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Linear Time Algorithms for Finding Articulation and Hinge Vertices of Circular Permutation Graphs

Hirotoshi HONMA[†a)], **Kodai ABE**[††], ***Members***, **Yoko NAKAJIMA**[†], ***Nonmember***, ***and*** **Shigeru MASUYAMA**[†††], ***Member***

SUMMARY Let $G_s = (V_s, E_s)$ be a simple connected graph. A vertex $v \in V_s$ is an articulation vertex if deletion of v and its incident edges from G_s disconnects the graph into at least two connected components. Finding all articulation vertices of a given graph is called the articulation vertex problem. A vertex $u \in V_s$ is called a hinge vertex if there exist any two vertices x and y in G_s whose distance increase when u is removed. Finding all hinge vertices of a given graph is called the hinge vertex problem. These problems can be applied to improve the stability and robustness of communication network systems. In this paper, we propose linear time algorithms for the articulation vertex problem and the hinge vertex problem of circular permutation graphs.

key words: *design and analysis of algorithms, articulation vertices, hinge vertices, circular permutation graphs*

1. Introduction

Let $G_s = (V_s, E_s)$ be a simple connected graph with $|V| = n$ and $|E| = m$. A vertex $v \in V_s$ is an *articulation vertex* if the deletion of v and its incident edges from G_s disconnects the graph into at least two connected components. A graph with no articulation vertex is called a *biconnected graph*. Finding all articulation vertices of a given graph is called the articulation vertex problem. An $O(n + m)$ time algorithm exists for solving the articulation vertex problem in simple graphs by using the traditional depth-first spanning tree method [1]. Moreover, efficient parallel algorithms for finding articulation vertices, bridges, and biconnected components in general graphs are given in [2], [3].

A vertex $u \in V_s$ is called a *hinge vertex* if there exist any two vertices x and y in G_s whose distance increase when u is removed. A graph without hinge vertices is called a *self-repairing graph*. Articulation vertices are a special case of hinge vertices in that the removal of an articulation vertex u changes the finite distance of some nonadjacent vertices x and y to infinity. Finding all hinge vertices of a given graph is called the hinge vertex problem. There exists an $O(n^3)$ time algorithm for solving the hinge vertex problem of a simple graph. These problems can be applied to improve the stability and robustness of communication network systems [4].

In many cases, more efficient algorithms can be developed by restricting the classes of graphs. For instance, for *permutation graphs*, Ibarra and Zheng [5] proposed an $O(\log n)$ time parallel algorithm using $O(n/\log n)$ processors for the articulation vertex problem, while Ho et al. [6] presented an $O(n)$ time algorithm for the hinge vertex problem on permutation graphs, whose minor error was corrected by [7]. Furthermore, for *interval graphs*, Sprague and Kulkarni [8] proposed an $O(\log n)$ time parallel algorithms with $O(n/\log n)$ processors for the articulation vertex problem, and Hsu et al. [9] presented an $O(n)$ time algorithm for the hinge vertex problem. Kao and Horng [10] proposed optimal $O(\log n)$ time parallel algorithms with $O(n/\log n)$ processors for finding all articulation vertices, bridges, and biconnected components of *circular-arc graphs*, which are a superclass of interval graphs.

Let $V_p = [1, 2, \ldots, n]$ be a vertex set and $P = [p(1), p(2), \ldots, p(n)]$ be a permutation of V_p. A permutation graph G_p is visualized by its corresponding *permutation model* M_p, which consists of two horizontal parallel lines called the *top channel* and *bottom channel*, respectively. Place the vertices $1, 2, \ldots, n$ on the top channel, ordered from left to right, and similarly, place $p(1), p(2), \ldots, p(n)$ on the bottom channel. Next, for each $i \in V_p$, draw a straight line from i on the top channel to i on the bottom channel. Then, an edge (i, j) in G_p exists if and only if lines i and j intersect in M_p. In this paper, "line" and "vertex" are used interchangeably. An example of a permutation model M_p and its corresponding permutation graph G_p is shown in Fig. 1. Permutation graphs are an important subclass of perfect graphs, and they are used for modeling practical problems in many areas, such as biology, genetics, very large scale integration (VLSI) design, and network planning [11].

Circular permutation graphs properly contain a set of permutation graphs as a subclass. Rotem and Urrutia first introduced circular permutation graphs and provided an $O(n^{2.376})$ time recognition algorithm [12]. Lou and Sarrafzadeh showed that circular permutation graphs and their models have several applications in VLSI layout design [13]. They presented an $O(\min(\delta n \log\log n, n \log n) + |E|)$ time algorithm for finding a maximum independent set of a circular permutation model, where δ is the minimum

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†The author is with the Department of Information Engineering, Kushiro National College of Technology, Kushiro-shi, 084–0916 Japan.
††The author is with the Department of Intelligent Interaction Technologies, University of Tsukuba, Tsukuba-shi, 305–8577 Japan.
†††The author is with the Department of Computer Science and Engineering, Toyohashi University of Technology, Toyohashi-shi, 441–8580 Japan.
a) E-mail: honma@info.kushiro-ct.ac.jp
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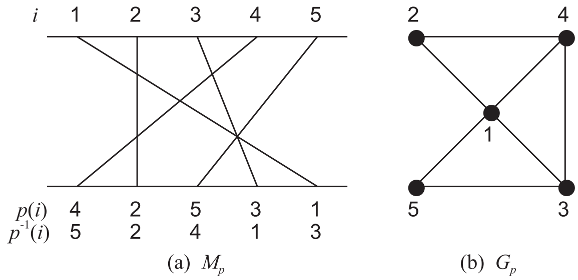


Fig. 1 Permutation model M_p and graph G_p.

degree of vertices in the corresponding circular permutation graph. Furthermore, they presented an $O(n \log\log n)$ time algorithm for finding the maximum clique and the chromatic number of a circular permutation model. Subsequently, the recognition algorithm was improved in $O(m + n)$ time by Sritharan [14].

In this paper, we propose linear time algorithms for the articulation and the hinge vertex problems in circular permutation graphs. Both of them can run in $O(n)$ time. The rest of this paper is organized as follows. Section 2 describes some definitions of circular permutation graphs and models. Section 3 introduces the extended circular permutation model and its properties. Sections 4 and 5 consider algorithms that address both articulation and the hinge vertex problem and the complexity of these algorithms. Section 6 concludes this paper.

2. Circular Permutation Model and Graph

We first illustrate the *circular permutation model* before defining the circular permutation graph. There exist inner and outer circles C_1 and C_2 with radii $r_1 < r_2$. Let $CP = [cp(1), cp(2), \ldots, cp(n)]$ be a permutation of integer sequence $[1, 2, \ldots, n]$. Furthermore, $cp^{-1}(i)$, $1 \leqslant i \leqslant n$, denotes the position of the number i in CP. Consecutive integers i, $1 \leqslant i \leqslant n$, are set to be counter-clockwise on C_1. Similarly, $cp(i)$, $1 \leqslant i \leqslant n$, is set to be counter-clockwise on C_2. For each i, $1 \leqslant i \leqslant n$, draw a chord joining the two i's, one on C_1 and the other on C_2, denoted as *chord i*. The geometric representation described above is called a circular permutation model CM. Figure 2 illustrates an example of CM with 12 chords constructed by $CP = [11, 1, 5, 10, 2, 7, 6, 9, 4, 8, 3, 12]$. This model is considered to be *proper* if any two chords i and j intersect at most once in the CM. In this paper, we consider only proper circular permutation graphs and models, and therefore, the word "proper" is omitted henceforth.

Next, we introduce circular permutation graphs. An undirected graph G is a circular permutation graph if it can be represented by the following circular permutation model CM: each vertex of the graph corresponds to a chord in the annular region between two concentric circles C_1 and C_2, and two vertices are adjacent in G if and only if their corresponding chords intersect exactly once [12]. Figure 3 illustrates the circular permutation graph G corresponding to

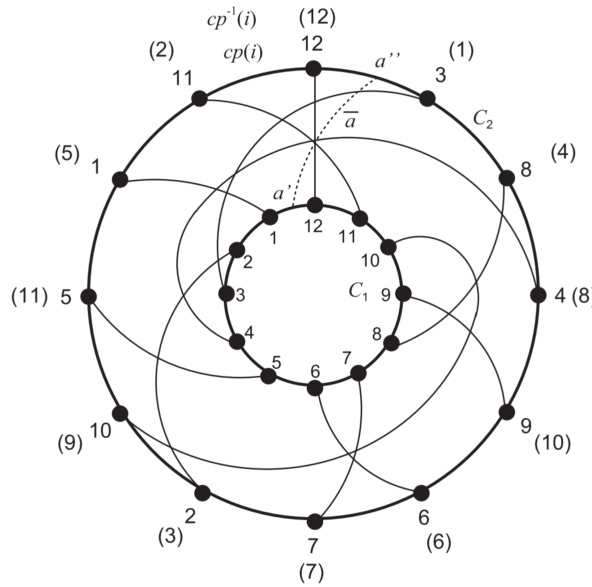


Fig. 2 Circular permutation model CM.

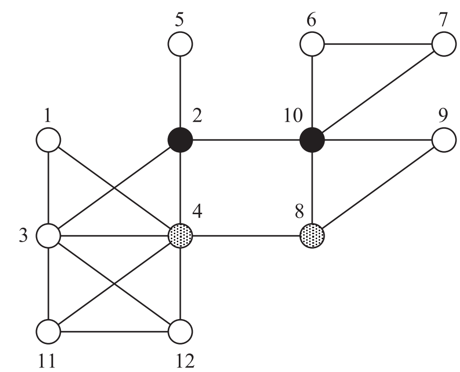


Fig. 3 Circular permutation graph G.

CM shown in Fig. 2. In this example, $\{2, 10\}$ is an articulation vertex set and $\{2, 4, 8, 10\}$ is a hinge vertex set.

Next, we consider a fictitious chord $\overline{a}$ which connects the point a' that is placed between 1 and 12 on C_1 and point a'' on C_2. A chord that intersects $\overline{a}$ is called a *feedback chord*. The set of all feedback chords is denoted by F. Moreover, a set of feedback chords that intersect $\overline{a}$ in clockwise is defined as F^-, and a set of feedback chords that intersect $\overline{a}$ counterclockwise is defined as F^+. We must place point a'' on C_2 so that $|F^-| = |F^+|$ is satisfied. In the example shown in Fig. 2, point a'' is placed between 3 and 12 on C_2. Consequently, $F = \{3, 4, 11, 12\}$, $F^- = \{3, 4\}$ and $F^+ = \{11, 12\}$. If a fictitious chord $\overline{a}$ exists that does not intersect any chord in CM, a model formed by opening CM along $\overline{a}$ is equivalent to a permutation model. This problem can be solved by applying Ibarra et al.'s algorithm [5] because this problem is the same as that of permutation graphs. In this paper, we assume that any fictitious chord intersects at least one chord.

3. Extended Circular Permutation Model

In this section, we introduce an *extended circular permutation model ECM* that is constructed from a *CM*.

Let n be the number of chords in *CM*. First, a point a' is fixed between 1 and n on C_1. Next, we consider a fictitious chord $\overline{a}$ with $|F^-| = |F^+|$. In Fig. 2, we obtain $|F^-| = |F^+| = 2$ by placing point a'' between 3 and 12 on C_2. *ECM* is formed by opening *CM* along $\overline{a}$. *ECM* consists of two horizontal parallel lines L_1 and L_2, called top and bottom channels, respectively. The top channel L_1 is assigned the consecutive number i, $-n+1 \leqslant i \leqslant 2n$, from left to right. The bottom channel L_2 is assigned $p(i)$, $-n+1 \leqslant i \leqslant 2n$, from left to right. Here, $p(i)$, $1 \leqslant i \leqslant n$, on L_2, is assigned a cp value on C_2 in the counter-clockwise direction from point a''. Next, $p(i)$, $1 \leqslant i \leqslant n$, changes to $p(i)-n$ if $i \in F^+$. Furthermore, $p(i)$, $1 \leqslant i \leqslant n$, changes to $p(i)+n$ if $i \in F^-$. We execute $p(i-n) = p(i) - n$ and $p(n+i) = p(i) + n$ for $1 \leqslant i \leqslant n$. For each $-n+1 \leqslant i \leqslant 2n$, a straight line is drawn from i on L_1 to i on L_2. After executing the above process, *ECM* is constructed from *CM*. Figure 4 illustrates *ECM* constructed from *CM* shown in Fig. 2. Here, $p^{-1}(i)$ denotes the position of i on L_2.

Circular permutation and circular-arc graphs are circular versions of permutation and interval graphs, respectively. Moreover, as mentioned in Sect. 1, circular permutation and circular-arc graphs are superclasses of permutation and interval graphs, respectively. Efficient algorithms have been developed that address various problems concerning permutation and circular-arc graphs. However, in general, problems for circular graphs tend to be more difficult than those for non-circular graphs. One of the reasons is that we can not uniquely determine the starting position of an algorithm for a circular graph due to the existence of feedback elements although it can be fixed for a non-circular graphs.

For several problems, we can develop circular versions of the existing algorithms by constructing extended intersection models of the problems. By using extended intersection models, we can determine a start position of algorithm uniquely and apply partially the algorithms of the non-circular versions. For instance, this method has been applied to develop efficient algorithms for the shortest path query problem [9], [15], the articulation vertex problem [10] on circular-arc graphs, maximum clique and chromatic number problems [13], the spanning forest problem [16] on circular-permutation graphs. In this paper, we use *ECM* to construct efficient algorithms for articulation and hinge vertex problems.

Property 1 stated below, can be derived in a straightforward manner from the processes of constructing *ECM*.

Property 1: Lines $i-n$, i, and $i+n$ in *ECM* correspond to the vertex i in G.

Two vertices i and j are adjacent in a circular permutation graph if and only if their corresponding chords intersect exactly once in *CM*. When two chords i and j $(i < j)$ intersect in *CM*, we distinguish the following three cases:

Case 1: $i \in F^-$ or $j \in F^+$

In this case, lines j and $i+n$ intersect in *ECM* with lines $i+n$ and j, respectively.

Case 2: $i \in F^+$ and $j \in F^-$

This case is infeasible because it implies that chords i and j intersect twice in *CM*.

Case 3: Remaining conditions for i and j

In these cases, lines i and j intersect in *ECM*.

Based on the above mentioned information, we can state Property 2 as follows:

Property 2: Let i and j $(i < j)$ be two vertices in G. Then, vertex i is adjacent to j if and only if lines i and j, or lines i and $j-n$, or lines $i+n$ and j intersect in *ECM*.

Some notations that form the basis of the algorithms in Sects. 4 and 5 are defined as follows: The set of all lines that intersect line i in *ECM* is denoted by $N(i)$. In addition, $N[i] = N(i) \cup \{i\}$. For line i in *ECM*, the following functions are defined: $TR(i) = \max\{\, j \mid j \in N[i]\}$ and $STR(i) = \max\{\, j \mid j \in (N[i] \setminus TR(i)) \cup \{i\}\}$. $D_R(i) = \{\, k \mid STR(i) < k < TR(i)\}$. $TL(i) = \min\{\, j \mid j \in N[i]\}$ and $STL(i) = \min\{\, j \mid j \in (N[i] \setminus TL(i)) \cup \{i\}\}$. $D_L(i) = \{\, k \mid TL(i) < k < STL(i)\}$. $BR(i) = k$ such that $p^{-1}(k) = \max\{\, p^{-1}(j) \mid j \in N[i]\}$. $BL(i) = k$ such that $p^{-1}(k) = \min\{\, p^{-1}(j) \mid j \in N[i]\}$. $A(i)$ and $B(i)$ for line i are defined as follows: $A(i) = |\{\, j \mid j \leqslant i, p^{-1}(j) > i\}|$ and $B(i) = |\{\, j \mid j > i, p^{-1}(j) \leqslant i\}|$. Table 1 shows $TR(i)$, $STR(i)$, $D_R(i)$, $TL(i)$, $STL(i)$, $D_L(i)$, $BR(i)$, $BL(i)$, $A(i)$, and $B(i)$ for *ECM* shown in Fig. 4.

4. Articulation Vertex Algorithm

In this section, we present an algorithm AVC that finds all articulation vertices of a circular permutation graph. Let *ECM* be an extended circular permutation model constructed from *CM*. We say a *path* exists between i and j if either line i directly intersects line j, or there exist lines $k_1, k_2, \ldots, k_s$ in *ECM* such that line i intersects k_1, k_1 intersects $k_2, \ldots, k_{s-1}$ intersects k_s, and k_s intersects line j. Moreover, two lines i and j in *ECM* belong to the same *line component* if there exists a path between i and j. In Fig. 4, line 8 is a cut line for lines 10 and 11.

4.1 Properties of Articulation Vertex

Ibarra and Q. Zheng [5] provided Lemma 1, which is a necessary and sufficient condition for the articulation vertex in a permutation graph G_p.

Lemma 1 ([5]): Let G_p be a permutation graph corresponding to a permutation model M_p. A vertex v is an articulation vertex of G_p if and only if there exists an integer i $(1 \leqslant i \leqslant n)$ such that either of the following conditions holds in M_p:

(1) $v = TR(p(i))$ for $B(i) = 1$, $A(i-1) = 1$, and $p(i) < i$,
(2) $v = BR(i)$ for $A(i) = 1$, $B(i-1) = 1$, and $p^{-1}(i) < i$.

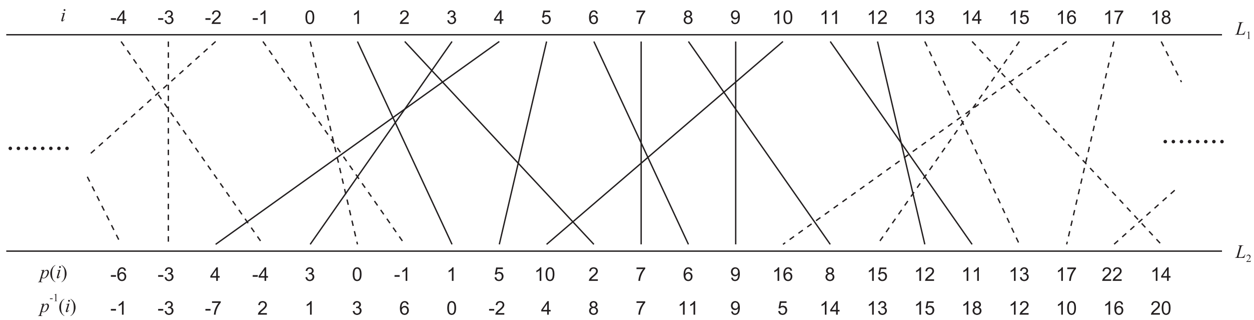


Fig. 4 Extended circular permutation model *ECM*.

Table 1 Example of $TR(i)$, $STR(i)$, $TL(i)$, $STL(i)$, $BR(i)$, $BR(i)$, $A(i)$ and $B(i)$.

i	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$p(i)$	−3	4	−4	3	0	−1	1	5	10	2	7	6	9	16	8	15	12	11	13	17
$p^{-1}(i)$	−3	−7	2	1	3	6	0	−2	4	8	7	11	9	5	14	13	15	18	12	10
$TR(i)$	−2	−2	4	4	4	10	4	4	5	10	10	16	10	10	16	16	16	22	16	16
$STR(i)$	−3	−2	3	3	3	5	3	4	5	7	7	10	9	10	15	15	15	17	15	16
$D_R(i)$						6…9				8,9	8,9	11…15								
$TL(i)$	−4	−10	−1	−1	1	2	−1	−4	2	6	6	8	8	2	11	11	13	14	11	8
$STL(i)$	−3	−6	−1	0	1	2	0	−1	5	6	7	8	9	6	11	12	13	14	14	14
$D_L(i)$								−3,−2	3,4					3…5						
$BR(i)$	−4	−4	−1	−1	1	2	2	2	2	6	6	8	8	8	11	11	13	14	12	11
$BL(i)$	−2	−2	4	4	4	4	4	4	5	10	10	10	10	10	16	16	16	16	16	16
$A(i)$				2	2	2	1	1	1	1	1	1	1	1	1	2				
$B(i)$				2	2	2	1	1	1	1	1	1	1	1	1	2				
$av_1(i)$										10		10			16					
$av_2(i)$								2	2					8						

Let $G = (V, E)$, $|V| = n$ be a circular permutation graph corresponding to a circular permutation model *CM*, and *ECM* be an extended circular permutation model constructed from *CM*. Hence, Lemmas 2 and 3 follow from Lemma 1.

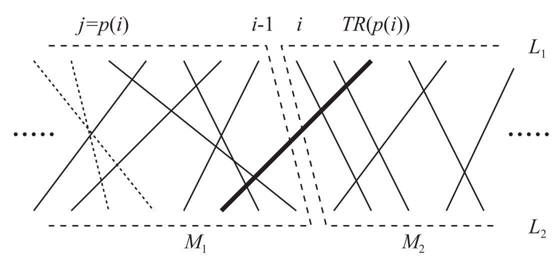
Lemma 2: $TR(p(i))$ is a cut line for $i-1$ and i in *ECM* if $B(i) = 1$, $A(i-1) = 1$ and $p(i) < i$.

(Proof) By Lemma 1–(1), the elimination of line $TR(p(i))$ from *ECM* disconnects it into at least two line components when $B(i) = 1$, $A(i-1) = 1$ and $p(i) < i$. Assume that *ECM* is divided into two line components, namely M_1 and M_2 by removing line $TR(p(i))$ (Fig. 5). We show that M_1 and M_2 include lines $i-1$ and line i, respectively.

From condition $A(i-1) = 1$, *ECM* has a line $j(\leqslant i-1)$ with $p^{-1}(j) > i-1$. We assume that $p^{-1}(j) > i$. There exists some line $r(\leqslant i-1)$ with $p^{-1}(r) = i$ from condition $p(i) < i$. It follows $A(i-1) \geqslant 2$ and contradicts the hypothesis of $A(i-1) = 1$. Hence, such a line r does not exist. This implies that $p^{-1}(j) = i$ and line j has maximum p^{-1} value in M_1.

According to Lemma 1–(1), only line $TR(p(i))$ connects M_1 and M_2. Furthermore, by the condition $B(i) = 1$, $TR(p(i)) > i$ and $p^{-1}(TR(p(i))) < i$. Since *ECM* is constructed under the condition that $|F^-| = |F^+|$, there are i positions from 1 to i on L_2. However, $i+1$ positions are required from 1 to i on L_2 when $p^{-1}(i) < i$. This is the contradiction of the pigeonhole principle. Thus, $p^{-1}(i) > i$.

Hence, for two lines $i-1$ and i, $p^{-1}(i-1) < i$ and $p^{-1}(i) > i$, respectively. Furthermore, line $p(i)$ has maximum p^{-1} value in M_1 and only line $TR(p(i))$ connects M_1 and M_2. Hence, M_1 and M_2 include lines $i-1$ and i, respectively. □



TR(*p*(*i*)) is a cut line for *i*-1 and *i*

Fig. 5 Example of Lemma 2.

Lemma 3: $BR(i)$ is a cut line for i and $i+1$ in *ECM* if $A(i) = 1$, $B(i-1) = 1$, and $p^{-1}(i) < i$.

(Proof) Lemma 3 is symmetric to Lemma 2. Hence, its proof is similar to that of Lemma 2. □

Lemma 4: Let $G = (V, E)$ be a circular permutation graph corresponding to *ECM*. A vertex v is an articulation vertex of G if and only if elimination of line v disconnects *ECM* into at least three line components.

(Proof) Sufficiency of this condition obviously holds; thus, we only prove necessity. Consider a case of where *ECM* is divided into just two line components M_1 and M_2 by re-

Algorithm 1: Algorithm AVC

Input: $CP = \{p(1), p(2), \ldots, p(n)\}$ of a circular permutation graph G.
Output: Articulation vertices of G.
(Step 1)
Construct ECM and compute $p^{-1}(i)$;
(Step 2)
Compute $TR(i)$, $BR(i)$ for i in ECM;
(Step 3)
Compute $A(i)$ and $B(i)$ for i in ECM;
(Step 4)
/* Compute $av_1(i)$ */ ;
for *each* $1 \leqslant i \leqslant n$ **do**
  if *(*$B(i) = 1$ *and* $A(i-1) = 1$ *and* $p(i) < i$*)* **then**
  $av_1(i) = TR(p(i))$;
end
(Step 5)
/* Compute $av_2(i)$ */;
for *each* $1 \leqslant i \leqslant n$ **do**
  if *(*$A(i) = 1$ *and* $B(i-1) = 1$ *and* $p^{-1}(i) < i$*)* **then**
  $av_2(i) = BR(i)$;
end
(Step 6)
for *each* $1 \leqslant i \leqslant n$ **do**
  Normalize $av_1(i)$;
  Normalize $av_2(i)$;
end
(Step 7)
if $av_1(i)$ *has at least two same values for* $1 \leqslant i \leqslant n$ **then**
$av_1(i)$ is an articulation vertex ;
if $av_2(i)$ *has at least two same values for* $1 \leqslant i \leqslant n$ **then**
$av_2(i)$ is an articulation vertex ;

Function Normalize v{
if $v < 1$ **then** $v := v + n$;
if $v > n$ **then** $v := v - n$;
return v ;
}

moving line v from ECM. M_1 includes some copies of lines that are in M_2, and M_2 includes some copies of lines that are on M_1 subject to conditions $F \neq \emptyset$ and $|F^-| = |F^+|$. Thus, ECM is divided into M_1 and M_2, but a graph corresponding to $M_1 \cup M_2$ is connected.

In the following lemma, assume that ECM is divided into $k(\geqslant 3)$ line components $M_1, M_2, \ldots, M_k$ when line v is removed from ECM. Here, M_1 includes some copies of lines that are in M_k, and M_k also includes some copies of lines that are in M_1. Thus, the subgraph corresponding to $M_1 \cup M_k$ is connected. Hence, $G - v$ is a graph with $k - 1$ connected components $(M_2, \ldots, M_{k-1}, M_1 \cup M_k)$. That is, $G - v$ is disconnected. □

Lemma 5: Let $G = (V, E)$ be a circular permutation graph corresponding to ECM. A vertex v is an articulation vertex of G if and only if there exist at least two identical values of v for $1 \leqslant i \leqslant n$ such that either of the following two conditions holds in ECM;

(1) $v = TR(p(i))$ for $B(i) = 1$ and $A(i-1) = 1$ and $p(i) < i$.
(2) $v = BR(i)$ for $A(i) = 1$ and $B(i-1) = 1$ and $p^{-1}(i) < i$.

(Proof) Assume that condition (1) holds for i_1 and i_2, i.e., $v = TR(p(i_1)) = TR(p(i_2))$ for $1 \leqslant i_1 < i_2 \leqslant n$. By Lemma 2, v is a cut line for $i_1 - 1$ and i_1, and is also a cut line for $i_2 - 1$ and i_2. Hence, the elimination of line v disconnects ECM into three line components M_1, M_2, and M_3 that include $i_1 - 1$, i_1, and i_2, respectively. By Lemma 4, G is disconnected because ECM is divided into at least three line components. Thus, vertex v is an articulation vertex of G. In a similar manner, we can prove case (2). □

We show an example in which vertex 10 is recognized as an articulation vertex by applying Lemma 5. In Fig. 4, when $i = 6$, $B(i) = 1$, $A(i-1) = 1$, and $p(i) = 2 < i$, and consequently, $v = TR(p(i)) = TR(p(6)) = 10$. Similarly, when $i = 8$, $B(i) = 1$, $A(i-1) = 1$, and $p(i) = 6 < i$, and thus, $v = TR(p(i)) = TR(6) = 10$ holds true. Thus, we can obtain 10 as the articulation vertex because the values $(v = 10)$ appear for $i = 6$ and 8.

4.2 Analysis of Algorithm AVC

The algorithm used to find all articulation vertices of a circular permutation graph is described formally in Algorithm AVC.

Next, we analyze the complexity of Algorithm AVC. In Step 1, we construct a circular permutation model ECM that can be executed in $O(n)$ time. In Step 2, $TR(i)$ and $BR(i)$ are computed. In Step 3, $A(i)$ and $B(i)$ are obtained. The above preprocessing steps take $O(n)$ time [5]. Steps 4–6 compute $av_1(i)$ and $av_2(i)$ by applying Lemma 5 and they run in $O(n)$ time. By applying Step 6 of Algorithm AVC, we obtain $av_1(i)$ and $av_2(i)$ in Table 1. After executing Step 7, all articulation vertices of a circular permutation graph are correctly found. In Table 1, each of av_1 and av_2 has two identical values, 10 and 2, respectively. Thus, vertices 10 and 2 are articulation vertices. Hence, we obtain the following theorem:

Theorem 1: Algorithm AVC can solve the articulation vertex problem of circular permutation graph in $O(n)$ time.

5. Hinge Vertex Algorithm

In this section, we present Algorithm HVC for finding all hinge vertices of circular permutation graphs. A vertex u is considered to be a hinge vertex if there exist any two vertices x and y in G whose distance increase by removing u.

5.1 Properties of Articulation Vertex

The following Lemma 6 proposed by Chang et al. [17] characterizes the hinge vertices of a simple graph G_s.

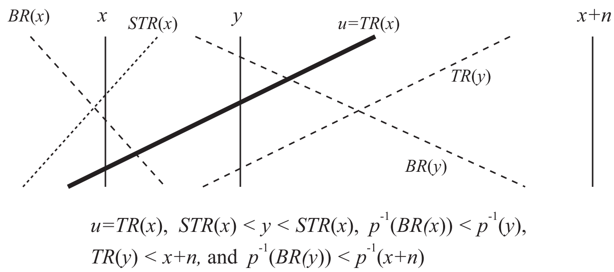


Fig. 6 Example of Lemma 8.

Lemma 6 ([17]): For a simple graph G_s, a vertex u is a hinge vertex of G_s if and only if there exist two nonadjacent vertices $x < y$ such that u is the only vertex adjacent to both x and y in G_s.

Lemma 7 provides the necessary and sufficient condition for hinge vertices in a permutation graph presented by Ho et al. [6].

Lemma 7 ([6]): Let G_p be a permutation graph corresponding to a permutation model M_p. A vertex u is a hinge vertex of G_p if and only if there exist two vertices $x < y$; such that either of the following conditions holds in M_p:

(1) $u = TR(x)$ for $y \in D_R(x)$ and $p^{-1}(BR(x)) < p^{-1}(y)$,
(2) $u = TL(y)$ for $x \in D_L(y)$ and $p^{-1}(x) < p^{-1}(BL(y))$.

Let $G = (V, E)$, $|V| = n$ be a circular permutation graph corresponding to a circular permutation model CM, and ECM be an extended circular permutation model constructed from CM. Lemmas 8 and 9 follow from Lemmas 6 and 7, respectively.

Lemma 8: A vertex $u = TR(x)$ is a hinge vertex of G if there exist two vertices $x < y \in V$ satisfying $y \in D_R(x)$, $p^{-1}(BR(x)) < p^{-1}(y)$, $TR(y) < x + n$, and $p^{-1}(BR(y)) < p^{-1}(x + n)$ in ECM.

(Proof) (⇒) If u is a hinge vertex of G, by Lemma 7, $u = TR(x)$, $STR(x) < y < TR(x)$, and $p^{-1}(BR(x)) < p^{-1}(y)$ in ECM. This indicates that line x does not intersect line y and u is the only line intersecting both lines x and y in ECM (Fig. 6). Assume that $TR(y) > x + n$ or $p^{-1}(BR(y)) > p^{-1}(x + n)$. If $TR(y) > x + n$, the line $TR(y)$ intersects both lines y and $x + n$. Note that line $x + n$ is a copy of line x. That is, both lines $x + n$ and x correspond to the same vertex x. This contradicts the assumption that u is the only vertex adjacent to vertices x and y in G. Furthermore, if $p^{-1}(BR(y)) > p^{-1}(x + n)$, line $BR(y)$ intersects both y and $x + n$. This is found to be contradictory to the assumption. Thus, necessity is satisfied.
(⇐) By Lemma 7, if $u = TR(x)$, $y \in D_R(x)$, and $p^{-1}(BR(x)) < p^{-1}(y)$, u is the only line that intersects both lines x and y in ECM. Furthermore, as $TR(y) < x + n$ and $p^{-1}(BR(y)) < p^{-1}(x + n)$, no line intersects both lines y and $x + n$. This implies that vertex u is the only vertex adjacent to both vertices x and y in G. Therefore, sufficiency is satisfied. □

Algorithm 2: Algorithm HVC

Input: $CP = \{p(1), p(2), \ldots, p(n)\}$ of a circular permutation graph G.
Output: Hinge vertices of G.
(Step 1)
Construct ECM and compute $p^{-1}(i)$;
(Step 2)
Compute $TR(i)$, $STR(i)$, $BR(i)$ for $1 \leqslant i \leqslant n$;
(Step 3)
Compute $TL(i)$, $STL(i)$, $BL(i)$ for $1 \leqslant i \leqslant n$;
(Step 4) /* Compute hinge vertices */ ;
for *each* $y \in D_R(x)$ **do**
  if $p^{-1}(BR(x) < p^{-1}(y)$, $TR(y) < x + n$ *and* $p^{-1}(BR(y)) < p^{-1}(x + n))$ **then** Normalize $TR(x)$ to obtain the hinge vertex ;
end
(Step 5)
for *each* $x \in D_L(y)$ **do**
  if $p^{-1}(x) < p^{-1}(BL(y))$, $y - n < TL(x)$, *and* $p^{-1}(y - n) < p^{-1}(BL(x))$ **then** Normalize $TL(y)$ to obtain the hinge vertex ;
end

Function Normalize v\{
if $v < 1$ **then** $v := v + n$;
if $v > n$ **then** $v := v - n$;
return v ;
\}

Lemma 9: A vertex $u = TL(y)$ is a hinge vertex of G if there exist two vertices $x < y \in V$ satisfying $x \in D_L(y)$, $p^{-1}(x) < p^{-1}(BL(y))$, $y - n < TL(x)$, and $p^{-1}(y - n) < p^{-1}(BL(x))$ in ECM.

(Proof) Lemma 9 is symmetric to Lemma 8. Hence, its proof is similar to that of Lemma 8. □

We show an example where vertex 4 is recognized as a hinge vertex by applying Lemma 8. In Fig. 4, for $x = 8$ and $y = 13$, $y = 13 \in D_R(x) = \{11, 12, 13, 14, 15\}$, $p^{-1}(BR(x)) = 14 < p^{-1}(y) = 15$, $TR(y) = 16 < (x+n) = 20$, and $p^{-1}(BR(y)) = 15 < p^{-1}(x + 2) = 23$ hold. Hence, $TR(x) = TR(8) = 16$ is a hinge vertex for 8 and 13 by Lemma 8. Normalization indicates that vertex 4 is a hinge vertex for 8 and 1.

5.2 Analysis of Algorithm HVC

The algorithm for finding all articulation vertices of a circular permutation graph is described formally in Algorithm HVC.

Next, we analyze the complexity of Algorithm HVC. In Step 1, we construct a circular permutation model ECM that can be executed in $O(n)$ time. $TR(i)$, $STR(i)$, and $BR(i)$ are computed in Step 2. $TL(i)$, $STL(i)$, and $BL(i)$ are computed in Step 3. Preprocessing steps 2 and 3 take $O(n)$ time [17]. Steps 4 and 5 find all hinge vertices by applying

Lemmas 8 and 9, respectively, and they run in $O(n)$ time. After executing Step 5, all hinge vertices of a circular permutation graph are correctly found. Hence, we have the following theorem:

Theorem 2: Algorithm HVC can solve the hinge vertex problem of a circular permutation graph in $O(n)$ time.

6. Concluding Remarks

In this paper, we proposed an algorithm that runs in $O(n)$ time to find all articulation vertices of a circular permutation graph. Our algorithm is constructed by employing Ibarra's algorithm [5]. Furthermore, we presented an algorithm that runs in $O(n)$ time to find all hinge vertices of a circular permutation graph. Our algorithm partially uses Ho's algorithm [6]. In future, we will continue this research by extending the results to other classes of graphs.

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Hirotoshi Honma was born in 1967. He received the B.E., M.E. and D.E. degrees in Engineering from Toyohashi University of Technology, in 1990, 1992 and 2009, respectively. He joined the Department of Information Engineering, Kushiro National College of Technology in 1992. He became an associate professor in 2001. His research interest includes computational graph theory and parallel algorithms. He is a member of IPSJ and ORSJ.

Kodai Abe was born in Kushiro in 1988 and graduated from the Advanced Course of Electronic and Information Systems Engineering Kushiro National College of Technology in 2011. He entered the Department of Intelligent Interaction Technologies, University of Tsukuba in 2011. His current research interests include graph theory. He is a member of ORSJ.

Yoko Nakajima graduated from the Electronic Engineering Course, Kushiro National College of Technology in 1989. She joined the Department of Information Engineering, Kushiro National College of Technology in 1989. Currently, she is an Assistant Professor of the Information Engineering Course, Kushiro National College of Technology. Her research interests include natural language processing. She is a member of IPSJ.

Shigeru Masuyama is a Professor at the Department of Computer Science and Engineering, Toyohashi University of Technology. He received the B.E., M.E. and D.E. degrees in Engineering (Applied Mathematics and Physics) from Kyoto University, in 1977, 1979 and 1983, respectively. He was with the Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University from 1984 to 1989. He joined the Department of Knowledge-Based Information Engineering, Toyohashi University of Technology in 1989. His research interest includes computational graph theory, combinatorial optimization, distributed algorithms and natural language processing. Dr. Masuyama is a member of the OR society of Japan, the Information Processing Society of Japan, the Institute of Systems, Control, Information Engineers of Japan and the Association for Natural Language Processing of Japan, etc.