LETTER *Special Section on Foundations of Computer Science — Foundations of Computer Science Supporting the Information Society —*

Calculation Solitaire is NP-Complete

Chuzo IWAMOTO[†a)], ***Member and*** **Tatsuya IDE**[†], ***Nonmember***

SUMMARY Calculation is a solitaire card game with a standard 52-card deck. Initially, cards A, 2, 3, and 4 of any suit are laid out as four foundations. The remaining 48 cards are piled up as the stock, and there are four empty tableau piles. The purpose of the game is to move all cards of the stock to foundations. The foundation starting with A is to be built up in sequence from an ace to a king. The other foundations are similarly built up, but by twos, threes, and fours from 2, 3, and 4 until a king is reached. Here, a card of rank i may be used as a card of rank $i+13j$ for $j \in \{0, 1, 2, 3\}$. During the game, the player moves (i) the top card of the stock either onto a foundation or to the top of a tableau pile, or (ii) the top card of a tableau pile onto a foundation. We prove that the generalized version of Calculation Solitaire is NP-complete.

key words: *calculation solitaire, card game, NP-complete*

1. Introduction

Calculation is a solitaire card game with a standard 52-card deck [1]. Here, cards A and J,Q,K are regarded as cards of rank 1 and 11, 12, 13, respectively. Initially, cards A, 2, 3, and 4 of any suit are laid out as four foundations (see Fig. 1). The remaining 48 cards are piled up as the stock, and there are four empty tableau piles. The purpose of the game is to move all cards of the stock to foundations (see Fig. 3). The foundation starting with A (called the A-foundation) is to be built up in sequence from an ace to a king. The other foundations are similarly built up, but by twos, threes, and fours from 2, 3, and 4, until a king is reached. Here, a card of rank i may be used as a card of rank $i+13j$ for $j \in \{0, 1, 2, 3\}$. During the game (see Fig. 2), the player moves (i) the top card of the stock either onto a foundation or to the top of a tableau pile, or (ii) the top card of a tableau pile onto a foundation.

In Fig. 2, the top card 4 of the stock can be moved onto the 2-foundation. If the second card 3 of the stock is moved to a tableau pile, then the third card 8 can be moved to the 4-foundation. Similarly, cards 5 and Q in the stock are moved to a tableau pile and the 4-foundation, respectively. Then, the card 3 in a tableau pile can be moved to the 4-foundation.

In this paper, we consider the generalized version of Calculation Solitaire. The generalized $(s \times k)$-card deck includes s suits, and each suit includes k ranks, where k is a prime number. The instance of the *Generalized Calculation Solitaire Problem* is a mid-play configuration consisting of s foundations, at most s tableau piles, and a stock (see Fig. 6). The problem is to decide whether the player can move all cards of the stock and tableau piles to foundations.

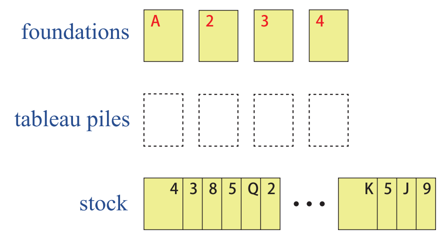

Fig. 1 Initial configuration of Calculation.

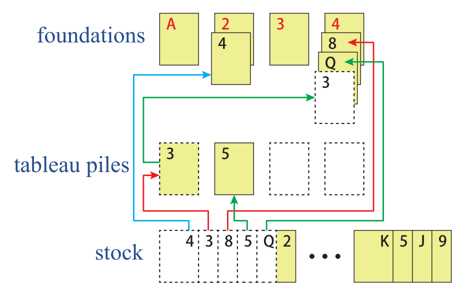

Fig. 2 A mid-play configuration.

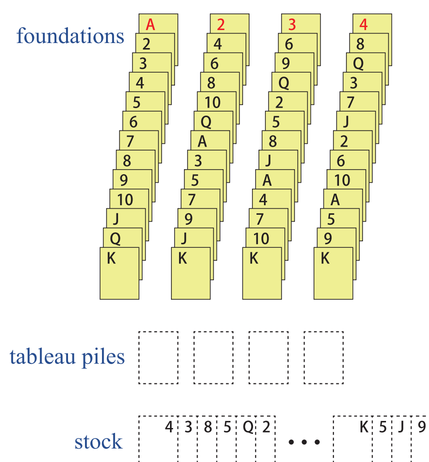

Fig. 3 Final configuration.

Theorem 1: The Generalized Calculation Solitaire Problem is NP-complete.

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[†]The authors are with Graduate School of Advanced Science and Engineering, Hiroshima University, Higashihiroshima-shi, 739–8527 Japan.
a) E-mail: chuzo@hiroshima-u.ac.jp

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It is not difficult to show that the problem belongs to NP, since the player moves at most $s(k-1)$ cards from the stock to foundations directly or by way of tableau piles.

There has been a lot of papers on the computational complexities of solitaire games. For example, FreeCell [5], Klondike [6], and Spider Solitaire [7] are known to be NP-complete. Recently, Arena and Ianni proved the NP-completeness of Scorpion Solitaire [2].

2. Reduction from 3SAT to Generalized Calculation

The definition of 3SAT is mostly from [4]. Let $U = \{x_1, x_2, \dots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. An instance of 3SAT is a collection $C = \{c_1, c_2, \dots, c_m\}$ of clauses over U such that $|c_j| = 3$ for each $c_j \in C$. The 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. This problem is known to be NP-complete. Without loss of generality, we can assume that the number n of variables is even.

We present a polynomial-time transformation from an arbitrary instance C of 3SAT to a mid-play configuration of Calculation such that C is satisfiable if and only if all cards of the stock and tableau piles can be moved to foundations.

First of all, the 1-foundation starts with a sequence of cards $1, 2, \dots, 5$ (see Fig. 4), and the 2-foundation is a single card 2. The stock starts with a sequence of cards $7, 9, \dots, 2i+5, \dots, 2n+5$.

Consider variable x_i for each $i = 1, 2, \dots, n$. Suppose that the 1-foundation is currently a sequence of cards of rank $1, 2, \dots, 2i+3$, and the top of the stock is a card $2i+5$ (see Fig. 4). Variable x_i is transformed to two tableau piles, which start with two gray cards $2i+4$ with labels $x_i = 1$ and $x_i = 0$, followed by blue and red cards with labels x_i and $\overline{x_i}$, respectively. The number of blue (resp. red) cards is q if literal x_i (resp. $\overline{x_i}$) appears q times in C. (In Fig. 6, two gray cards 6 are followed by one blue card 14 and two red cards 26,38 because literals x_1 and $\overline{x_1}$ appear once and twice in C, respectively.)

In Fig. 4, one of the two gray cards $2i+4$ is moved onto the top card $2i+3$ of the 1-foundation. Then, card $2i+5$ of the stock can be moved to the 1-foundation. (In Fig. 6, one of the two gray cards 6 is moved onto card 5 of the 1-foundation. Then, card 7 of the stock can be moved to the 1-foundation.) Repeating this procedure for $i = 1, 2, \dots, n$, we can assign either 1 or 0 to each variable $x_i \in \{x_1, x_2, \dots, x_n\}$.

The 2-foundation will be used and explained later. Consider the 3rd through $(n+2)$-th foundations of Fig. 4. A white card $2i+4$ has already been placed on card $i+2$ of the $(i+2)$-foundation for each $i \in \{1, 2, \dots, n\}$. Thanks to those foundations, a gray card $2i+4$ may not move to

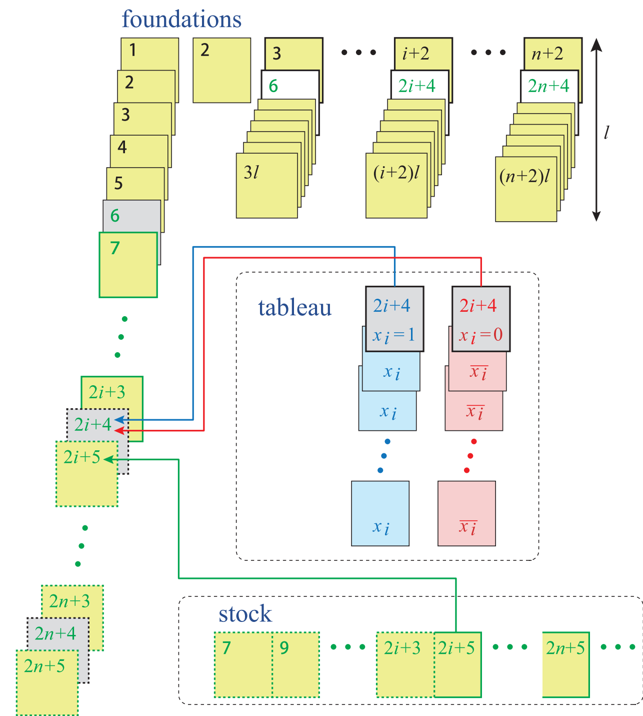


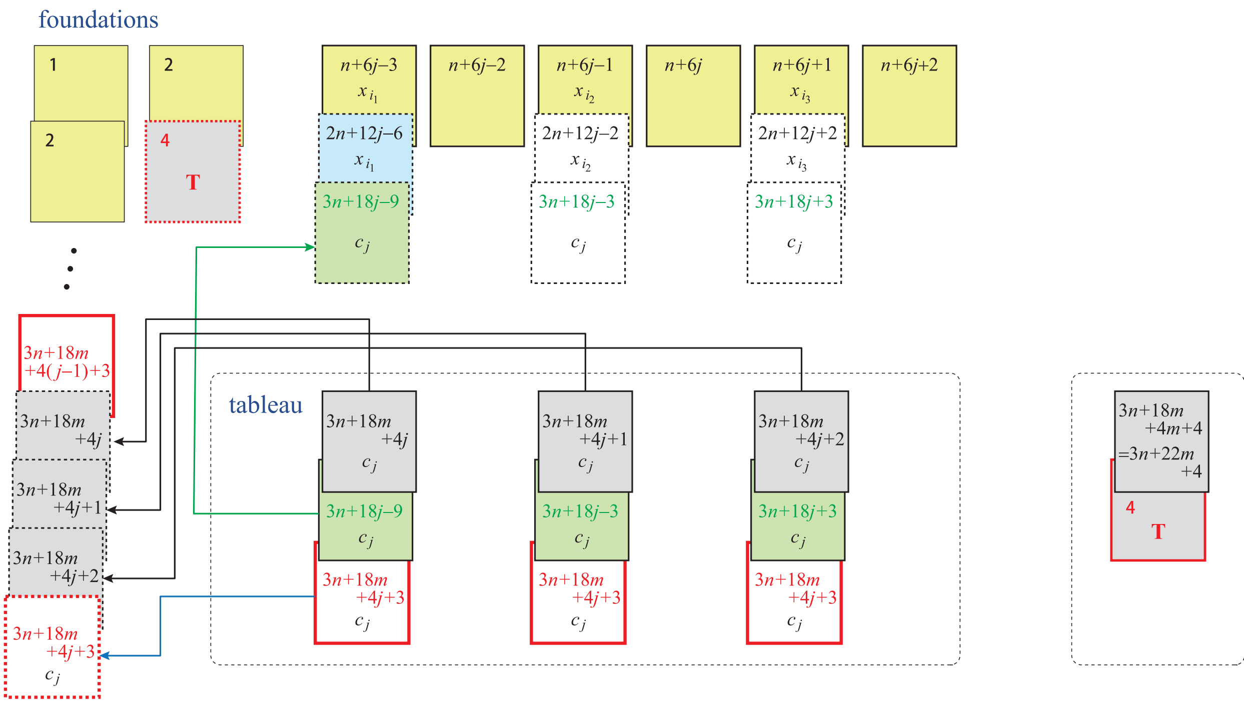
Fig. 4 Variable Gadget.

an unintended foundation. (The number l of layers of those foundations will be fixed later.)

Figure 5 (a) is a clause gadget for $c_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$, where each of x_{i_1}, x_{i_2}, and x_{i_3} is a positive or negative literal. Each clause c_j is transformed to six foundations and three tableau piles. The six foundations start with cards $n+6j-3$ through $n+6j+2$ (see also cards $7, 8, \dots, 12$ when $n = 4$ and $j = 1$ in Fig. 6). Three cards $n+6j-3$, $n+6j-1$, and $n+6j+1$ will be followed by cards $2n+12j-6$, $2n+12j-2$, and $2n+12j+2$ with labels x_{i_1}, x_{i_2}, and x_{i_3}, respectively. (In Fig. 6, cards 7,9,11 will be followed by cards 14,18,22 with labels $x_1, \overline{x_2}, x_3$, since $c_1 = \{x_1, \overline{x_2}, x_3\}$.)

Consider literal x_{i_1} in c_j (see card $n+6j-3$ in Fig. 5 (a)). Note that the card $2n+12j-6$ with label x_{i_1} has not yet been placed on the $(n+6j-3)$-foundation. (i) That card $2n+12j-6$ is placed under the tableau card with *label* $x_{i_1} = 1$ if x_{i_1} is a *positive* literal. (See Fig. 6. Blue card 14 with *positive* literal x_1 is placed under gray card 6 with label $x_1 = 1$, where $n = 4$, $j = 1$, and $2n+12j-6 = 14$.) (ii) That card $2n+12j-6$ is placed under the tableau card with *label* $x_{i_1} = 0$ if x_{i_1} is a *negative* literal. (See Fig. 6. Red cards 26,38 with *negative* literal $\overline{x_1}$ are placed under gray card 6 with label $x_1 = 0$, where $n = 4$, $j = 2, 3$, and $2n+12j-6 = 26, 38$. Note that literal $\overline{x_1}$ appears in c_2, c_3.)

Consider three tableau piles of Fig. 5 (a), where gray cards $3n+18m+4j$, $3n+18m+4j+1$, and $3n+18m+4j+2$ are placed in the top layer (see gray cards 70,71,72 when $n = 4$, $m = 3$, and $j = 1$ in Fig. 6). The second layer of the tableau piles are green cards $3n+18j-9$, $3n+18j-3$, and $3n+18j+3$ of label c_j, which may possibly move to the corresponding foundations (see green cards 21,27,33 in



(a) (b)

Fig. 5 (a) Clause gadget for $c_j = (x_{i_1}, x_{i_2}, x_{i_3})$. (b) Target card 4 in the rightmost tableau pile.

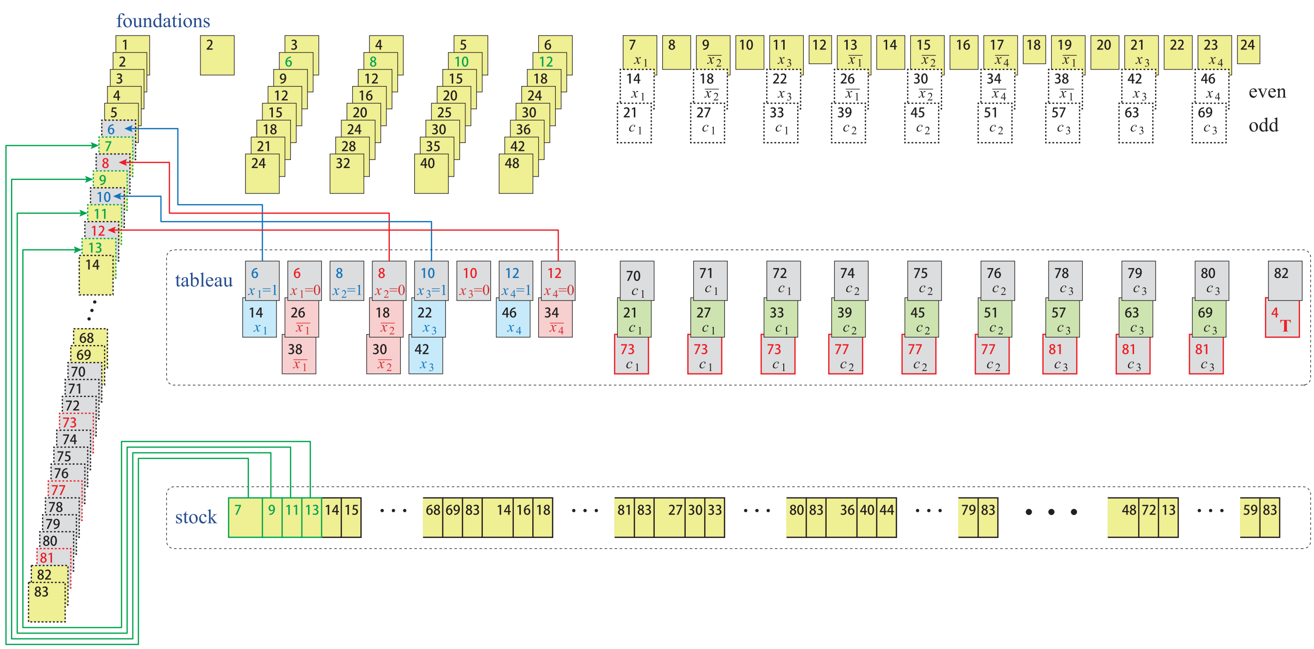


Fig. 6 Calculation transformed from 3SAT when $n = 4$, $m = 3$, and $s = 2 + n + 6m = 24$. The set of clauses is $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, \overline{x_2}, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, and $c_3 = \{\overline{x_1}, x_3, x_4\}$.

Figs. 6 and 7). The third layer of the tableau piles are three cards of the same rank $3n + 18m + 4j + 3$ (see three gray cards of rank 73 in Fig. 6). In this paragraph, we used cards of ranks $3n + 18m + 4j$ through $3n + 18m + 4j + 3$. This is because we had already used a card of rank $3n + 18j + 3$ in the $(n+6j+1)$-foundation (see Fig. 5 (a)). Thus, cards in the corresponding tableau must start with rank $3n+18m+4j$ for $j \in \{1, 2, \dots, m\}$ so that $3n + 18m + 4j$ when $j = 1$ is greater than $3n + 18j + 3$ when $j = m$.

Suppose that the top of the 1-foundation is a card $3n + 18m + 4(j-1) + 3$ (see Fig. 5 (a)). Then, we can move three gray cards $3n+18m+4j$, $3n+18m+4j+1$, and $3n+18m+4j+2$

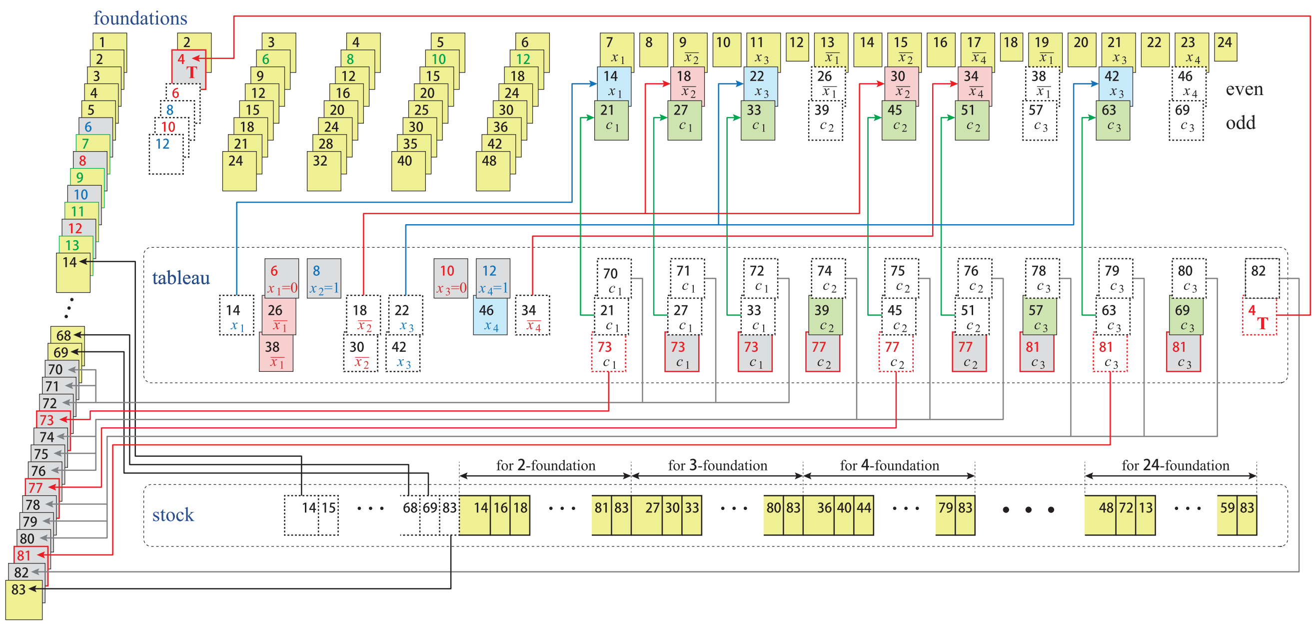


Fig. 7 The target card 4 with label **T** is moved onto the 2-foundation if and only if 3SAT C is satisfiable.

from tableau piles to the 1-foundation in that order. If at least one of the three green cards $3n+18j-9$, $3n+18j-3$, and $3n+18j+3$ is moved from a tableau pile to the corresponding foundation, then we can move a card $3n+18m+4j+3$ to the 1-foundation (see card 73 in the 1-foundation of Fig. 7). We repeat this procedure for $j = 1, 2, \ldots, m$.

Figure 5 (b) is a tableau pile, consisting of card $3n + 22m + 4$ and the *target* card 4 with label **T** (see cards 82 and 4 in Fig. 6). Consider the case $j = m$ in Fig. 5 (a). If one of the three cards $3n+18m+4j+3$ ($= 3n+22m+3$) is moved to the 1-foundation, then card $3n+22m+4$ of Fig. 5 (b) can also be moved to the 1-foundation. Finally, the target card 4 can be moved to the 2-foundation (see cards 82 and 4 in Fig. 7).

Now we fix the value l in Fig. 4 so that l satisfies $(n + 2)l \geq 2n + 12m + 2$. (In Fig. 7, see cards 48 and 46, where $48 = (n + 2)l \geq 46 = 2n + 12m + 2$ when $l = 8$.) Thanks to those l layers, blue and red cards in Fig. 6 may not move to unintended foundations.

The number s of suits is fixed to be $s = 2 + n + 6m$ (see 24 foundations of Fig. 6). The number k of ranks is fixed to be the minimum prime number $p \geq 3n + 22m + 4$. (In Fig. 6, $k = p = 83 \geq 3n + 22m + 4 = 82$.) From Bertrand-Chebyshev theorem [3], for any integer $n \geq 4$, there always exists a prime number p with $n < p \leq 2n$.

Figure 6 is the mid-play configuration of Calculation transformed from 3SAT $C = \{c_1, c_2, c_3\}$, where $c_1 = \{x_1, \overline{x_2}, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, and $c_3 = \{\overline{x_1}, x_3, x_4\}$. In Fig. 6, we can move eight cards 6,7; 8,9; 10,11; 12,13 from tableau piles and the stock to the 1-foundation, which corresponds to a truth assignment $(x_1, x_2, x_3, x_4) \in \{0, 1\}^4$.

In Fig. 7, three blue cards and three red cards can be moved from tableau piles to the 7th through 24th foundations (see cards 14; 18,30; 22,42; 34). Also, cards $14, 15, \ldots, 69$ of the stock can be moved to the 1-foundation.

Consider 3×3 cards with label c_1 in tableau piles (in Fig. 6, see cards 70,71,72; 21,27,33; and 73,73,73). As shown in Fig. 7, we can move (i) cards 70,71,72 to the 1-foundation, (ii) cards 21,27,33 to the 7th through 11th foundations, and (iii) one of the three cards 73 to the 1-foundation. After repeating this procedure for every $c_j \in \{c_1, c_2, c_3\}$, we can move the card 82 to the 1-foundation. Finally, we can move the target card 4 to the 2-foundation.

Once the target card 4 with label **T** is placed on the 2-foundation (see Fig. 7), gray cards 6,8,10,12 can be moved to the 2-foundation. The remaining red, blue, and green cards in tableau piles can be moved to the 7th through 24th foundations (see cards 26,38,46 and 39,57,69). The remaining part of the stock contains all cards for the 2nd through 24th foundations, where the six gray cards 73, 73, 77, 77, 81, 81 are excluded from the stock of Fig. 7.

In Fig. 6, we use only every other foundation among the 7th through 24th foundations. This is because we want blue and red cards to be of even rank, and green cards to be of odd rank. (In Figs. 6 and 7, the $7, 9, \ldots, 23$rd foundations have cards $14, 18, \ldots, 46$ in the second layer, and cards $21, 27, \ldots, 69$ in the third layer.) Since every green card in the tableau has odd rank, it cannot be moved to the second layer of the $7, 9, \ldots, 23$rd foundations.

From this construction, the instance C of 3SAT is satisfiable if and only if all cards in the stock and tableau piles can be moved to foundations. From Figs. 6 and 7, one can see that $(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$ satisfies all clauses of C.

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