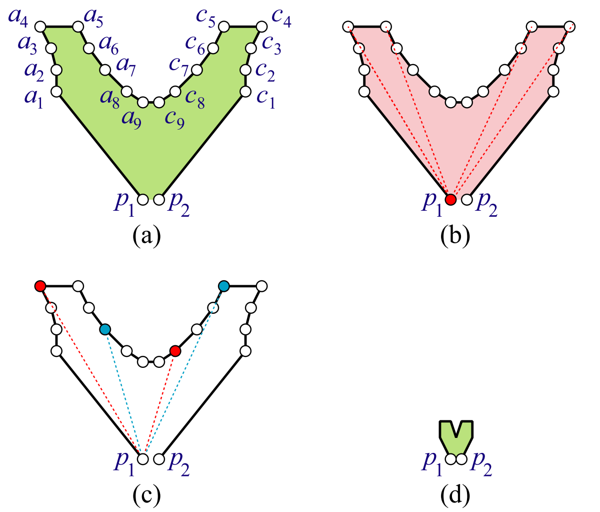


Fig. 3 (a) A *bowl*-shaped gadget [2]. (b) A guard on p_1 or p_2 can see all the vertices of $\{a_1, a_2, \ldots, a_9\} \cup \{c_1, c_2, \ldots, c_9\}$. (c) If no guards are placed on vertices of $\{p_1, p_2\}$, then there exists no conflict-free 2-color guard-set. Thus, a guard must be placed on at least one of p_1 and p_2 (see (b)). (d) is a simplified illustration of (a), where two vertices p_1 and p_2 are called *door vertices* of the bowl.

p_2 simultaneously (see Fig. 3 (c)). From the properties (i) and (ii), one can see that, *in any conflict-free* 2*-coloring of a bowl-shaped gadget, there is a guard placed on* p_1 *or* p_2 *(or both)* (see Fig. 3 (b)). In the following, we use a simplified illustration shown in Fig. 3 (d) as a bowl-shaped gadget. Vertices p_1 and p_2 are called *door vertices* of the bowl. The distance between p_1 and p_2 is assumed to be so tiny that there is no accidental visibility between a vertex inside the bowl and a vertex outside of the bowl.

In Fig. 4 (a), the green area surrounded by a 10-vertex chain from a_1 to a_2 is called a *pocket* gadget. Vertices d_1 and d_2 can see all the 10 vertices of the pocket, but neither a_1 nor a_2 does so. If a pair of red and blue guards are placed on door vertices of bowls (see Fig. 4 (b)), then they can see the inside of the pocket in the conflict-free condition. On the other hand, Fig. 4 (c) is an illegal 2-coloring because of the following reason. In order to guard the inside of the pocket in the conflict-free condition, we need a *single* blue guard on the 10-vertex chain. However, the 10-vertex chain contains no single vertex which can see every point of the inside of the pocket. Figure 4 (d) is a simplified illustration of Fig. 4 (a).

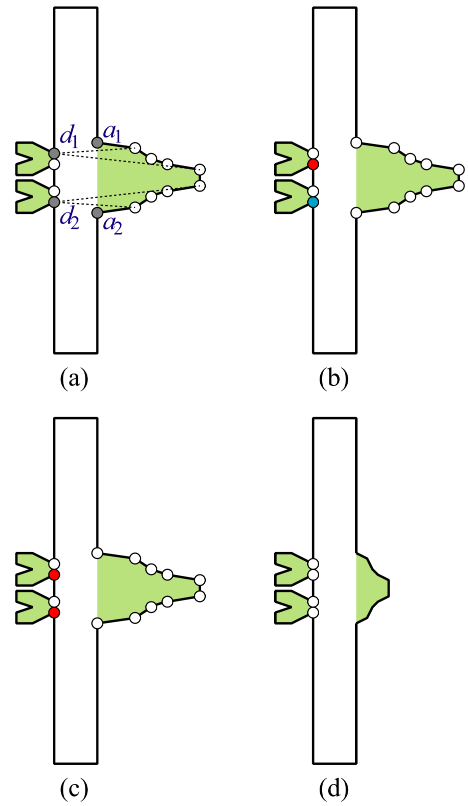


Fig. 4 (a) The green area surrounded by a 10-vertex chain from a_1 to a_2 is called a *pocket*-gadget [2]. (b) A pair of red and blue guards on door vertices of bowls can see the inside of the pocket in the conflict-free condition. (c) is an illegal 2-coloring. (d) is a simplified illustration of (a).

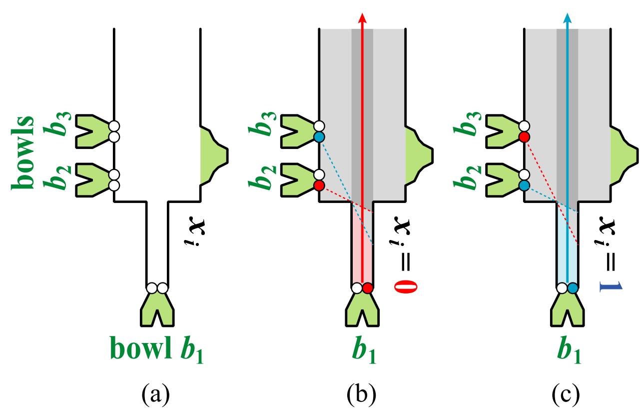


Fig. 5 (a) Variable gadget. (b) Conflict-free guarding when $x_i = 0$. (c) Conflict-free guarding when $x_i = 1$. (c) is obtained from (b) by switching red and blue guards.

3.3 Transformation from an Instance of Clause-Linked Planar 3SAT to a Polygon with Holes

We present a polynomial-time transformation from an arbitrary instance of clause-linked planar 3SAT C to a polygon with holes such that C is satisfiable if and only if there is a conflict-free 2-chromatic vertex-guard set which together observe every point in the polygon.

Each variable $x_i \in \{x_1, x_2, \ldots, x_n\}$ is transformed into the variable gadget as illustrated in Fig. 5. In the gadget, there are three bowl-shaped gadgets b_1 and b_2, b_3 and one pocket gadget. From the reasons given in Sect. 3.2, a pair of red and blue guards must be placed on two of the door vertices of bowls b_2, b_3 (see Figs. 5 (b) and 5 (c)). A door vertex of bowl b_1 emits a beam of red or blue light upward. A red and blue beams in Figs. 5 (b) and 5 (c) correspond to the assignment $x_i = 0$ and $x_i = 1$, respectively.

In Fig. 5 (b), a dark gray area is observed by two red guards and one blue guard, and a light gray area is observed

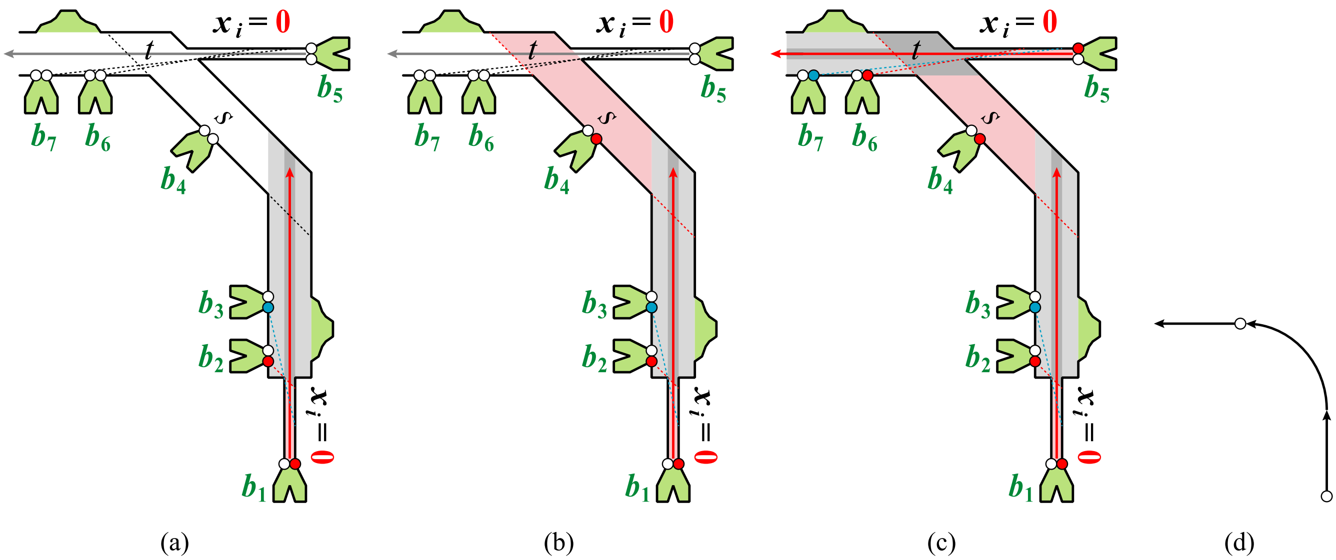


Fig. 6 (a) Left-turn gadget when $x_i = 0$. (b) Area s is observed by a red guard. (c) A red guard and a pair of red and blue guards are placed on b_5 and b_6, b_7, respectively. (d) is a simplified illustration of a left-turn gadget. A left-turn gadget when $x_i = 1$ is obtained from (c) by switching red and blue guards.

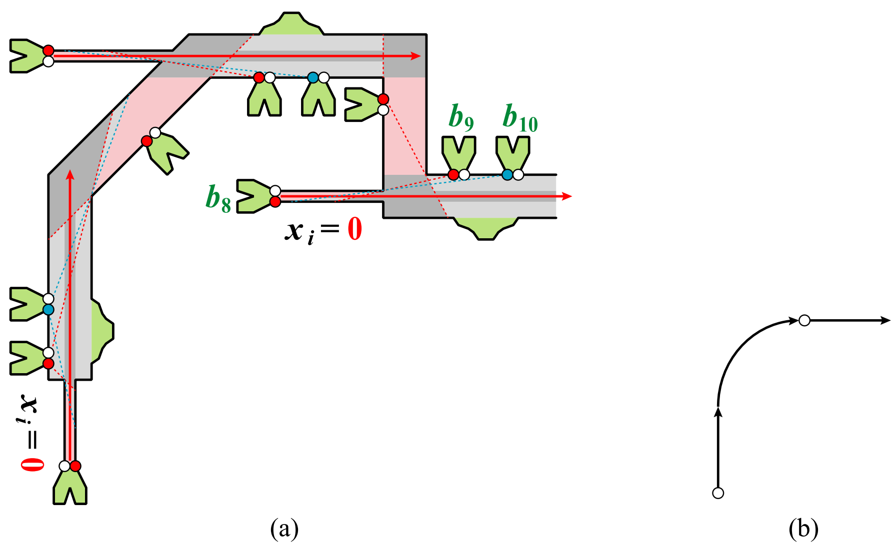


Fig. 7 (a) Right-turn gadget when $x_i = 0$. (b) Simplified illustration.

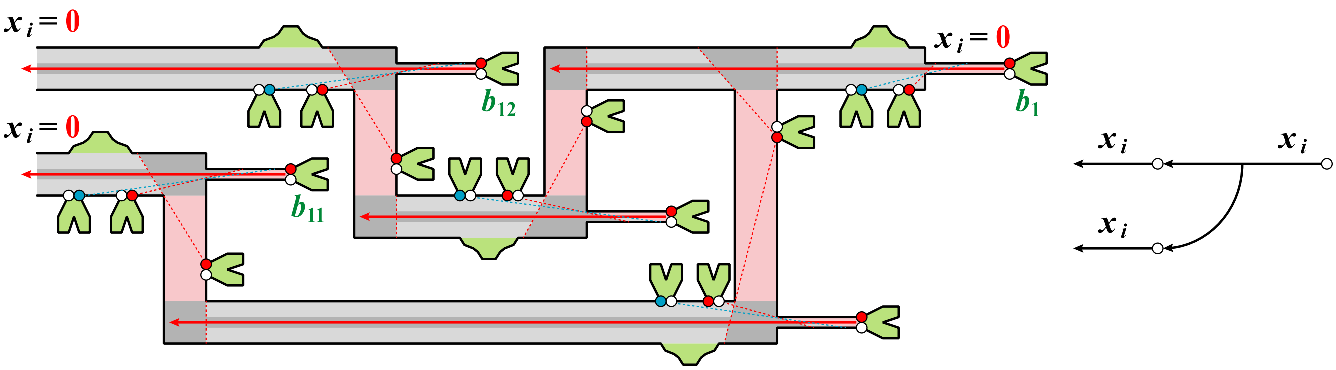
by one red guard and one blue guard. Note that Fig. 5 (c) is obtained from Fig. 5 (b) by switching red and blue guards. In Figs. 6–8 and 9 (a), we present figures only for $x_i = 0$.

Figure 6 is a left-turn gadget. (a) Suppose that bowl b_1 emits a beam of red light (see Fig. 6 (a)). Since the dark gray area is observed by two red guards and one blue guard, area s must be observed by a red guard (see Fig. 6 (b)). (b) Now, area t is observed by a red guard of b_4, and t is also seen by door vertices of b_5 and b_6, b_7. (c) In order to satisfy the conflict-free condition, we must place a red guard on b_5 and a pair of red and blue guards on b_6 and b_7, respectively. (d) is a simplified illustration of a left-turn gadget.

Figure 7 is a right-turn gadget. Bowl b_8 emitting a red beam is used so that a pair of bowls (see b_9, b_{10}) and a pocket are located on the left and right sides of the beam, respectively. (Bowls b_{11} and b_{12} in Fig. 8 (a) are used for the same purpose.) Figure 8 is a branching gadget. If bowl b_1 emits a red beam, then b_{11} and b_{12} also emit red beams.

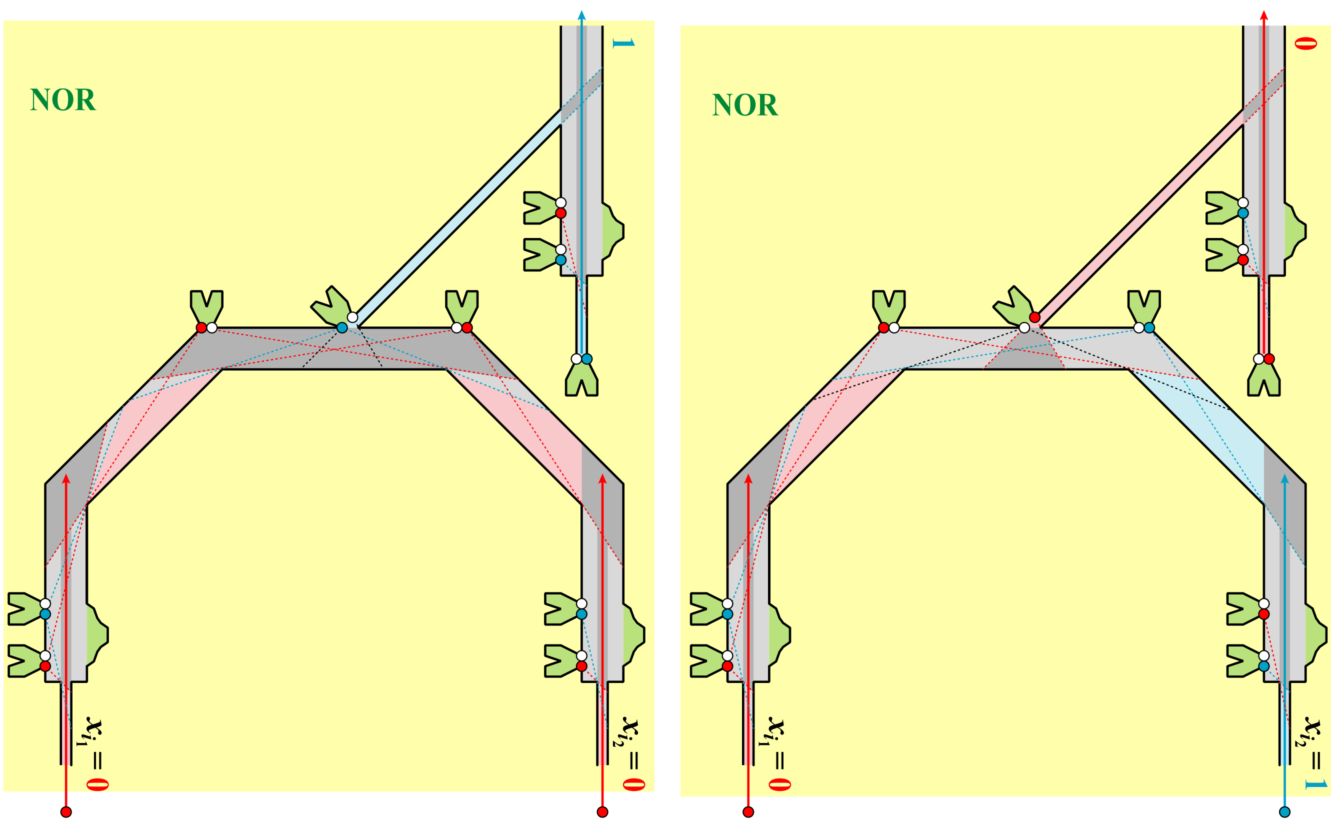
Figure 9 is a NOR gadget. If $x_{i_1} = x_{i_2} = 0$ (see Fig. 9 (a)), then the NOR gadget will emit a blue beam (= value 1) upward. By switching red and blue guards in Fig. 9 (a), one can see that the NOR gadget outputs 0 if $x_{i_1} = x_{i_2} = 1$. (Figure 9 (b) is explained later; the case $x_{i_1} \neq x_{i_2}$ will have to be treated carefully.)

A NOT gadget (see Fig. 10 (a)) is obtained by connecting a branching gadget and a NOR gadget. In the NOT gadget, the input is $x_i = 0$ if and only if the output is $\overline{x_i} = 1$. An OR gadget (see Fig. 10 (b)) is obtained by connecting a



(a) (b)

Fig. 8 (a) Branching gadget when $x_i = 0$. (b) is a simplified illustration.



(a) (b)

Fig. 9 NOR gadget. (a) If $x_{i_1} = x_{i_2} = 0$, this gadget outputs value 1 upward. By switching red and blue guards, one can see that the gadget outputs 0 if $x_{i_1} = x_{i_2} = 1$. (b) When $x_{i_1} \neq x_{i_2}$, the gadget *can* output 0 (see the body text for details).

NOR gadget and a NOT gadget. Note that, if $x_{i_1} = x_{i_2} = 0$, the OR gadget outputs 0. By switching red and blue guards, one can see that the OR gadget outputs 1 if $x_{i_1} = x_{i_2} = 1$. (The case $x_{i_1} \neq x_{i_2}$ is explained later.)

A clause gadget $c_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ (see Fig. 11) contains three OR gadgets. If $x_{i_1} = x_{i_2} = x_{i_3} = 0$, the clause gadget outputs value $c_j = 0$. By switching red and blue guards, one can see that the clause gadget outputs $c_j = 1$ if $x_{i_1} = x_{i_2} = x_{i_3} = 1$. (The remaining cases are explained in the next paragraph.)

Consider a NOR gadget when $x_{i_1} \neq x_{i_2}$ (see Fig. 9 (b)). In this case, there exists a conflict-free 2-chromatic guard set (see red and blue guards in Fig. 9 (b)) so that the clause gadget emits a red beam (= value 0) upward. Namely, the NOR gadget *can* output 0 when $x_{i_1} \neq x_{i_2}$. Thus, in Fig. 10 (b), the OR gadget *can* output 1 when $x_{i_1} \neq x_{i_2}$. Hence, in Fig. 11, a clause gadget *can* output $c_j = 1$ if at least one of x_{i_1}, x_{i_2}, and x_{i_3} is 1. In summary, if $(x_{i_1}, x_{i_2}, x_{i_3}) = (0, 0, 0)$ (resp.

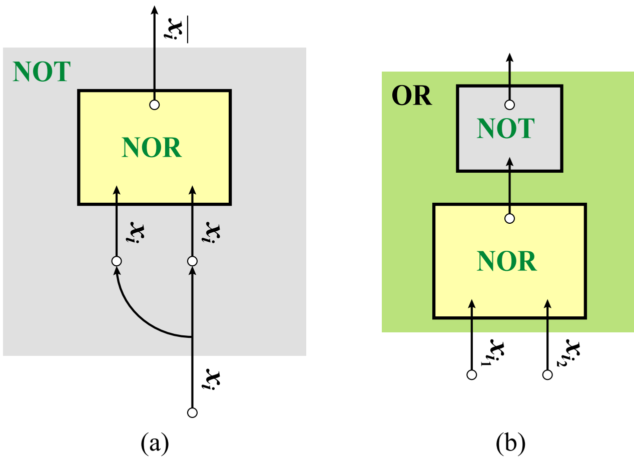


Fig. 10 (a) NOT gadget. The input is $x_i = 0$ if and only if the output is $\overline{x_i} = 1$. (b) OR gadget. If $x_{i_1} = x_{i_2} = 0$, the OR gadget outputs 0. By switching red and blue guards, one can see that the OR gadget outputs 1 if $x_{i_1} = x_{i_2} = 1$. When $x_{i_1} \neq x_{i_2}$, the OR gadget *can* output 1 (see the body text for details).

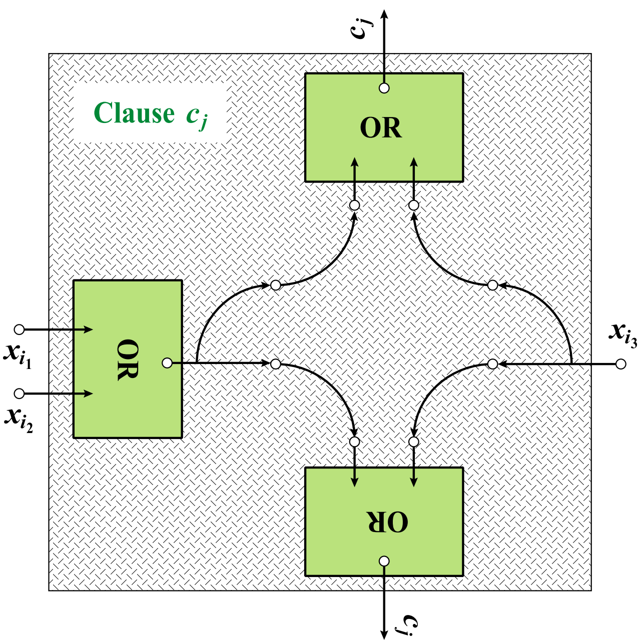


Fig. 11 Clause gadget $c_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. If $x_{i_1} = x_{i_2} = x_{i_3} = 0$, then the clause gadget outputs $c_j = 0$. By switching red and blue guards, one can see that the clause gadget outputs $c_j = 1$ when $x_{i_1} = x_{i_2} = x_{i_3} = 1$. On the other hand, if at least one of x_{i_1}, x_{i_2}, and x_{i_3} is 1, then the clause gadget *can* output $c_j = 1$.

$(1,1,1)$) then the clause gadget *must* output $c_j = 0$ (resp. $c_j = 1$) (see the previous paragraph), and if $(x_{i_1}, x_{i_2}, x_{i_3}) \notin \{(0,0,0),(1,1,1)\}$ then the clause gadget *can* output $c_j = 1$. (In Fig. 13, if $(x_1, x_2, x_3, x_4) = (1,1,0,1)$, there exists a conflict-free 2-chromatic guard set so that all of c_1, c_2, and c_3 output 1. Here, $(1,1,0,1)$ satisfies the 3SAT instance given in the caption.)

Figure 12 is an XNOR gadget, which connects clause gadgets c_j and c_{j+1} for every $j \in \{0,1,\ldots,m-1\}$ (see also

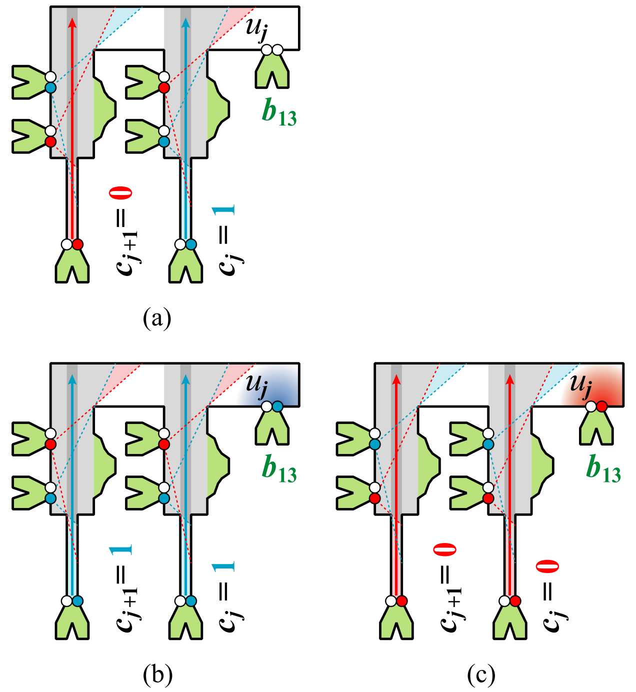


Fig. 12 XNOR gadget. This gadget connects clause gadgets c_j and c_{j+1}. (a) is an invalid placement of red and blue guards. (b,c) If clauses c_j and c_{j+1} have value 1 (resp. value 0), then area u_j can be observed by a blue guard (resp. red guard).

Fig. 13). Figure 12 (a) is an invalid placement of red and blue guards, since we cannot place neither a red guard nor a blue guard on a door vertex of bowl b_{13} in order to observe area u_j. On the other hand, if both clauses c_j and c_{j+1} have value 1 (resp. value 0), then area u_j can be observed by a blue guard (resp. red guard) (see Figs. 12 (b,c)).

Consider a Clause-Linked Planar 3SAT $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, \overline{x_2}, x_3\}$, $c_2 = \{x_1, x_2, \overline{x_4}\}$, and $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$. Figure 2 is a planar embedding of the graph G, which corresponds to C. (It is known that there is a linear-time algorithm for generating a planar embedding of a planar graph [3]. Therefore, any instance of planar 3SAT can be transformed to an embedding of the corresponding graph in polynomial time.)

Figure 13 is a sketch of a polygon P with holes transformed from Clause-Linked Planar 3SAT C. Here, polygon P can be obtained from G by replacing vertices with gadgets of Figs. 5, 11 and by replacing edges with gadgets of Figs. 6–8, 10, 12. In Fig. 13, $c_0 = \{x_0, x_0, x_0\}$ is a dummy clause, where x_0 is a dummy variable.

The transformation from Clause-Linked Planar 3SAT C to polygon P can be done in polynomial time. This is because the size of polygon P (see Fig. 13) is linearly bounded by the size of planar graph G (see Fig. 2). More precisely, a bowl-shape gadget in Fig. 3 contains 20 vertices, and a pocket-gadget in Fig. 4 is a 10-vertex chain. The remaining gadgets of Figs. 5–12 contains a constant number of bowl-shaped gadgets and pocket gadgets. Since each vertex of any gadget is assumed to have integral coordinates,

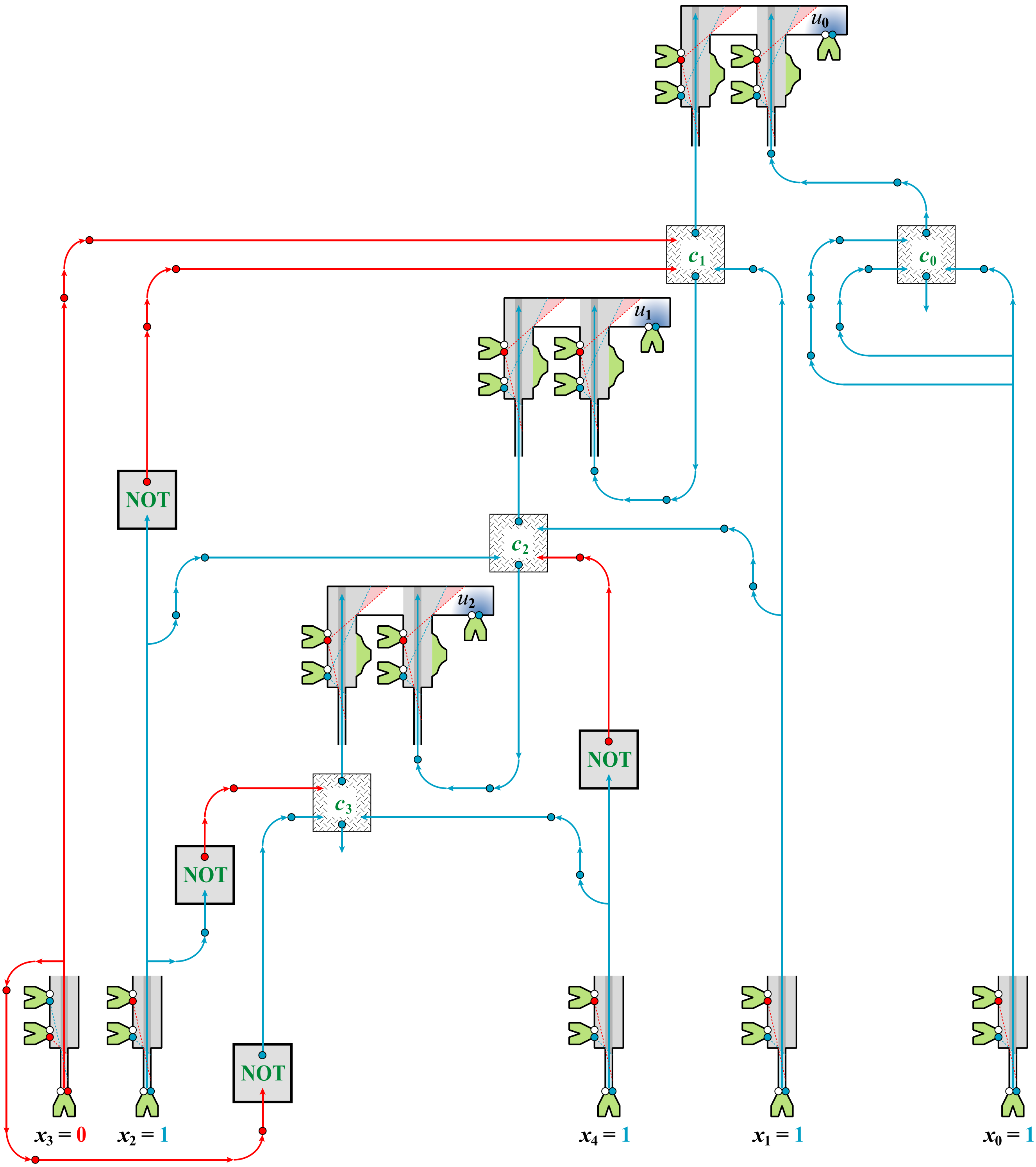


Fig. 13 Sketch of polygon P with holes transformed from $U = \{x_1, x_2, x_3, x_4\}$ and $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, \overline{x_2}, x_3\}$, $c_2 = \{x_1, x_2, \overline{x_4}\}$, and $c_3 = \{\overline{x_2}, \overline{x_3}, x_4\}$. In this figure, $c_0 = \{x_0, x_0, x_0\}$ is a dummy clause, where x_0 is a dummy variable. The polygon with holes constructed according to this figure can be guarded by red and blue guards in the conflict-free condition. From the positions of red and blue guards, one can see that $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$ satisfies all the clauses.

the transformation from Clause-Linked Planar 3SAT C to the planar layout of polygon P can be done in polynomial time.

Lemma 1: *The instance C of 3SAT is satisfiable if and only if there exists a conflict-free 2-chromatic vertex-guard set for the polygon P with holes.*

Proof. (⇒) Suppose that the instance C of 3SAT is satisfiable. In Fig. 13, clause gadget c_0 can emit a blue beam upward, since the dummy clause $c_0 = \{x_0, x_0, x_0\}$ is satisfied if the dummy variable $x_0 = 1$. Then, area u_0 can be observed by a blue guard if c_1 is satisfied. Suppose that c_0 and c_1 are satisfied. Then, area u_1 can be observed by a blue guard if c_2 is satisfied. By continuing this observation, one can see that all areas $u_0, u_1, \dots, u_{m-1}$ can be observed by blue guards if all of $c_1, c_2, \dots, c_m$ are satisfied.

(⇐) Suppose that the instance C of 3SAT is not satisfiable. Consider an arbitrary assignment $(b_1, b_2, \dots, b_n) \in \{0,1\}^n$ for $(x_1, x_2, \dots, x_n)$. Since C is not satisfiable, there exists at least one clause $c_j = \{x_{h_1}, x_{h_2}, x_{h_3}\}$ such that $x_{h_1} = x_{h_2} = x_{h_3} = 0$ when the assignment is $(b_1, b_2, \dots, b_n)$. Here, each of x_{h_1}, x_{h_2}, and x_{h_3} is a positive or negative literal. Furthermore, for the same assignment $(b_1, b_2, \dots, b_n)$, there exists at least one clause $c_k = \{x_{l_1}, x_{l_2}, x_{l_3}\}$ such that $x_{l_1} = x_{l_2} = x_{l_3} = 1$ because of the following reason: Assume for contradiction that there is no $c_k = \{x_{l_1}, x_{l_2}, x_{l_3}\}$ such that $x_{l_1} = x_{l_2} = x_{l_3} = 1$. Then, every clause contains at least one literal x whose value is 0. Now, consider the "inverted" assignment $(\overline{b_1}, \overline{b_2}, \dots, \overline{b_n})$. For the inverted assignment, every clause contains at least one literal of value $\overline{x} = 1$. This implies that C is satisfiable, a contradiction.

Therefore, in any unsatisfiable instance C of 3SAT, there are two clauses $c_j = \{x_{h_1}, x_{h_2}, x_{h_3}\}$ and $c_k = \{x_{l_1}, x_{l_2}, x_{l_3}\}$ such that $x_{h_1} = x_{h_2} = x_{h_3} = 0$ and $x_{l_1} = x_{l_2} = x_{l_3} = 1$ for every assignment $(b_1, b_2, \dots, b_n) \in \{0,1\}^n$. If $j < k$, there exists an integer $j' \in \{j, j+1, \dots, k-1\}$ such that the area $u_{j'}$ is observed by neither a blue guard nor a red guard (see Fig. 12 (a)). The case $k < j$ is similar. This completes the proof of Lemma 1. □

4. Conclusion

In this paper, we studied a chromatic variant of the art gallery problem under the conflict-free conditions. It was shown that the *vertex-to-point* conflict-free chromatic art gallery problem for polygons with holes is NP-hard when the number of colors is two. Proving the NP-hardness of the problem under the *point-to-point* condition remains an open problem.

References

[1] A. Bärtschi and S. Suri, "Conflict-free chromatic art gallery coverage," Algorithmica, vol.68, no.1, pp.265–283, 2014. DOI: 10.1007/s00453-012-9732-5
[2] O. Çağırıcı, S.K. Ghosh, P. Hliněný, and R. Roy, "On conflict-free chromatic guarding of simple polygons," arXiv:1904.08624, 26 pages, 2019.
[3] N. Chiba, T. Nishizeki, S. Abe, and T. Ozawa, "A linear algorithm for embedding planar graphs using PQ-trees," J. Comput. Syst. Sci., vol.30, no.1, pp.54–76, 1985. DOI: 10.1016/0022-0000(85)90004-2
[4] V. Chvátal, "A combinatorial theorem in plane geometry," J. Comb. Theory B., vol.18, no.1, pp.39–41, 1975.
[5] S.J. Eidenbenz, C. Stamm, and P. Widmayer, "Inapproximability results for guarding polygons and terrains," Algorithmica, vol.31, no.1, pp.79–113, 2001.
[6] L.H. Erickson and S.M. LaValle, "A chromatic art gallery problem," Technical report, Department of Computer Science, University of Illinois at Urbana-Champaign, 2011.
[7] S.P. Fekete, S. Friedrichs, M. Hemmer, J.B.M. Mitchell, and C. Schmidt, "On the chromatic art gallery problem," Proc. 30th Canadian Conference on Computational Geometry, Halifax, pp.73–79, 2014.
[8] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, NY, 1979.
[9] F. Hoffman, "On the rectilinear art gallery problem," M. Paterson (eds.), ICALP 1990, LNCS, vol.443, pp.717–728, Springer, Berlin, Heidelberg, 1990.
[10] F. Hoffmann, K. Kriegel, S. Suri, K. Verbeek, and M. Willert, "Tight bounds for conflict-free chromatic guarding of orthogonal art galleries," Comp. Geom. Theor. Appl., vol.73, pp.24–34, 2018.
[11] C. Iwamoto and T. Ibusuki, "Computational complexity of the chromatic art gallery problem for orthogonal polygons," IEICE Trans. Fundamentals, vol.E104-A, no.9, pp.1108–1115, 2021. DOI: 10.1007/978-3-030-39881-1_13
[12] M.J. Katz and G.S. Roisman, "On guarding the vertices of rectilinear domains," Comp. Geom. Theor. Appl., vol.39, no.3, pp.219–228, 2008.
[13] D.T. Lee and A.K. Lin, "Computational complexity of art gallery problems," IEEE Trans. Inform. Theory, vol.32, no.2, pp.276–282, 1986.
[14] J. O'Rourke, Art Gallery Theorems and Algorithms, Oxford University Press, New York, 1987.
[15] A. Pilz, "Planar 3-SAT with a clause-variable cycle," Proc. 16th Scandinavian Symposium and Workshops on Algorithm Theory, Malmö, pp.31:1–31:13, 2018.
[16] F.P. Preparata and M.I. Shamos, Computational Geometry: An Introduction, Springer-Verlag, New York, 1985.
[17] D. Schuchardt and H.-D. Hecker, "Two NP-hard art-gallery problems for ortho-polygons," Math. Logic Quar., vol.41, no.2, pp.261–267, 1995.

Chuzo Iwamoto received the B.Eng. degree from Yamaguchi University in 1990, and M.Eng. and Dr. Eng. degrees from Kyushu University in 1992 and 1995, respectively. From 1995 to 1997, he was a lecturer at Kyushu Institute of Design. In 1997, he joined Hiroshima University as an associate professor, and he is currently a professor of the Graduate School of Advanced Science and Engineering, Hiroshima University.

Tatsuaki Ibusuki received the Bachelor of Liberal Arts and Master of Engineering from Hiroshima University in 2019 and 2021, respectively. He is currently working in IIC Partners CO., LTD.