LETTER

Fast Hypercomplex Polar Fourier Analysis

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SUMMARY Hypercomplex polar Fourier analysis treats a signal as a vector field and generalizes the conventional polar Fourier analysis. It can handle signals represented by hypercomplex numbers such as color images. Hypercomplex polar Fourier analysis is reversible that means it can reconstruct image. Its coefficient has rotation invariance property that can be used for feature extraction. However in order to increase the computation speed, fast algorithm is needed especially for image processing applications like realtime systems and limited resource platforms. This paper presents fast hypercomplex polar Fourier analysis based on symmetric properties and mathematical properties of trigonometric functions. Proposed fast hypercomplex polar Fourier analysis computes symmetric points simultaneously, which significantly reduce the computation time.

key words: *fast hypercomplex polar Fourier analysis, hypercomplex polar Fourier analysis, Fourier analysis*

1. Introduction

Fourier analysis has been widely used and is still under active research in image processing, signal processing and many engineering fields [1]. By representing image as hypercomplex numbers, especially the quaternions discovered by Hamilton [2],hypercomplex Fourier transform is proposed for color image processing [3]. The relationship between right-side quaternion Fourier transform and left-side quaternion Fourier transform is established [4]. Based on hypercomplex Fourier transform, effective algorithms for motion estimation in color image sequences are studied [5]. Quaternionic Gabor filters are designed to combine the color channels and the orientations in the image plane [6].

Inspired from these findings, we have studied hypercomplex polar Fourier analysis. By introducing a hypercomplex number, hypercomplex polar Fourier analysis [8] treats a signal as a vector field and generalizes the polar Fourier analysis [7]. Hypercomplex polar Fourier analysis can handle color image. With orthogonality, it can decompose and reconstruct color image. The coefficients hold rotation invariant property. With these properties, it can be widely used as an image processing tool. Unfortunately, hypercomplex polar Fourier analysis involve many Bessel function and trigonometric computations, for which no fast method has been reported. Therefore, reduction of the computation time is very significant.

This paper focuses on fast hypercomplex polar Fourier analysis. Fast and compact method to compute the coefficients of hypercomplex polar Fourier analysis is proposed by using mathematical properties of trigonometric functions and points relationships in multi-spectral images. The two dimensional basis function of hypercomplex polar Fourier analysis has symmetry properties with respect to the x axis, y axis, $y = x$ line, $y = -x$ line and origin that can be used for fast computation. The computational complexity can be reduced by calculating half of the first quadrant. For image processing applications, computation time is important factor. Using the proposed method, only one eighth computational time is needed.

The organization of this paper is as follows. The basic theory of hypercomplex polar Fourier analysis including mathematics definitions are provided in Sect. 2. Section 3 presents the proposed method in detail. Experiments are designed to demonstrate effectiveness of the proposed method in Sect. 4. Finally, Sect. 5 concludes this study.

2. Background

2.1 Hypercomplex Number

As a type of hypercomplex number and generalization of complex number, the quaternion, its properties and applications have been studied [9]. In signal and image processing, quaternion number based methods are actively researched [3]–[6], [10].

Complex number has two components, the real part and imaginary part. Quaternion has one real part and three imaginary parts. Given $a, b, c, d \in \mathbb{R}$, a quaternion $q \in \mathbb{H}$ ($\mathbb{H}$ denotes Hamilton) is defined as

$$q = \mathcal{S}(q) + \mathcal{V}(q), \mathcal{S}(q) = a, \mathcal{V}(q) = bi + cj + dk \quad (1)$$

where $\mathcal{S}(q)$ is scalar part and $\mathcal{V}(q)$ is vector part. i, j, k are imaginary operators obeying the following rules

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, ij = -ji = k, \\ &jk = -kj = i, ki = -ik = j, \end{aligned} \quad (2)$$

The norm of quaternion q is

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}. \quad (3)$$

Quaternion q is named as unit quaternion if it is in set

$$\mathbb{U} = \{q | q \in \mathbb{H}, \|q\| = 1\}. \quad (4)$$

Manuscript received July 25, 2011.

Manuscript revised December 25, 2011.

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DOI: 10.1587/transinf.E95.D.1166

If quaternion q in following set,

$$\mathbb{P} = \{q | q \in \mathbb{H}, \mathcal{S}(q) = 0\}, \tag{5}$$

it is called pure quaternion. The quaternions belonging to set

$$\mathbb{S} = \{q | q \in \mathbb{U}, q \in \mathbb{P}\}, \tag{6}$$

are called unit pure quaternion. Euler formula holds for hypercomplex numbers,

$$e^{\mu\phi} = \cos(\phi) + \mu \sin(\phi) \tag{7}$$

Color image can be represented in as hypercomplex number form [3]

$$f(x, y) = f_R(x, y)i + f_G(x, y)j + f_B(x, y)k, \tag{8}$$

where $f_R(x, y)$, $f_G(x, y)$ and $f_B(x, y)$ are the red, green and blue components.

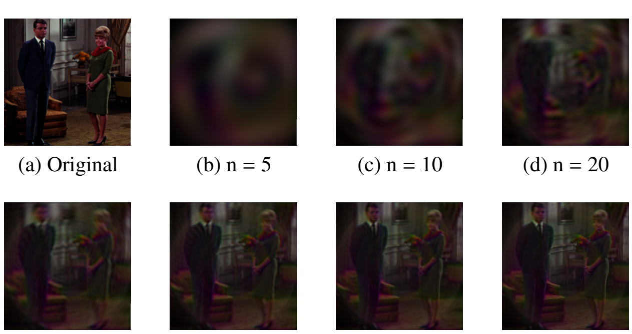
2.2 HyperComplex Polar Fourier Analysis

Given a 2D function $f(x, y)$, it can be transformed from cartesian coordinate to polar coordinate $f(r, \varphi)$, where r and φ denote radius and azimuth respectively. Hypercomplex Polar Fourier analysis involves points within the largest inner circle of the image. After normalization, it is defined on the unit circle that $r \leq 1$ and can be expanded with respect to the basis function. Hypercomplex polar Fourier analysis is defined as

$$f(r, \varphi) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} HP_{nm} R_{nm}(r) e^{\mu m \varphi}, \tag{9}$$

where μ is unit pure quaternion and is defined as $\mu = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$, and the coefficient is

$$\begin{aligned} HP_{nm} &= \frac{1}{\sqrt{2\pi}} \int_0^1 \int_0^{2\pi} R_{nm}(r) f(r, \varphi) e^{-\mu m \varphi} r dr d\varphi \\ &= \frac{1}{\sqrt{2\pi}} \int_0^1 \int_0^{2\pi} R_{nm}(r) f(r, \varphi)(\cos m\varphi - \mu \sin m\varphi) r dr d\varphi \end{aligned} \tag{10}$$



(a) Original (b) n = 5 (c) n = 10 (d) n = 20

(e) n = 40 (f) n = 60 (g) n = 80 (h) n = 100

Fig. 1 Hypercomplex polar Fourier analysis.

where

$$R_{nm}(r) = \frac{1}{\sqrt{N_n^{(m)}}} J_m(x_{mn} r), \tag{11}$$

in which J_m is the m-th order first class Bessel series [11], and $N_n^{(m)}$ can be deduced by imposing boundary conditions according to the Sturm-Lioville (S-L) theory [12]. With zero-value boundary condition,

$$N_n^{(m)} = \frac{1}{2} J_{m+1}^2 (x_{mn}), \tag{12}$$

in which x_{mn} is the nth positive root for $J_m(x)$.

The coefficient HP_{nm} is rotation invariant. Hypercomplex polar Fourier analysis is reversible. As Fig. 1 shown with n increases bigger, more and more detail part of the image can be obtained.

3. Fast Hypercomplex Polar Fourier Analysis

This section presents fast hypercomplex polar Fourier analysis. From Eq. (10), for same radius r, the different integrand part for each point is $f(r, \varphi)(\cos m\varphi - \mu \sin m\varphi)$. As shown in Fig. 2, point (x, y) is a point in first quadrant below $y = x$, has seven other symmetric points with respect to x axis, y axis, $y = x$, $y = -x$ and origin.

Mappings between polar and cartesian coordinates are show in Table 1.

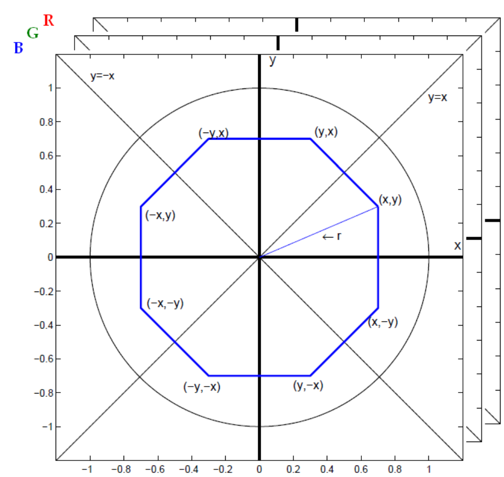


Fig. 2 Symmetric points in multi-spectral image.

Table 1 (r,θ) and its symmetric points.

Polar Coordinate	Cartesian Coordinate
(r, θ)	(x, y)
(r, $\frac{\pi}{2} - \theta$)	(y, x)
(r, $\frac{\pi}{2} + \theta$)	(−y, x)
(r, $\pi - \theta$)	(−x, y)
(r, $\pi + \theta$)	(−x, −y)
(r, $\frac{3\pi}{2} - \theta$)	(−y, −x)
(r, $\frac{3\pi}{2} + \theta$)	(y, −x)
(r, $2\pi - \theta$)	(x, −y)

$$G_m(x,y)=\begin{cases}(f(x,y)+f(y,x)+f(-y,x)+f(-x,y)\\+f(-x,-y)+f(-y,-x)+f(y,-x)+f(x,-y))cos(m\varphi) & \text{if } mod(m,4)=0\\(f(x,y)-f(-x,y)-f(-x,-y)+f(x,-y))cos(m\varphi)\\+(f(y,x)-f(-y,x)-f(-y,-x)+f(y,-x))sin(m\varphi) & \text{if } mod(m,4)=1\\(f(x,y)-f(y,x)-f(-y,x)+f(-x,y)\\+f(-x,-y)-f(-y,-x)-f(y,-x)+f(x,-y))cos(m\varphi) & \text{if } mod(m,4)=2\\(f(x,y)-f(-x,y)-f(-x,-y)+f(x,-y))cos(m\varphi)\\-(f(y,x)-f(-y,x)-f(-y,-x)+f(y,-x))sin(m\varphi) & \text{if } mod(m,4)=3\end{cases} \tag{22}$$

$$H_m(x,y)=\begin{cases}(f(x,y)-f(y,x)+f(-y,x)-f(-x,y)\\+f(-x,-y)-f(-y,-x)+f(y,-x)-f(x,-y))sin(m\varphi) & \text{if } mod(m,4)=0\\(f(x,y)+f(-x,y)-f(-x,-y)-f(x,-y))sin(m\varphi)\\+(f(y,x)+f(-y,x)-f(-y,-x)-f(y,-x))cos(m\varphi) & \text{if } mod(m,4)=1\\(f(x,y)+f(y,x)-f(-y,x)-f(-x,y)\\+f(-x,-y)+f(-y,-x)-f(y,-x)-f(x,-y))sin(m\varphi) & \text{if } mod(m,4)=2\\(f(x,y)+f(-x,y)-f(-x,-y)-f(x,-y))sin(m\varphi)\\-(f(y,x)+f(-y,x)-f(-y,-x)-f(y,-x))cos(m\varphi) & \text{if } mod(m,4)=3\end{cases} \tag{23}$$

Within period 2π, $sin(\varphi)$ and $cos(\varphi)$ functions are periodic functions. Periods for $sin(m\varphi)$ and $cos(m\varphi)$ are $2\pi/m$. Derived from the periodic and symmetric properties of trigonometric functions that used in FFT [13], mathematical relationships for trigonometric functions exist with respect to different m. If l is divided by 4 with remainder 1 that means $mod(l,4)=1$, following relationship for sine function can be deduced

$$sin\left(l\left(\frac{\pi}{2}-\theta\right)\right)=cos(l\theta), \tag{13}$$

$$sin\left(l\left(\frac{\pi}{2}+\theta\right)\right)=cos(l\theta), \tag{14}$$

$$sin\left(l\left(\pi-\theta\right)\right)=sin(l\theta), \tag{15}$$

$$sin\left(l\left(\pi+\theta\right)\right)=-sin(l\theta), \tag{16}$$

$$sin\left(l\left(\frac{3\pi}{2}-\theta\right)\right)=-cos(l\theta), \tag{17}$$

$$sin\left(l\left(\frac{3\pi}{2}+\theta\right)\right)=-cos(l\theta), \tag{18}$$

$$sin\left(l\left(2\pi-\theta\right)\right)=-sin(l\theta). \tag{19}$$

Similar relationships also exist for cosine function and other l values. For the eight symmetric points on the same radius r, if coefficients can be calculated simultaneously, then the computation time for trigonometric function and Bessel function can be reduced.

Based on foregoing discussion, fast hypercomplex polar Fourier analysis is given by

$$\begin{array}{c}FastHP_{nm}=\dfrac{1}{\sqrt{2\pi}}\displaystyle\iint_D R_{nm}(\sqrt{x^2+y^2})\\(G_m(x,y)-\mu H_m(x,y))dxdy\end{array}, \tag{20}$$

where

$$D=\{(x,y)|0\le x\le 1, 0\le y\le x, 0\le x^2+y^2\le 1\}\ , \tag{21}$$

and $G_m(x,y)$ and $H_m(x,y)$ are shown in Eq. (22), (23).

With same result, proposed method can share computation between symmetric points that substantially cut down the time in order to obtain the final result. The proposed method is unrelated to image content. Experiments are designed and results are given in following section.

4. Experimental Results

The performance of the proposed fast hypercomplex polar Fourier analysis in computation reduction is validated through comparative experiments using different images. Images with different content are tested for test to illustrate the efficiency and feasibility of the proposed method over direct computation. PC environment (Celeron 1.86 GHz, 2 G Memory) is used to perform the experiments. Algorithms are implemented by C++. GNU Scientific Library [14] is used for Bessel function.

4.1 Synthetic Images

In this experiment, synthetic images are generated using the formula ,

$$\begin{array}{l}f(i,j)=round[random(N,N)],\\0\le f(i,j)\le 255, \forall i,j,\end{array} \tag{24}$$

Table 2 CPU elapsed time for synthetic images.

Resolution	Direct Method	Proposed Method	Ratio
64*64	0.119	0.016	0.133
128*128	0.494	0.068	0.138
256*256	1.981	0.274	0.138
512*512	8.029	1.102	0.137

Fig. 3 Standard images.

Table 3 CPU elapsed time for test standard images.

Coefficients	Direct Method	Proposed Method	Ratio
5	0.965	0.130	0.135
10	1.924	0.262	0.136
20	3.888	0.532	0.137
30	5.884	0.804	0.137
40	7.846	1.066	0.136

where $f(i, j)$ is the function which its pixels integer in values, $N \times N$ the image resolution and i, j are the indices of the image pixels. In this experiment, largest value V of $f(i, j)$ equals 255. Color image is generated.

These synthetic images are varied in resolution and content. Hypercomplex polar Fourier analysis is applied to the synthetic images. Direct calculations use Eq. (10). The proposed methods use Eq. (20). The number of coefficients computed for fast hypercomplex polar Fourier analysis is 20. With same computation result, but two methods take different running time. Their computation performances in terms of the average CPU elapsed time are given in Table 2. The results show that computation time is greatly reduced. Experimental results on real images are given in next subsection.

4.2 Real Images

Test data set consists of standard images as shown in Fig. 3. With different number of coefficients computed, the performances in terms of CPU elapsed time are given in Table 3. From the result, the proposed method is effective and is unrelated to number of coefficients and image content. By sharing computation between symmetric points, fast hypercomplex polar Fourier analysis significantly boost the speed.

5. Conclusions

In this letter, fast hypercomplex polar Fourier analysis is proposed based on previous work. By using the symmetric properties and mathematical properties of trigonometric functions, the proposed method calculates one eighth of Bessel functions and trigonometric functions. That means, for two dimensional images proposed method largely reduces the computation time. Experimental results are given on different images to illustrate the effectiveness. Image processing applications that need fast hypercomplex polar Fourier analysis will benefit from this work.

Acknowledgements

This work was supported in part by Grant-in-Aid (No.21500181) for Scientific Research by the Ministry of Education, Science and Culture of Japan.

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