PAPER

Voronoi Game on a Path

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SUMMARY The Voronoi game is a two-person perfect information game modeling a competitive facility location. The original version of the game is played on a continuous domain. Only two special cases (1-dimensional case and 1-round case) have been extensively investigated. Recently, the discrete Voronoi game of which the game arena is given as a graph was introduced. In this note, we give a complete analysis of the discrete Voronoi game on a path. There are drawing strategies for both the first and the second players, except for some trivial cases.

key words: *Discrete Voronoi game, combinatorial game theory*

1. Introduction

The Voronoi game is an idealized model for a competitive facility location. It was proposed by Ahn, Cheng, Cheong, Golin, and Oostrum [1]. The Voronoi game is played on a bounded continuous arena by two players. Two players $\mathcal{B}$ (black) and $\mathcal{W}$ (white) put marks on a predetermined number t of points alternately, and the continuous field is subdivided according to the *nearest neighbor rule*. At the final step, the player who dominates larger area wins. There is a web site introducing the game [6]. You can play the game on the site.

The Voronoi game is a natural game, but the general case seems to be very hard to analyze from the theoretical point of view. Hence, Ahn et al. investigated the case where the game field is a bounded 1-dimensional continuous domain in [1]. On the other hand, Cheong, Har-Peled, Linial, and Matoušek [2], and Fekete and Meijer [4] dealt with a 2-dimensional case, but they restricted the games to be one-round; $\mathcal{B}$ puts his t marks first, and next $\mathcal{W}$ puts his t marks.

Recently, a natural discrete model was proposed independently by Teramoto, Demaine, and Uehara [5] and Dürr and Thang [3]. The discrete Voronoi game is played on a given finite graph G, instead of a bounded continuous arena. Each vertex of G can be assigned to nearest vertices occupied by $\mathcal{B}$ or $\mathcal{W}$, according to the *nearest neighbor rule*. (Hence, a vertex can be "tie" when it has the same distance from a vertex occupied by $\mathcal{B}$ and another vertex occupied by $\mathcal{W}$.) Finally, the player who dominates larger area (or a larger number of vertices) wins.

Teramoto et al. showed that the discrete Voronoi game is PSPACE-complete for general case, and NP-complete for one-round case. They also investigated complete k-ary trees, and showed that the first player $\mathcal{B}$ has an advantage for the discrete Voronoi game on a complete k-ary tree, when the tree is sufficiently large (comparing to t). Their results for a complete k-ary tree require some conditions for k, n, and t. Hence, one of the simplest case cannot be captured; what if G is a path?

In this note, we show a complete analysis of the discrete Voronoi game on a path. We state that it always ends up in a draw except for some trivial cases. We give the more precise statement below.

Theorem 1: None of the players $\mathcal{B}$ and $\mathcal{W}$ has a winning strategy of the discrete Voronoi game on a path, except the case that the length of the path is even and the number of the rounds is equal to one. In this exceptional case, $\mathcal{B}$ has a trivial winning strategy.

The proof of the theorem has two parts. In one part, we show that the first player $\mathcal{B}$ has no winning strategy (except the trivial case). Then in the other part we show that the second player $\mathcal{W}$ also has no winning strategy. These complete the proof.

2. Definitions

In this section, we formulate the discrete Voronoi game on a graph. We denote a Voronoi game by $VG(G, t)$, where $G = (V, E)$ is a simple undirected graph, and t is the number of rounds.

The two players, $\mathcal{B}$ (black) and $\mathcal{W}$ (white), alternately occupy an *empty* vertex on the graph G. The empty vertex is a vertex not occupied yet. This implies that $\mathcal{B}$ and $\mathcal{W}$ cannot occupy the same vertex, and each player cannot occupy the same vertex twice or more. Hence, it is implicitly assumed that $0 < 2t \leq |V|$ holds.

Let B_i (resp. W_i) be the set of vertices occupied by player $\mathcal{B}$ (resp. $\mathcal{W}$) at the end of the i-th round. We define the distance $d(v, w)$ between two vertices v and w as the number of edges on the shortest path between them. Each vertex of G can be assigned to the nearest vertices occupied by $\mathcal{B}$ and $\mathcal{W}$, according to the *nearest neighbor rule*. We define a *dominated set* $\mathcal{V}(A, B)$ (or *Voronoi regions*) of a subset $A \subset V$ against a subset $B \subset V$, where $A \cap B = \emptyset$ as

$$\mathcal{V}(A, B) = \left\{ u \in V \;\middle|\; \min_{v \in A} d(u, v) < \min_{w \in B} d(u, w) \right\} \setminus A.$$

The dominated sets $\mathcal{V}(B_i, W_i)$ and $\mathcal{V}(W_i, B_i)$ represent the

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sets of vertices dominated at the end of the i-th round by $\mathcal{B}$ and $\mathcal{W}$, respectively. Let $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}_{\mathcal{W}}$ denote $\mathcal{V}(B_t, W_t)$ and $\mathcal{V}(W_t, B_t)$, respectively. Since some vertices can be "tie" when they have the same distance from a vertex occupied by $\mathcal{B}$ and another vertex occupied by $\mathcal{W}$, there may exist the set N_i of the *neutral* vertices, $N_i := \{u \in V \mid \min_{v \in B_i} d(u, v) = \min_{w \in W_i} d(u, w)\}$. Clearly, each vertex in N_i belongs neither to $\mathcal{V}(B_i, W_i)$ nor to $\mathcal{V}(W_i, B_i)$.

The player who dominates larger number of vertices at the end of the final round wins, in the discrete Voronoi game. More precisely, $\mathcal{B}$ *wins* if $|\mathcal{V}_{\mathcal{B}}| > |\mathcal{V}_{\mathcal{W}}|$, and $\mathcal{W}$ wins (or $\mathcal{B}$ *loses*) if $|\mathcal{V}_{\mathcal{B}}| < |\mathcal{V}_{\mathcal{W}}|$. The game is *draw* otherwise.

We denote by P_n a path of length n. Note that the length of a path is the number of edges in it, not the number of vertices in it. We denote the vertices of P_n by $v_0, v_1, \ldots, v_n$, and we denote the edges of P_n by $e_1, e_2, \ldots, e_n$, where an edge e_i connects v_{i-1} and v_i.

We call a subpath P of P_n with at least one empty vertex as a *segment*, if both the ends of P are occupied, and all the other vertices are empty. We sometimes call a subpath P of P_n a segment, too, if P's one end is empty v_0 or empty v_n, the other end is occupied, and the other vertices are empty. A segment whose ends are occupied by $\mathcal{B}$ is a *black segment*, and a segment whose ends are occupied by $\mathcal{W}$ is a *white segment*. A segment whose ends are occupied by $\mathcal{B}$ and $\mathcal{W}$ is *neutral*. Clearly, a black segment contributes to $\mathcal{V}_{\mathcal{B}}$, and a white segment contributes to $\mathcal{V}_{\mathcal{W}}$, while a neutral segment contributes neither to $\mathcal{V}_{\mathcal{B}}$ nor to $\mathcal{V}_{\mathcal{W}}$.

We first observe the following trivial cases.

Observation 1:
(1) $VG(G = (V, E), t)$ is draw if $|V| = 2t$.
(2-1) $\mathcal{B}$ wins on $VG(P_n, 1)$ if n is even.
(2-2) $VG(P_n, 1)$ is draw if n is odd.

Note that P_n, a path of length n, has $n+1$ vertices. Thus, the number of vertices in P_n is even, if n is odd.
Proof: (1): Every vertex is finally occupied.
(2-1): The first player $\mathcal{B}$ occupies the center vertex $v_{n/2}$; then the second player $\mathcal{W}$ should lose.
(2-2): If the first player $\mathcal{B}$ does not occupy one of two center vertices $v_{(n-1)/2}$ and $v_{(n-1)/2+1}$, the second player $\mathcal{W}$ can win by occupying one of the two center vertices that is closer to the vertex occupied by the first player. Thus, the first player occupies one of the two center vertices, then the second player occupies the other one, and the game ends in a draw. □

3. $\mathcal{B}$ has No Winning Strategy Except the Trivial Case

Assume that $t > 1$. We show that $\mathcal{B}$ has no winning strategy. In order to show this, we show that $\mathcal{W}$ has a drawing strategy. The idea is simple. If $\mathcal{W}$ occupies the vertex symmetric to that last occupied by $\mathcal{B}$ in each round, then clearly $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}_{\mathcal{W}}$ are symmetric, and thus, $|\mathcal{V}_{\mathcal{B}}| = |\mathcal{V}_{\mathcal{W}}|$ holds. Therefore, we have the following lemma.

Lemma 2: $\mathcal{B}$ has no winning strategy of the game $VG(P_n, t)$, when t is greater than one, and n is odd.

When n is even (P_n has odd number of vertices), $\mathcal{B}$ can occupy the center vertex. In this case, $\mathcal{W}$ cannot occupy the symmetric vertex. We consider this case next. First, we prove the following lemma.

Lemma 3: $\mathcal{W}$ has a drawing strategy on the $VG(P_n, t)$, when $B_1 = \{v_0\}$, and $W_1 = \{v_n\}$.

Proof: At the beginning of the round 2, the number of black segments is zero. Occupying a vertex, $\mathcal{B}$ can increase the number of black segments by at most one. When $\mathcal{B}$ increased the number of black segments, $\mathcal{W}$ can decrease it by occupying a vertex in a black segment. Therefore, the number of black segments can be always zero at the end of $\mathcal{W}$'s move.

Clearly, vertices in neutral segments do not contribute $\mathcal{V}_{\mathcal{B}}$ nor $\mathcal{V}_{\mathcal{W}}$. Vertices in white segments only contribute $\mathcal{V}_{\mathcal{W}}$, and does not contribute $\mathcal{V}_{\mathcal{B}}$. Therefore, $\mathcal{B}$ cannot make $\mathcal{V}_{\mathcal{B}}$ positive, since the number of black segments is zero. Hence $\mathcal{B}$ cannot win. □

We can prove the following lemma in the similar manner.

Lemma 4: Suppose that $|\mathcal{V}(B_{k-1}, W_{k-1})| \leq |\mathcal{V}(W_{k-1}, B_{k-1})|$ holds for some positive $k \leq t$. If $\mathcal{B}$ occupies, in round k, a vertex in a neutral segment P, $\mathcal{W}$ can keep $|\mathcal{V}(B_k, W_k)| \leq |\mathcal{V}(W_k, B_k)|$.

Proof: Occupying a vertex in a neutral segment, $\mathcal{B}$ can increase the number of black segments at most one. If $\mathcal{B}$ made a new black segment, $\mathcal{W}$ can decrease the number of black segments by occupying a vertex in the black segment made in the round. Hence, $\mathcal{W}$ can make all the segments in P at the end of the round k be neutral. Thus, $|\mathcal{V}(B_k, W_k)| \leq |\mathcal{V}(W_k, B_k)|$ holds.

Note that, in the case that the number of empty vertices in P is exactly one at the beginning of the round k, $\mathcal{W}$ cannot occupy any vertex in P. However, in this case, $\mathcal{W}$ can occupy a vertex in some black segment. If there is no black segment, $|\mathcal{V}(B_{k-1}, W_{k-1})| = |\mathcal{V}(B_k, W_k)| = 0$. Therefore, $|\mathcal{V}(B_k, W_k)| \leq |\mathcal{V}(W_k, B_k)|$ holds. □

Next, we consider the case that $\mathcal{B}$ does not occupy the center vertex $v_{n/2}$ at the first round.

Lemma 5: $\mathcal{W}$ has a drawing strategy on the $VG(P_n, t)$, when $t > 1$, n is even, and $B_1 \neq \{v_{n/2}\}$.

Proof: $\mathcal{W}$ continues the symmetric strategy until $\mathcal{B}$ occupies the center vertex. Let k be the round at that $\mathcal{B}$ occupies the center vertex.

Clearly, $|\mathcal{V}(B_{k-1}, W_{k-1})| = |\mathcal{V}(W_{k-1}, B_{k-1})|$ holds. Moreover, $v_{n/2}$ is in a neutral segment P at the beginning of the round k. Thus, $\mathcal{W}$ can keep $|\mathcal{V}(B_k, W_k)| \leq |\mathcal{V}(W_k, B_k)|$ by Lemma 4.

At the round $k' > k$, if $\mathcal{B}$ occupies an empty vertex not contained in P, $\mathcal{W}$ takes the symmetric strategy. Of course, $|\mathcal{V}(B_{k'}, W_{k'})| \leq |\mathcal{V}(W_{k'}, B_{k'})|$ holds. If $\mathcal{B}$ occupies an empty vertex contained in P, $\mathcal{W}$ can keep $|\mathcal{V}(B_{k'}, W_{k'})| \leq |\mathcal{V}(W_{k'}, B_{k'})|$ by Lemma 4.

Note that, it is possible that $\mathcal{W}$ cannot take the symmetric strategy, since the method in Lemma 4 sometimes forces $\mathcal{W}$ to occupy a vertex not in P. However, in this case, occupying a vertex in a black segment, $\mathcal{W}$ can keep $|\mathcal{V}(B_{k'}, W_{k'})| \leq |\mathcal{V}(W_{k'}, B_{k'})|$, since the reason why $\mathcal{W}$ cannot take the symmetric strategy is always that the vertex is already occupied by $\mathcal{W}$. □

The last case that we have to consider is that $\mathcal{B}$ occupies the center vertex at the first round.

Lemma 6: $\mathcal{W}$ has a drawing strategy on the $VG(P_n, t)$, when $t > 1$, n is even, and $B_1 = \{v_{n/2}\}$.

Proof: $\mathcal{W}$ occupies $v_{n/2+1}$ at the first round. If $\mathcal{B}$ occupies a vertex whose symmetric vertex is empty, $\mathcal{W}$ occupies the symmetric vertex, and changes to the strategy of Lemma 5. Otherwise, $\mathcal{W}$ occupies the vertex adjacent to the vertex $\mathcal{B}$ occupied last, unless it is not the last round. At the last round, $\mathcal{W}$ occupies the vertex v_0 or v_n to win. Then, this strategy is a drawing strategy. □

From Lemmas 2, 5, and 6, we have the following theorem.

Theorem 7: $\mathcal{B}$ has no winning strategy on $VG(P_n, t)$, when t is greater than one.

4. $\mathcal{W}$ has No Winning Strategy

The basic idea is that, if $\mathcal{B}$ can occupy the vertices at regular intervals (see Fig. 1), $\mathcal{B}$ does not lose. Note that we represent vertices by boxes in figures in this paper.

Lemma 8: $\mathcal{W}$ cannot win on $VG(P_n, t)$, if $n = t\times(l+1)-1$, and

$$B_t = \{v_{\lfloor l/2 \rfloor}, v_{\lfloor l/2 \rfloor+(l+1)}, v_{\lfloor l/2 \rfloor+2(l+1)}, \ldots, v_{\lfloor l/2 \rfloor+(t-1)(l+1)} = v_{n-\lceil l/2 \rceil}\},$$

for some positive integer l.

Proof: Since all the moves of $\mathcal{B}$ are specified, we can regard the game as one-round game. Regarding the two end segments as one segment, there are t black segments of length l, at the end of the $\mathcal{B}$'s round. Occupying exactly one empty vertex of each segment, $\mathcal{W}$ can make the $\mathcal{B}$'s outcome $|\mathcal{V}_{\mathcal{B}}| - |\mathcal{V}_{\mathcal{W}}|$ at most one. Note that $\mathcal{W}$ should occupy $v_{\lfloor l/2 \rfloor+(t-1)(l+1)+1}$ in the end segment. Otherwise, $\mathcal{W}$ loses. In this case, since there is no white segment, $\mathcal{W}$ cannot win. Precisely, there can be one white segment at the end. However, there is a longer or the same size black segment at the opposite end. If $\mathcal{W}$ wants to win, $\mathcal{W}$ should occupy two vertices in a black segment to make a white segment. However, to do so, the number of empty vertices in the white segment is at most $l-2$, and $\mathcal{W}$ have to give up a black segment of l empty vertices. Thus, $\mathcal{W}$ cannot win. □

In fact, $\mathcal{W}$ can easily avoid this strategy, since $\mathcal{W}$ can occupy some of the vertices that $\mathcal{B}$ wants to occupy. However, there is a good break-though for $\mathcal{B}$. Let n be $t \times (l+2) - 1$. Then $\mathcal{B}$ can occupy one of $v_{\lfloor l/2 \rfloor}$ and $v_{\lfloor l/2 \rfloor+1}$, one of $v_{\lfloor l/2 \rfloor+(l+2)}$ and $v_{\lfloor l/2 \rfloor+(l+2)+1}$, $\ldots$, one of $v_{\lfloor l/2 \rfloor+(t-1)(l+2)}$ and $v_{\lfloor l/2 \rfloor+(t-1)(l+2)+1}$ (see Fig. 2). Thus, we have the lemma below.

Lemma 9: $\mathcal{W}$ cannot win on $VG(P_n, t)$, if $n = t\times(l+2)-1$ for some positive integer l.

Proof: $\mathcal{B}$ can occupy one of $v_{\lfloor l/2 \rfloor}$ and $v_{\lfloor l/2 \rfloor+1}$, one of $v_{\lfloor l/2 \rfloor+(l+2)}$ and $v_{\lfloor l/2 \rfloor+(l+2)+1}$, $\ldots$, one of $v_{\lfloor l/2 \rfloor+(t-1)(l+2)}$ and $v_{\lfloor l/2 \rfloor+(t-1)(l+2)+1}$. If $\mathcal{W}$ occupies one vertex in each black segment, $\mathcal{W}$ cannot make a white segment. If $\mathcal{W}$ occupies two vertices in a black segment, $\mathcal{W}$ can make a white segment of at most l empty vertices. However, in this case, $\mathcal{W}$ have to give up a black segment of at least l empty vertices. Therefore, $\mathcal{W}$ cannot win. □

The lemma above deals with a very special case. In most cases, there is no integer l satisfying the condition. We separate the other cases into the following two cases

1. $0 < (n+1) \bmod t \leq t/2$.
2. $(n+1) \bmod t > t/2$.

Now, we consider the case 1. Let $r = (n+1) \bmod t$, and let $l = (n-r+1)/t-2$. The idea is that, we alternately arrange $l+1$ empty vertices and l empty vertices r times (with separators of two vertices), and then we arrange the remaining l empty vertices $t-2r-1$ times. Precisely, when $t = 2$, and $r = 1$ hold, we only arrange $l+1$ empty vertices on the center. See Fig. 3. In order to avoid the complex indices, we denote the vertices that $\mathcal{B}$ wants to occupy by $w_1, w_2, \ldots, w_{2t}$ ($w_1 = v_{\lfloor l/2 \rfloor}, \ldots, w_{2t} = v_{n-\lceil l/2 \rceil}$). We define a function $p(i)$ as follows.

$$p(i) = \begin{cases} i-1, & \text{when } i \text{ is even,} \\ i+1, & \text{when } i \text{ is odd.} \end{cases}$$

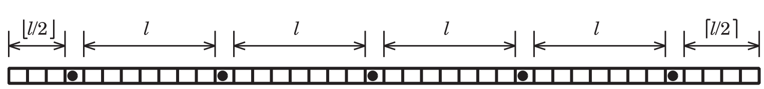

Fig. 1 Let l, t, and n be 7, 5, and $t \times (l+1) - 1 = 39$, respectively. If $\mathcal{B}$ can occupy these vertices, $\mathcal{B}$ does not lose on $VG(P_n, t)$. Note that we illustrate a vertex by a box.

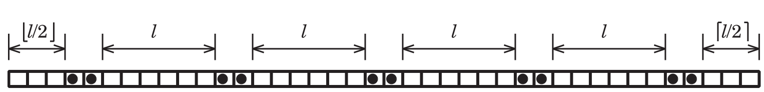

Fig. 2 $\mathcal{B}$ can occupy one of each vertex pair.

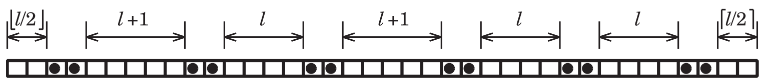

Fig. 3 $\mathcal{B}$'s strategy in the case $0 < (n+1) \bmod t \leq t/2$.

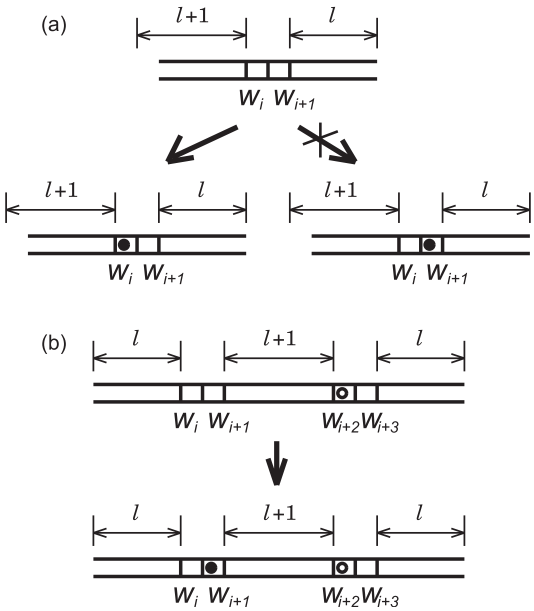

Fig. 4 The strategies changed.

Then, w_i and $w_{p(i)}$ are adjacent vertex pair, and $\mathcal{B}$ has to occupy one of them.

Lemma 10: $\mathcal{W}$ cannot win on $VG(P_n, t)$, if $0 < (n + 1) \bmod t \leq t/2$.

Proof: The basic strategy is the same as that of Lemma 9. First, $\mathcal{B}$ occupies w_{2t-1}. If $\mathcal{W}$ occupies w_i, and if $w_{p(i)}$ is empty, then $\mathcal{B}$ occupies $w_{p(i)}$. Otherwise, $\mathcal{B}$ occupies w_j for some $1 \leq j \leq 2t$ such that $w_{p(j)}$ is empty. However, with this strategy, $\mathcal{B}$ may lose, since $\mathcal{W}$ can make r white segments of $l + 1$ empty vertices. Therefore, $\mathcal{B}$ has to slightly change the strategy.

The problem is that $\mathcal{W}$ can make a white segment of $l + 1$ empty segments. This occurs when $\mathcal{W}$ occupies both w_i and w_{i+1} such that there are $l + 1$ empty vertices between w_i and w_{i+1}. In order to avoid this situation, $\mathcal{B}$ must change the strategy. See Fig. 4.

(a) $\mathcal{B}$ does not occupy w_i, if w_i and $w_{p(i)}$ are empty, and $w_{p(i)}$ is adjacent to $l + 1$ empty vertices.
(b) $\mathcal{B}$ does not occupy w_i, if $w_{p(i)}$ is adjacent to $l + 1$ empty vertices, and $\mathcal{W}$ occupied $w_{p(i)}$. Instead of occupying w_i, $\mathcal{B}$ occupies the vertex w_j, such that there are $l + 1$ segment between w_j and $w_{p(i)}$. If w_j is already occupied by $\mathcal{B}$, $\mathcal{B}$ occupies w_i. Note that w_j is not occupied by $\mathcal{W}$, since if so, $w_{p(i)}$ has to be occupied by $\mathcal{B}$. Further, $w_{p(j)}$ is not occupied by $\mathcal{B}$, since the strategy above forbids $\mathcal{B}$ to occupy $w_{p(j)}$.

If $\mathcal{B}$ takes this strategy, $\mathcal{W}$ cannot make a white segment of $l + 1$ empty vertices anymore. The new strategy (b) allowed $\mathcal{W}$ to occupy both w_i and $w_{p(i)}$. However, $\mathcal{W}$ has no advantage to do so. We explain the reason below. We can assume that $i = p(i) - 1$ without loss of generality. Occupying $w_{p(i)}$, $\mathcal{W}$ can increase the number of white segments of length equal to l by at most one. However, to do so, $\mathcal{W}$ can occupy one less vertices outside the path from w_i to w_{i+2} compared the case that $\mathcal{W}$ occupies a vertex outside the path. Therefore, the number of black segments of length more than or equal to l at the end of the game is exactly greater than that of white segments, outside the path. Therefore, $\mathcal{B}$ can occupy a vertex inside the path to win. □

Finally, we consider the last case, case 2. The idea is very similar to that of the case 1. We alternately arrange black segments of $l + 1$ empty vertices and those of l empty vertices r times, and then we arrange the remaining black segments of $l + 1$ empty vertices.

Lemma 11: $\mathcal{W}$ cannot win on $VG(P_n, t)$, if $(n+1) \bmod t > t/2$.

Proof: Using the same strategy as the case 1, $\mathcal{B}$ can keep the number of black segments of l empty vertices less than that of white segments of l empty vertices. Therefore, $\mathcal{V}_{\mathcal{B}}$ is never less than $\mathcal{V}_{\mathcal{W}}$. □

From Lemmas 9, 10, and 11, we have the theorem below.

Theorem 12: $\mathcal{W}$ has no winning strategy on $VG(P_n, t)$.

From Theorems 7 and 12, we have the main theorem (Theorem 1).

5. Concluding Remarks

We solved an unsolved case of the discrete Voronoi game which cannot be captured in the framework stated in [5]. $VG(P_n, t)$ is draw, with a trivial exception. Our proof can be easily applied to show that $VG(C_n, t)$ is draw, where C_n is a cycle of length n. It is a challenging open problem to determine if there is a polynomial time algorithm for Voronoi game on a tree determining which player will win. Voronoi game on a grid graph also seems interesting, since it models more "realistic" situation than the ordinary Voronoi game.

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