LETTER

Finding the Minimum Number of Face Guards is NP-Hard*

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SUMMARY We study the complexity of finding the minimum number of face guards which can observe the whole surface of a polyhedral terrain. Here, a face guard is allowed to be placed on the faces of a terrain, and the guard can walk around on the allocated face. It is shown that finding the minimum number of face guards is NP-hard.
key words: *face guards, polyhedral terrains, NP-hard*

1. Introduction

The art gallery problem is to determine the minimum number of guards who can observe the interior of a gallery. Chvátal [4] proved that $\lfloor n/3 \rfloor$ guards are the lower and upper bounds for this problem; namely, $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary for observing the interior of an n-vertex simple polygon.

The decision version of the problem is to decide whether, given a polygon and an integer k, the polygon can be guarded with k or fewer guards. This problem is known to be NP-hard [12], [13].

In three dimensions, a similar visibility problem has been considered for n-vertex triangulated polyhedral terrains. It is known that $\lfloor n/2 \rfloor$ is both the lower bound [3] and the upper bound [2] of *vertex guards* of a polyhedral terrain. Here, a vertex guard is a guard that is only allowed to be placed at the vertices of a terrain. Also, it is known that the minimum vertex-guard problem is NP-hard [5].

An *edge guard* is a guard that is only allowed to be placed on the edges of a terrain, and the edge guard can move between the endpoints of the edge. For the edge guarding problem, it is known that (i) the lower bound is $\lfloor (4n-4)/13 \rfloor$ [3], (ii) the upper bound is $\lfloor n/3 \rfloor$ [2], and (iii) the minimum edge-guard problem is NP-hard [1].

The authors studied the *face guarding problem*, where a face guard is allowed to be placed on the faces of a terrain, and the face guard can walk around only on the allocated face. A face guard can observe the allocated face and its adjacent faces. Here, two faces are said to be adjacent if they share a vertex.

The face guarding problem is motivated by applications in guarding bordering territories. In the real world, a territorial owner keeps watch over neighboring lands not only from an edge (borderline) or a vertex (corner), but also from all his territory.

It was shown that $\lfloor n/3 \rfloor$ is the lower bound and $\lfloor (2n-5)/7 \rfloor$ is the upper bound for the number of face guards of an n-vertex triangulated polyhedral terrain [9]. Recently, the same authors improved both lower and upper bounds to $\lfloor (n-1)/3 \rfloor$ [10].

In this paper, we study the decision version of the face guarding problem. First, we will show that it is NP-hard to decide whether there exists a triangular-face set of size k that covers all triangular faces of a planar graph. Then, we show that finding the minimum number of face guards in a triangulated polyhedral terrain is NP-hard.

2. Definitions and Results

Let G be a planar graph. A face of G is called *triangular* if it is bounded by three edges. Let F be the set of all faces of G, and let $H \subseteq F$ be the set of all triangular faces. A set $H' \subseteq H$ is said to *cover* G if every face in $\boldsymbol{H}$ shares a vertex with a triangular face in H'.

The instance of the *triangular-face covering problem* is a planar graph G and a positive integer k. The problem asks whether there exists a triangular-face set of size k that covers G.

The definitions of polyhedral terrains and visibility are mostly from [3]. A *polyhedral terrain* is a polyhedral surface in three dimensions such that its intersection with any vertical line is either a point or empty. A polyhedral terrain is *triangulated* if each of its faces is a triangle.

Two points x and y of a terrain are said to be *visible* if the line segment xy does not contain any points below the terrain. A point x of a terrain is said to be *visible from a face* f if there exists a point y on the face f such that x and y are visible. A set of faces is said to *cover* a terrain if every point of the terrain is visible from one of these faces.

The instance of the *geometric face guarding problem* is a triangulated polyhedral terrain T and a positive integer k. The problem asks whether there exists a face set of size k that covers T. Now we are ready to present the main results.

Theorem 1: *The triangular-face covering problem for planar graphs is NP-hard.*

Theorem 2: *The geometric face guarding problem for triangulated polyhedral terrains is NP-hard.*

The proof of Theorem 1 is given in Sect. 3. Theorem 2 can

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be obtained from Theorem 1 by a transformation from a planar graph to a terrain given in Sect. 4. By Theorem 2, one can see that finding the minimum number of face guards in a triangulated polyhedral terrain is NP-hard.

3. Proof of Theorem 1

3.1 PLANAR 3SAT

The definition of PLANAR 3SAT is mostly from [LO1] on page 259 of [8]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \ldots, c_j, \ldots, c_m\}$ of clauses over U such that (i) $|c_j| = 3$ for each $c_j \in C$ and (ii) the bipartite graph $B = (V, E)$, where $V = U \cup C$ and E contains exactly those pairs $\{x, c\}$ such that either literal x or $\overline{x}$ belongs to the clause c, is planar.

The PLANAR 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. This problem is known to be NP-hard. For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$ provide an instance of PLANAR 3SAT. For this instance, the answer is "yes", since there is a truth assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfying all clauses. It is known that PLANAR 3SAT is NP-complete even if each variable occurs exactly once in positive and exactly twice in negation [6], [11].

3.2 Transformation from a 3SAT-Instance to a Graph

We construct a polynomial-time transformation from an arbitrary instance C of PLANAR 3SAT to a planar graph G and an integer k such that C is satisfiable if and only if G has a triangular-face set of size k that covers G.

Each variable $x_i \in \{x_1, x_2, \ldots, x_n\}$ is transformed to graph G_{x_i} of Fig. 1 (a). This graph is composed of seven triangular faces, denoted by $p_i, q_i, r_i, s_i, t_i, a_i, b_i$. Each clause $c_j \in \{c_1, c_2, \ldots, c_m\}$ is transformed to triangle c_j of three vertices and three edges (see c_1, c_2, c_3, c_4 in Fig. 2). Vertices u_i, v_i and w_i, y_i in Fig. 1 will be used for the connections between G_{x_i} and c_j.

The graph G_{x_i} can be covered by a triangular-face set of size two. Let $O_i \subset \{p_i, q_i, r_i, s_i, t_i, a_i, b_i\}$ be such a set. This O_i has the following property. If $t_i \in O_i$ (see Fig. 1 (a)), then another face in O_i must be q_i in order to cover faces p_i, q_i, r_i, a_i. Thus, if $t_i \in O_i$, then any triangular face connected to G_{x_i} via a single vertex u_i or v_i is not covered by O_i (see triangular face c_1 in Fig. 2, which is not covered by $O_1 = \{q_1, t_1\}$). Later, one can see that $t_i \in O_i$ implies $x_i = 0$ and $\overline{x_i} = 1$, and $p_i \in O_i$ implies $x_i = 1$ and $\overline{x_i} = 0$.

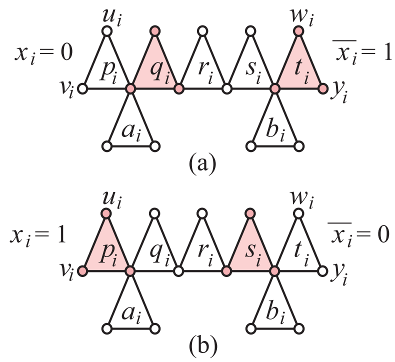

Fig. 1 Graph G_{x_i}. (a) If $t_i \in O_i$ then $q_i \in O_i$. (b) If $p_i \in O_i$ then $s_i \in O_i$.

If clause c_j contains literal x_i (resp. $\overline{x_i}$), then triangle c_j is connected to vertex u_i or v_i (resp. w_i or y_i). For example, in Fig. 2, since literal $\overline{x_1}$ is contained in c_2 and c_3, triangles c_2, c_3 are connected to vertices y_1, w_1. Finally, let $k = 2n$.

3.3 Necessary and Sufficient Conditions

In this section, we show that all clauses $c_1, c_2, \ldots, c_m$ are satisfiable if and only if there is a triangular-face set of size k that covers G.

Assume that there is a truth assignment for $x_1, x_2, \ldots, x_n$ satisfying all the clauses. A triangular set O of size k covering G can be constructed as follows. For each $i \in \{1, 2, \ldots, n\}$, if $x_i = 0$ (resp. $x_i = 1$) in that assignment, then we select q_i, t_i (resp. p_i, s_i) as triangular faces in O. After this procedure, the size of O becomes k.

Since each of $\{q_i, t_i\}$ and $\{p_i, s_i\}$ cover all of the seven faces in G_{x_i} (see Fig. 1), O covers all faces of G_{x_i} for all $i \in \{1, 2, \ldots, n\}$. If literal x_i (resp. $\overline{x_i}$) satisfies clause c_j, then triangle $p_i \in O$ (resp. $t_i \in O$) covers triangle c_j. Therefore, if there is a truth assignment for $x_1, x_2, \ldots, x_n$ satisfying all the clauses, then there is a triangular set O of size k that covers G.

Assume that there is a triangular set O of size k that covers G. Each graph G_{x_i} is covered by two faces, and not by one face (even if all triangles connected to G_{x_i} are selected as O's faces, see Fig. 3). This implies that triangular set O of size k does not contain any triangle $c_j \in \{c_1, c_2, \ldots, c_m\}$, since $k = 2n$ (see k red triangles in Fig. 2).

If $t_i \in O$, then (i) triangles connected to w_i and y_i are covered (see triangles c_2 and c_3 covered by t_1 in Fig. 2) and (ii) triangles connected to u_i or v_i are not covered (see c_1). On the other hand, if $p_i \in O$, then (i) triangles connected to u_i and v_i are covered (see c_1 covered by p_2) and (ii) triangles connected to w_i or y_i are not covered (see c_2, c_4). Therefore, if there is a triangular set O of size k that covers G, then all the clauses are satisfiable. This completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section, we transform the planar graph G constructed

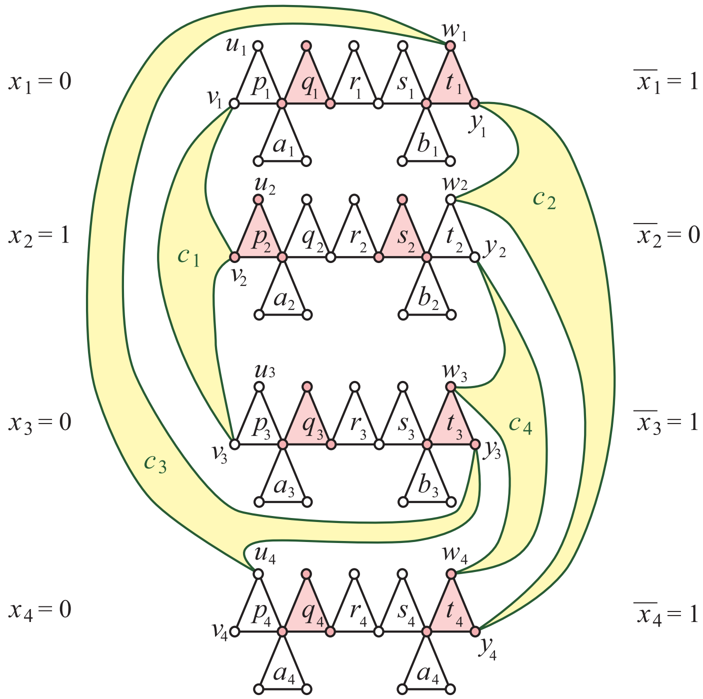

Fig. 2 Graph transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, x_3\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, \overline{x_4}\}$, $c_3 = \{\overline{x_1}, \overline{x_3}, x_4\}$, and $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. C is satisfiable, since there is a truth assignment $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ satisfying all clauses c_1, c_2, c_3, c_4.

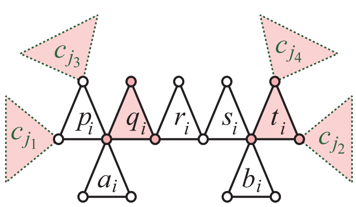

Fig. 3 Graph G_{x_i} is covered by two of G_{x_i}'s faces, and not by one face, even if all triangles $c_{j_1}, c_{j_2}, c_{j_3}, c_{j_4}$ are selected as O's faces.

in Sect. 3 into a triangulated polyhedral terrain T such that G can be covered by a triangular-face set O of size k if and only if all faces of T are visible from a face set of size $k+2l$, where l is the number of non-triangular faces of G.

From Fáry's theorem [7], we can assume that the planar graph constructed in Sect. 3 is embedded on the plane without crossings so that its edges are straight line segments. The following description is based on the idea of [1]. We triangulate every non-triangular face of G as follows. Let $z = (z_0, z_1, \ldots, z_{s-1})$ be a face in G of size $s \geq 4$ (see Fig. 4). We split the face z into two faces by adding edges $(z_0, a), (a, b), \ldots, (f, g), (g, z_{\lceil s/2 \rceil})$. Inside the upper and lower faces, we add triangles $\triangle(stu)$ and $\triangle(vwx)$, respectively. Then, we add the following edges:

$$(s, a), (s, b), (u, b), (u, c), (u, d), (u, e), (u, f), (t, f), (t, g);$$
$$(s, z_0), (s, z_1), \ldots, (s, z_{\lceil s/4 \rceil}); \ (t, z_{\lceil s/4 \rceil}), \ldots, (t, z_{\lceil s/2 \rceil});$$
$$(v, a), (v, b), (w, b), (w, c), (w, d), (w, e), (w, f), (x, f), (x, g);$$
$$(x, z_{\lceil s/2 \rceil}), \ldots, (x, z_{\lceil 3s/4 \rceil}); \ (v, z_{\lceil 3s/4 \rceil}), \ldots, (v, z_{s-1}), (v, z_0).$$

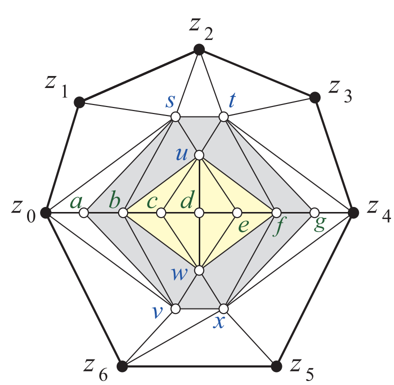

Fig. 4 Face of size $s = 7$ is triangulated.

Let G' be the planar graph obtained by applying the above triangulation procedure for every non-triangular face of G.

As an upper bound, every face inside cycle $z_0, z_1, \ldots, z_{s-1}, z_0$ can be covered by two faces (see $\triangle(stu)$ and $\triangle(vwx)$ in Fig. 4). As a lower bound, at least two faces are required to cover eight faces inside cycle b, u, f, w, b, and two such faces must be inside cycle a, s, t, g, x, v, a. Therefore, all triangular faces of G have already been covered by the triangular-face set O constructed in Sect. 3 (see red faces of Fig. 2), and all triangular faces constructed in this section are covered by $2l$ faces inside cycles a, s, t, g, x, v, a (see yellow and grey faces of Fig. 4). By this construction, there is a triangular-face set of size $k + 2l$ that covers G' if and only if there is a truth assignment for $x_1, x_2, \ldots, x_n$ satisfying all the clauses.

Finally, we construct a triangulated *convex* terrain T whose underlying graph is G'. Here, a terrain is said to be *convex* if every point of the terrain is also a point on the boundary of the convex hull of the vertices of the terrain. A face guard on a convex terrain can only observe the allocated face and its adjacent faces. Thus, upper and lower bounds of face guards used to guard a triangulated convex terrain coincide with those of triangular faces used to cover the corresponding triangulated plane graph.

It is known that, given a convex terrain T_1, a vertex v_1 can always be added to T_1 such that the resulting object T_1' is a convex terrain, which is the same as T_1 except for vertex v_1 and the faces adjacent to v_1 [3]. Therefore, we can construct a triangulated convex terrain T whose underlying graph is G' by translating all vertices of G' along the z direction one by one. By this construction, there is a triangular-face set of size $k+2l$ that covers G' if and only if there is the corresponding face guard set of the same size that covers T. This completes the proof of Theorem 2.

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References

[1] V.H.F. Batista, F.L.B. Ribeiro, and F. Protti, "On the complexity of the edge guarding problem," Proc. 26th European Workshop on Computational Geometry, pp.53–56, Dortmund, Germany, 2010.

[2] P. Bose, D. Kirkpatrick, and Z. Li, "Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces," Comput. Geom. Theory Appl., vol.26, pp.209–219, 2003.

[3] P. Bose, T. Shermer, G. Toussaint, and B. Zhu, "Guarding polyhedral terrains," Comput. Geom. Theory Appl., vol.7, pp.173–185, 1997.

[4] V. Chvátal, "A combinatorial theorem in plane geometry," J. Combin. Theory, ser.B, vol.18, pp.39–41, 1975.

[5] R. Cole and M. Sharir, "Visibility problems for polyhedral terrains," J. Symb. Comput., vol.7, no.1, pp.11–30, 1989.

[6] E. Dahlhaus, D.S. Johnson, C.H. Papadimitriou, P.D. Seymour, and M. Yannakakis, "The complexity of multiterminal cuts," SIAM J. Comput., vol.23, no.4, pp.864–894, 1994.

[7] I. Fáry, "On straight-line representation of planar graphs," Acta Sci. Math., vol.11, pp.229–233, 1948.

[8] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, New York, NY, USA, 1979.

[9] C. Iwamoto, J. Kishi, and K. Morita, "Lower bound of face guards of polyhedral terrains," J. Inf. Process., vol.20, no.2, pp.435–437, 2012.

[10] C. Iwamoto and T. Kuranobu, "Improved lower and upper bounds of face guards of polyhedral terrains," IEICE Trans. Inf. & Syst. (Japanese Edition), vol.J95-D, no.10, pp.1869–1872, Oct. 2012.

[11] J. Kratochvíl, "A special planar satisfiability problem and a consequence of its NP-completeness" Discrete Appl. Math., vol.52, pp.233–252, 1994.

[12] D.T. Lee and A.K. Lin, "Computational complexity of art gallery problems," IEEE Trans. Inf. Theory, vol.32, no.2, pp.276–282, 1986.

[13] J. O'Rourke, Art Gallery Theorems and Algorithms, Oxford University Press, New York, NY, USA, 1987.