

On the Non-existence of Rotation-Symmetric von Neumann Neighbor Number-Conserving Cellular Automata of Which the State Number is Less than Four

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SUMMARY A number-conserving cellular automaton (NCCA) is a cellular automaton such that all states of cells are represented by integers and the total number of its configuration is conserved throughout its computing process. In contrast to normal cellular automata, there are infinitely many assignments of states for NCCAs with a constant state number. As for von Neumann neighbor (radius one) NCCAs with rotation-symmetry, a local function can be represented by summation of four binary functions. In this paper, we show that the minimum size of state set of rotation-symmetric von Neumann neighbor NCCA is 5 by using this representation.

key words: cellular automata, number-conservation

1. Introduction

A number-conserving cellular automaton (NCCA) is a cellular automaton such that all states of cells are represented by integers and the total number of its configuration is conserved throughout its computing process. It can be thought as a kind of modelization of the physical phenomena, for example, for modeling fluid dynamics [4] and highway traffic flow [9].

Boccarda et al. [2]. studied number conservation of one-dimensional CAs on circular configurations based on a general theorem on additive conserved quantities by Hattori et al. [5]. Durand et al. [3]. studied the two-dimensional case and the relation between several boundary conditions. Although their theorems are useful for deciding given CAs are number-conserving or not, it is quite difficult to design NCCAs with complex transition rules.

As for von Neumann neighbor (radius one) NCCAs with rotation-symmetry, a local function can be represented by summation of four binary functions [10]. This indicates that the type of the neighborhood and the symmetry of the rule constrain the design of the rule for an NCCA. By using this framework, we constructed 14-state logically universal NCCA.

In the general case, 2-state logically universal CA was constructed by Banks [1]. But in the NCCAs case, it turned out that it is difficult to construct a small state universal NCCA for the constraint of rotation-symmetry. In this paper, we make clear that there is no rotation-symmetric von

Neumann neighbor NCCA of which the size (element number) of state set is less than four.

There is a study of enumeration of one dimensional radius one NCCAs for a constant size of state set by now. But it is not exhaustive listing for the size of state set. Because there are infinitely many assignments of states for NCCAs with a constant size of state set.

At first, we show another way of inspecting the number of state of NCCAs that the above representation is also useful for estimating the size of state set. To be more precise, we calculate state number of rotation-symmetric NCCA of which all state are represented by symbols. As a result, we show that the minimum size of state set of rotation-symmetric von Neumann neighbor NCCA is 5.

2. Definitions

Definition 2.1: A deterministic two-dimensional von Neumann neighbor cellular automaton is a system defined by $A = (\mathbf{Z}^2, Q, f, q)$, where \mathbf{Z} is the set of all integers, Q is a non-empty finite set of internal states of each cell, $f : Q^5 \rightarrow Q$ is a mapping called a local function and $q \in Q$ is a quiescent state that satisfies $f(q, q, q, q, q) = q$.

A configuration over Q is a mapping $\alpha : \mathbf{Z}^2 \rightarrow Q$ and the set of all configurations over Q is denoted by $\text{Conf}(Q)$, i.e., $\text{Conf}(Q) = \{\alpha | \alpha : \mathbf{Z}^2 \rightarrow Q\}$.

The function $F : \text{Conf}(Q) \rightarrow \text{Conf}(Q)$ defined as follow is called the global function of A .

$$\begin{aligned} \forall (x, y) \in \mathbf{Z}^2, \\ F(\alpha)(x, y) = f(\alpha(x, y), \alpha(x, y + 1), \alpha(x + 1, y), \\ \alpha(x, y - 1), \alpha(x - 1, y)). \end{aligned}$$

Definition 2.2: CA A is said to be number-conserving iff

$$\forall \mathbf{c} \in \text{Conf}(Q), \sum_{(x,y) \in \mathbf{Z}^2} \{F(\mathbf{c})(x, y) - \mathbf{c}(x, y)\} = 0.$$

Definition 2.3: We say that CA A is trivial if its global function F satisfies the following condition.

$$\forall \mathbf{c} \in \text{Conf}(Q), \mathbf{c} = F(\mathbf{c}).$$

We say that A is non-trivial if it is not trivial.

In this paper, we only consider CAs with finite configurations, i.e., the number of cells which states are not quiescent is finite. Next we define some symmetry conditions.

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Definition 2.4: CA A is said to be *rotation-symmetric* if its local function f satisfies the following condition.

$$\forall c, u, r, d, l \in Q, f(c, u, r, d, l) = f(c, r, d, l, u).$$

3. Von Neumann Neighbor Number-Conserving Cellular Automata

We showed a necessary and sufficient condition for a von Neumann neighbor CA to be number-conserving in [10].

Theorem 3.1: [10] A deterministic two-dimensional von Neumann neighbor CA $A = (\mathbf{Z}^2, Q, f, q)$ is number-conserving iff f satisfies

$$\begin{aligned} &\exists g_U, g_R, g_D, g_L, h_{UR}, h_{RD}, h_{DL}, h_{LU} : Q^2 \rightarrow \mathbf{Z}, \\ &\forall c, u, r, d, l \in Q, \\ &f(c, u, r, d, l) = c + g_U(c, u) + g_R(c, r) + g_D(c, d) \\ &\quad + g_L(c, l) + h_{UR}(u, r) + h_{RD}(r, d) + h_{DL}(d, l) \\ &\quad + h_{LU}(l, u), \\ &g_U(c, u) = -g_D(u, c), g_R(c, r) = -g_L(r, c), \\ &h_{UR}(u, r) = -h_{DL}(r, u), h_{RD}(r, d) = -h_{LU}(d, r). \end{aligned}$$

Next, we derived a necessary and sufficient condition for a rotation-symmetric CA to be number-conserving from the condition.

Corollary 3.1: [10] A deterministic two-dimensional rotation-symmetric von Neumann neighbor CA $A = (\mathbf{Z}^2, Q, f, q)$ is number-conserving iff f satisfies

$$\begin{aligned} &\exists g, h : Q^2 \rightarrow \mathbf{Z}, \forall c, u, r, d, l \in Q, \\ &f(c, u, r, d, l) = c + g(c, u) + g(c, r) + g(c, d) \\ &\quad + g(c, l) + h(u, r) + h(r, d) + h(d, l) + h(l, u), \\ &g(c, u) = -g(u, c), h(u, r) = -h(r, u). \end{aligned}$$

According to this condition, a local function of a rotation-symmetric NCCA is represented by summation of two binary function g and h . The binary function g indicates the number flow between two cells in a vertical or horizontal direction, and h does in a diagonal direction. In a vertical or horizontal flow, a value moves on two cells of which states are arguments of g . But in the diagonal flow case, cells on which a value moves don't correspond to arguments of h . This causes divergence of the state number.

In order to design a rotation-symmetric NCCA, we only have to define g and h . Although it may not be a CA for the divergence of its state number, there exists a procedure to modify these functions for construction of an NCCA after designing g and h . The procedure is following.

Procedure 3.1: [10]

1. Choose a *partial* state set \tilde{Q} of size $k(> 0)$ and design binary function $g(x, y)$ and $h(x, y)$ for $(x, y) \in \tilde{Q}^2$.
2. $Q' := \emptyset$. If $f(v) \notin \tilde{Q}$, then $Q' := Q' \cup f(v) (\forall v \in \tilde{Q}^5)$.
 $Q := \tilde{Q} \cup Q'$.
3. If $Q' = \emptyset$, then stop this procedure.

4. $S := \emptyset$. If $f(c, u, r, d, l) \notin Q$, then $S := S \cup \{(c, u, r, d, l) | (\forall c \in Q', \forall u, r, d, l \in Q)\}$.
5. If $S = \emptyset$, then stop this procedure.
6. $\forall (c, u, r, d, l) \in S$,
 - a. $T := \{(u, r), (r, d), (d, l), (l, u)\}$, $U := \{u, r, d, l\}$.
 - b. $g(c, i) := v_i$ and $g(i, c) := -v_i (\forall i \in U)$ as $\sum_{i \in U} v_i = -\sum_{w \in T} h(w)$.
7. $Q' := \emptyset$. $\forall (c, u, r, d, l) \in Q^5$,
 - a. $\xi := c + g(c, u) + g(c, r) + g(c, d) + g(c, l)$.
 - b. If $\xi \notin Q$, then $Q' := Q' \cup \{\xi\}$.
8. $Q := Q \cup Q'$. Goto line 4.

We will show an example of a construction of a rotation-symmetric NCCA according to this procedure.

Example 3.1: Let's consider an NCCA $A_{ex} = (\mathbf{Z}^2, Q_{ex}, f_{ex}, 0)$. We define $\tilde{Q}_{ex} = \{0, 4\}$, $g(0, 4) = 1$, $g(4, 0) = -1$, and other values of g and all values of h as 0. According to the line 2 of the Procedure 3.1, $Q'_{ex} = \{1, 2, 3\}$ and $Q_{ex} = \{0, 1, 2, 3, 4\}$. In line 4, S is \emptyset because all values of h are 0. Therefore this procedure stops at line 5. The size of state set is 5.

4. A Relationship between the Binary Function of a Rotation-Symmetric Number-Conserving Cellular Automaton and Its State-Number

Any rotation-symmetric NCCA can be represented by the functions g and h of Corollary 3.1. In this section, we show there is no assignment of values to these functions for the local function of an NCCA which state is less than 5 and the minimum assignment of these functions can produce 5-state NCCA rules.

Lemma 4.1: If we define $\tilde{Q} = \{a, b\}$ and $g(a, b) \neq 0$ for any a and $b(a < b)$ to design a non-trivial rotation-symmetric von Neumann neighbor NCCA, $|Q|$ of this NCCA is in the range [5, 10].

Proof. We set $g(a, b) = -g(b, a) = \alpha (\alpha \in \mathbf{N})$ and regard other value of g as 0. When we calculate f for all \tilde{Q}^5 according to the corollary 3.1, it becomes clear that $a + \alpha, a + 2\alpha, a + 3\alpha, a + 4\alpha, b - 4\alpha, b - 3\alpha, b - 2\alpha$ and $b - \alpha$ are also needed in addition to a and b . We have to define values of g of which arguments includes these new states. If we regard these values as 0, other states are not generated. In the case of $b - a > 8\alpha$, these numbers are different each other as followings (Fig. 1). In the case of $b - a \leq 8$, there are cases that $|Q|$ gets smaller than 10 because some of them overlap each other. When $b - a = 8\alpha, a + 4\alpha = b - 4\alpha$, the $|Q|$ is 9. In the same way, when $b - a = n\alpha (n = 1, 2, \dots, 7)$, there is $5 - |n - 4|$ overlaps. So the $|Q|$ is $5 + |n - 4|$. Therefore the $|Q|$ is in the range [5, 10]. \square



Fig. 1 $|Q|$ on a number line.

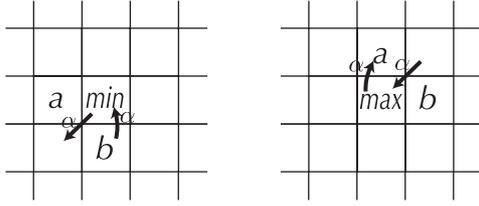


Fig. 2 Value flows on configurations.

Lemma 4.2: If we define $\tilde{Q} = \{a, b\}$ and $h(a, b) \neq 0$ for any a and $b (a < b)$ to design a non-trivial rotation-symmetric von Neumann neighbor NCCA, the minimum value of $|Q|$ is 5.

Proof. We set $h(a, b) = -h(b, a) = \alpha (\alpha \in \mathbf{N})$ and regard other value of h as 0. We also set the minimum and the maximum value of Q as min and max respectively. To prevent divergence of the state number, the two value $g(min, b) = \alpha, g(max, a) = -\alpha$ need to be defined at least (Fig. 2). When we calculate f for all \tilde{Q}^5 according to the corollary 3.1, it becomes clear that $Q = \{min, min + \alpha, min + 2\alpha, min + 3\alpha, min + 4\alpha, b - 4\alpha, b - 3\alpha, b - 2\alpha, b - \alpha, b, a, a + \alpha, a + 2\alpha, a + 3\alpha, a + 4\alpha, max - 4\alpha, max - 3\alpha, max - 2\alpha, max - \alpha, max\}$. These numbers must satisfy the following inequalities.

$$\begin{aligned} min &\leq a, b \leq max, min \leq b - 4\alpha, \\ a + 4\alpha &\leq max, 4\alpha \leq max - min. \end{aligned}$$

If $min = a$ and $max = b$, $Q = \{a, a + \alpha, a + 2\alpha, a + 3\alpha, a + 4\alpha, b - 4\alpha, b - 3\alpha, b - 2\alpha, b - \alpha, b\}$. In the same way of the proof of lemma 3.1, when $b - a = 4\alpha$, $|Q|$ is 5 and this is the minimum value of $|Q|$. If we calculate f for all Q^5 , we can verify the sufficiency of Q . \square

Theorem 4.1: The minimum value of $|Q|$ of a non-trivial rotation-symmetric von Neumann neighbor NCCA is 5.

Proof. According to lemma 4.1 and 4.2, if the number of non-zero values of function g or h is one, the $|Q|$ needs 5 at least. If the number of non-zero values of function g or h is more than two, $|\tilde{Q}|$ needs more than 2. Therefore, there is no possibility that $|Q|$ is below 5. \square

5. Conclusion

In this paper, we show that the minimum size of state set

of rotation-symmetric von Neumann neighbor NCCA is 5. Although the representation of a local function by binary functions was suggested for an easy construction method of a rotation-symmetric NCCA, it is also useful for estimating the lower bound of its size of state set. As the result, the minimum state set size of the rotation-symmetric von Neumann neighbor universal NCCA should be from 5 to 13 so far. But the careful inspection of the representation will reduce the range.

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