LETTER Random Walks on Stochastic and Deterministic Small-World Networks

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SUMMARY Many deterministic small-world network models have been proposed so far, and they have been proven useful in describing some real-life networks which have fixed interconnections. Search efficiency is an important property to characterize small-world networks. This paper tries to clarify how the search procedure behaves when random walks are performed on small-world networks, including the classic WS small-world network and three deterministic small-world network models: the deterministic small-world network created by edge iterations, the tree-structured deterministic small-world network, and the small-world network derived from the deterministic uniform recursive tree. Detailed experiments are carried out to test the search efficiency of various small-world networks with regard to three different types of random walks. From the results, we conclude that the stochastic model outperforms the deterministic ones in terms of average search steps.

key words: complex networks, small-world networks, random walks, search efficiency

1. Introduction

There are many models describing numerous networks. As a typical feature of the world-wide complex networks, the small-world phenomenon has been extensively studied in recent years [1]–[3]. Two typical properties that make smallworld networks different from other complex networks are their small average path length (APL) and large clustering coefficient [4]. The small-world network models can perfectly match the typical properties of real-life networks, which show that the average distance between any pair of nodes is relatively small although the size of the whole network is large, and nodes connected to one common node have a high probability to gather into a cluster.

In 1998, Watts and Strogatz [4] introduced a network model with small APL and large clustering coefficient, named WS small-world model, which was a groundbreaking research work on the properties of small-world networks. After that, many stochastic network models similar to the WS small-world model were proposed, e.g., the NW model by Newman and Watts [5], [6]. By calling models above "stochastic" we emphasize that these models propagate new nodes or edges by following a probabilistic rule. While randomness is a general characteristic of complex 1215

networks, it more or less masks some information about how the networks are constructed, making it difficult for us to understand the shape of the networks and distinguish nodes that are connected to a certain node. What's more, random models can't perfectly describe some real-life networks which have fixed interconnections [7]. Therefore, more and more research works have been carried out on generating small-world networks in deterministic manners. In 2000, Comellas et al. proposed the first deterministic small-world network [8]. In 2006, Zhang et al. presented a deterministic small-world network created by edge iterations [9]. Recently, Guo et al. presented a tree-structured deterministic small-world model [10], while Lu and Guo proposed a deterministic small-world model derived from the uniform recursive tree [11]. The most evident advantage provided by deterministic networks is that we can calculate their properties analytically.

Search strategies and techniques have taken a large part in the research of complex networks, and they have been widely used for finding the shortest path between two locations, and so on [12]. As a fundamental dynamic process, the random walk is a powerful tool to study the structure of complex networks [13]. Random walk in deterministically generated network receives increasing attention from research fields such as physics and biology [14], [15]. Because the main factor of structure affecting the speed of diffusion in deterministic network is not well understood, we try to study the search efficiency in such networks.

In this Letter, our aim is to experimentally compare the search efficiency between stochastic small-world networks and deterministic ones by applying three different random walk search strategies on them. Section 2 reviews the WS small-world model and three deterministic small-world networks. Section 3 gives the experimental results and compares the four small-world networks in terms of search efficiency. Section 4 draws a conclusion of the whole paper.

2. Overview of Small-World Networks

In this Letter, we choose the WS small-world network as the representative of stochastic networks. The WS small-world network model starts from a ring lattice with n nodes and k edges per node [4]. Each node connects to k/2 nodes on its right-hand side and k/2 nodes on its left-hand side. Then we rewire each edge with a probability p by holding one terminal node unchanged and select a new node as the other

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Fig.1 The first four steps of the iterative process of the deterministic small-world network created by edge iterations.



Fig. 2 The first four steps of the iterative process of the tree-structured deterministic small-world network.

terminal. When 0 , the WS network has a small average path length as well as a large clustering coefficient, which obviously reveals the small-world properties [4].

Now let us turn our hands to introducing some deterministic small-world networks. The deterministic smallworld network created by edge iterations is derived from a triangle whose three nodes connect one another [9]. This initial state is called N(0) and we denote the network after t steps of evolution by N(t). For $t \ge 1$, N(t) is obtained from N(t-1) by adding for each edge created at step t-1a new node and attaching it to both end nodes of the edge. Figure 1 shows the first four steps of the iterative process. The analytical results show that, for infinite *t*, the average node degree approximately equals a small value 4 and the clustering coefficient approaches a constant value 0.6931, which reveals that the network is a sparse graph with a high clustering coefficient. What's more, the average path length increases more slowly than $\ln(N_t)$, where N_t represents the number of nodes after t steps of iteration. Thus, this model is a deterministic small-world network.

Another deterministic small-world network proposed by Guo et al. [10] focuses on the iteration of a binary-tree structure. At each iteration, after adding two new branches from each node in the last layer as the usual binary-tree structure, we additionally add links between each pair of brother nodes and links between each grandfather node and its four grandson nodes. We add these links in order to transform the binary tree with a zero clustering coefficient into a network with a high clustering coefficient structure. Figure 2 shows the first four steps of the iterative process. After quantitative analysis of this network model, as the iteration goes on, the clustering coefficient approaches a high constant value 0.7333. The growth speed of the diameter has a linear relationship with the natural logarithm of the number of nodes. With these properties, this model matches the definition of small-world networks.

Apart from the specific iterative rules mentioned above, small-world networks can also be constructed by altering the



Fig. 3 The first four iterations of the growth process of the DURT based small-world network.

existing complex networks. The small-world network derived from the deterministic uniform recursive tree is a typical example [11]. At each iteration, this model adds some edges to the deterministic uniform recursive tree (DURT) with a simple rule in order to get a high clustering coefficient. Figure 3 shows the network obtained after the first four iterations, where the dashed arcs denote the links generated by the extra operation at each iteration. Analytical results show that it is a sparse network with a high clustering coefficient and a small diameter, satisfying the necessary properties of small-world networks.

3. Experimental Results

Deterministic small-world network models can be used to describe many real-world nonrandom networks, such as electronic circuits and communication networks [7]. So it is worthwhile to study the search efficiency of these deterministic networks. In many cases, it is impossible to provide global information about how the nodes connect to each other. In other words, the search strategies should be based only on the local information such as the identities and node degrees of the neighbors. Thus, the random walk process can accurately simulate this situation. Assuming that each node only knows those nodes connecting to it, to find the target node, we can adopt three types of random walks as follows:

(1) Unrestricted Random Walk (URW): at each step, the walker randomly selects one of the neighbors of the current node as the next destination with equal probability. This procedure is repeated until it finds any neighbor of the target node.

(2) No-retracing Random Walk (NRRW): at each step, the walker randomly selects one of the neighbors of the current node, excluding the one that has been visited at last step, as the next destination with equal probability. This process is repeated until it finds any neighbor of the target node.

(3) Self-avoiding Random Walk (SARW): at each step, the walker randomly selects one of the neighbors of the current node, excluding all the nodes that have been visited in previous steps, as the next destination with equal probability. This process is repeated until it finds any neighbor of the target node.

The search efficiency of networks closely relates to search strategies and network topology. Average search steps can generally characterize the efficiency, which can be defined as follows: in a network with N nodes, we randomly



Fig.4 Log-Log chart of average search steps versus the number of nodes for WS small-world networks.

choose a node v_i as the source node for N times. At each time, we apply the random walk from the source node v_i to any other node v_j , obtaining the number of search steps T_{ij} . Then the average search steps between arbitrary two nodes can be written as:

$$\bar{T} = \frac{1}{N(N-1)} \sum_{i,j} T_{ij}$$
 (1)

Now we turn to studying the performance of average search steps when we apply above three types of random walks on the four types of small-world models reviewed in Sect. 2. We first generate 10 WS small-world networks with a constant rewiring probability p = 0.1, which is a typical value for the model to reveal the properties of small-world. We set the numbers of nodes to be 8, 16, 32, 64, 128, 256, 384, 512, 768, and 1024, respectively. Figure 4 compares the results of three types of random walks on these 10 WS small-world networks. Here, the number of average steps is the average result over N(N - 1) runs. We can see that URW and NRRW have almost the same growth curve, while SARW has a significant improvement in search efficiency.

For the deterministic small-world network created by edge iterations, we create 10 networks corresponding to the first 10 iterations, with the numbers of nodes being 3, 6, 12, 24, 48, 96, 192, 384, 768 and 1536, respectively. Figure 5 compares the results of three types of random walks on these 10 deterministic small-world networks. We can see that the curve is similar to Fig. 4.

For the tree-structured deterministic small-world network, we take the first 9 iterations to generate 9 networks, with the numbers of nodes being 3, 7, 15, 31, 63, 127, 255, 511 and 1023, respectively. We show the chart of average search steps versus the number of nodes in Fig. 6, which is similar to Fig. 5.

For the small-world network derived from the deterministic uniform recursive tree (DURT), we iterate 10 times to generate 10 networks with the numbers of nodes being 2, 4, 8, 16, 32, 64, 128, 256, 512 and 1024, respectively. We show the chart of average search steps versus the number of nodes in Fig. 7, from which we can find that the curves



Fig. 5 Log-Log chart of average search steps versus the number of nodes for deterministic small-world networks created by edge iterations.



Fig. 6 Log-Log chart of average search steps versus the number of nodes for tree-structured deterministic small-world networks.



Fig.7 Log-Log chart of average search steps versus the number of nodes for DURT small-world networks.

for DURT small-world networks have the upward tendency similar to Figs. 5 and 6.

Now it is time to put all of the above results together and make a comparison. Here, we only compare the results under the SARW strategy among different kinds of smallworld networks. As shown in Fig. 8, we put four curves



Fig. 8 Comparisons of search efficiency among four kinds of smallworld networks under the SARW search strategy.

in the same coordinates. From bottom to up, we first meet the curve of WS small-world networks (labeled as "WS"), which means that the WS network requires the fewest average search steps, i.e., it has the highest search efficiency under the SARW strategy. Then we come to the curve of deterministic small-world networks created by edge iterations (labeled as "EI"), which shows that EI-based network is the best model among three types deterministic smallworld networks. However, it takes almost twice as many average search steps as the WS model at the network size 1000. Finally, we reach the two curves for the other two networks, i.e., the tree-structured deterministic small-world network (labeled as "TSD") and the DURT small-world network (labeled as "DURT"). They stand at the top of the chart and overlap nearly all the way up. At the network size 1000, they need almost triple as many average search steps as the WS small-world network. After the crosswise comparison, we can conclude that the stochastic WS small-world network has the most efficient search structure among all networks, while the deterministic small-world network created by edge iterations has the fewest average search steps among all deterministic networks.

4. Conclusions

In this Letter, we have experimentally studied the search efficiency of three types of random walks on the WS smallworld network and three deterministic small-world networks. From the experimental results, we can make the conclusion that the stochastic WS small-world network outperforms three deterministic small-world networks in term of search efficiency. In the future, our work will concentrate on proposing more practical deterministic small-world networks with optimized search efficiency compared to the WS network.

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