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Inapproximability of Maximum *r*-Regular Induced Connected Subgraph Problems*

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SUMMARY Given a connected graph $G = (V, E)$ on n vertices, the Maximum r-Regular Induced Connected Subgraph (r-MaxRICS) problem asks for a maximum sized subset of vertices $S \subseteq V$ such that the induced subgraph $G[S]$ on S is connected and r-regular. It is known that 2-MaxRICS and 3-MaxRICS are $\mathcal{NP}$-hard. Moreover, 2-MaxRICS cannot be approximated within a factor of $n^{1-\varepsilon}$ in polynomial time for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$. In this paper, we show that r-MaxRICS are $\mathcal{NP}$-hard for any fixed integer $r \geq 4$. Furthermore, we show that for any fixed integer $r \geq 3$, r-MaxRICS cannot be approximated within a factor of $n^{1/6-\varepsilon}$ in polynomial time for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$.

key words: *induced connected subgraph, regularity, $\mathcal{NP}$-hardness, inapproximability*

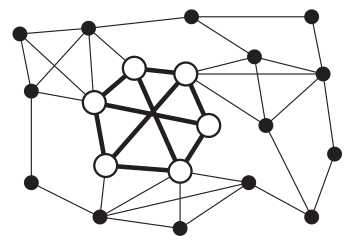

Fig. 1 An input graph G and a maximum induced connected 3-regular subgraph.

1. Introduction

In this paper we only consider simple, undirected, and unweighted graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges in G, respectively. A graph G_S is a subgraph of a graph G if $V(G_S) \subseteq V(G)$ and $E(G_S) \subseteq E(G)$. For a subset of vertices $S \subseteq V(G)$, by $G[S]$, we mean the subgraph of G induced on S, which is called the *induced subgraph*.

For the graph maximization problems, an algorithm ALG is called a σ-approximation algorithm and ALG's approximation ratio is σ if $OPT(G)/ALG(G) \leq \sigma$ holds for every input graph G, where $ALG(G)$ and $OPT(G)$ denote the objective function values of solutions obtained by ALG and an optimal algorithm, respectively.

The problem Maximum Induced Subgraph (MaxIS) for a fixed property Π is the following class of problems ([GT21] in [4]): Given a graph G, find the maximum number of vertices that induces a subgraph satisfying the property Π. The problem MaxIS is very universal; a lot of graph optimization problems can be formulated as MaxIS by specifying appropriately the property Π. For example, if the property Π is "G_S is bipartite", then we wish to find the largest induced bipartite subgraph of a given graph G. Therefore, MaxIS is one of the most important problems in the fields of graph theory and combinatorial optimization, and thus extensively studied in these decades. Unfortunately, however, it is well known that the MaxIS problem is intractable for a large class of interesting properties. For example, in [8], Lund and Yannakakis prove that the Maximum Induced Subgraph problem for the natural properties such as *planar*, *outerplanar*, *bipartite*, *complete bipartite*, *acyclic*, *degree-constrained*, *chordal*, and *interval* cannot be approximated within a factor of $n^{1-\varepsilon}$ in polynomial time for any positive constant ε unless $\mathcal{P} = \mathcal{NP}$, where n is the number of the vertices in the input graph.

1.1 Our Problems and Results

A graph is *r-regular* if the degree of every vertex is exactly r. The *regularity* of graphs must be one of the most basic properties. In this paper we consider the following variant of the MaxIS problem, i.e., the desired properties the induced subgraph must satisfy are *regularity* and *connectivity*:

> Maximum r-Regular Induced Connected Subgraph (r-MaxRICS)
>
> **Input:** A graph $G = (V, E)$.
> **Goal:** Find a maximum subset of vertices $S \subseteq V$ such that the induced subgraph $G[S]$ on S is connected and r-regular.

For example, if the graph illustrated in Fig. 1 is an input of 3-MaxRICS, then the subgraph induced by the "white" vertices has the maximum size of six.

Since a *clique* is connected and regular, the Maximum

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Clique problem is related to r-MaxRICS in some sense. The Maximum clique is very difficult even to approximate [6]. Clearly, however, the problem of finding a clique of a constant degree is solvable in polynomial time. On the other hand, r-MaxRICS is hard even if r is a small constant as follows: The problem 2-MaxRICS is known as Longest Induced Cycle or Chordless Cycle problem since an induced connected 2-regular subgraph means a chordless cycle in the input graph. In [7] Kann shows the following inapproximability of 2-MaxRICS:

Theorem 1 ([7]): 2-MaxRICS cannot be approximated in polynomial time within a factor of $n^{1-\varepsilon}$ for any constant $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$,where n is the number of vertices in the input graph.

In [3] Bonifaci, Di Iorio, and Laura consider the following problem and show its $\mathcal{NP}$-hardness:

Maximum Regular Induced Subgraph (MaxRIS)
Input: A graph $G = (V, E)$.
Goal: Find a maximum subset of vertices $S \subseteq V$ such that the induced subgraph $G[S]$ on S is regular.

Recall that, in our problem r-MaxRICS, the degree of the output subgraph must be r, but, MaxRIS does not specify the value of the degree. Thus, strictly speaking, MaxRIS is slightly different from r-MaxRICS, but the same reduction introduced in [3] shows the following intractability when $r = 3$:

Theorem 2 ([3]): 3-MaxRICS is $\mathcal{NP}$-hard.

However, it would be hard to show the hardness of approximating r-MaxRICS for $r \geq 3$ by using a similar reduction with small modification to the reduction in [3]. In this paper, by using a different *gap-preserving* reduction, we first show the following inapproximability of 3-MaxRICS.

Theorem 3: 3-MaxRICS cannot be approximated in polynomial time within a factor of $n^{1/6-\varepsilon}$ for any constant $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$,where n is the number of vertices in the input graph.

Furthermore, by using additional ideas to the reduction, we show the same inapproximability of r-MaxRICS for any fixed integer $r \geq 4$.

Corollary 1: For any fixed integer $r \geq 4$, r-MaxRICS cannot be approximated in polynomial time within a factor of $n^{1/6-\varepsilon}$ for any constant $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$,where n is the number of vertices in the input graph.

The proofs of Theorem 3 and Corollary 1 will be given in Sect. 3.

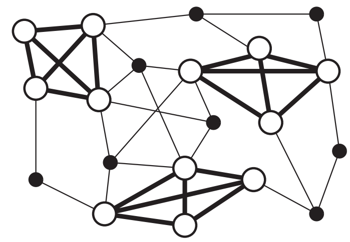

Fig. 2 An input graph G and a maximum induced 3-regular subgraph.

1.2 Related Work

Recently, the problem of finding a maximum induced subgraph having regularity is very popular. Many researchers study the following variant, that is, the connectivity property is not imposed on the induced subgraph.

Maximum r-Regular Induced Subgraph (r-MaxRIS)
Input: A graph $G = (V, E)$.
Goal: Find a maximum subset of vertices $S \subseteq V$ such that the induced subgraph $G[S]$ on S is r-regular.

For example, suppose that the input graph of 3-MaxRIS is illustrated in Fig. 2, which is the same as one in Fig. 1. Then, the three connected components induced by the “white” vertices has the maximum size of 12.

Now we do not require the connectivity constraint. Thus, the problems when $r = 0$ and $r = 1$ correspond to the well studied Maximum Independent Set and Maximum Induced Matching problems, respectively. The former problem is hard even to approximate [6]. The $\mathcal{NP}$-hardness of the latter problem is also shown in [1], [11]. In [10] Orlovich, Finke, Gordon, and Zverovich prove the Maximum Induced Matching cannot be approximated within a factor of $|V|^{1/2-\varepsilon}$ in polynomial time for any $\varepsilon > 0$. The parameterized complexity and exact exponential algorithms of r-MaxRIS are studied in [9] and [5], respectively. Very recently, in [2] Cardoso, Kamińsi, and Lozin prove that r-MaxRIS is $\mathcal{NP}$-hard for any value of $r \geq 3$. Motivated by this result, we investigate the complexity of the connected version problem r-MaxRICS for $r \geq 3$ in this paper.

2. Notation

By (u, v) we denote an edge with endpoints u and v. For a vertex u, the set of vertices adjacent to u in G is denoted by $N_G(u)$ or simply by $N(u)$, and $(u, N_G(u))$ denotes the set $\{(u, v) \mid v \in N_G(u)\}$ of edges. Let the degree of a vertex u be denoted by $deg(u)$, i.e., $deg(u) = |N(u)|$. A (simple) path P of length ℓ from a vertex v_0 to a vertex v_ℓ is represented as a sequence of vertices such that $P = \langle v_0, v_1, \cdots, v_\ell \rangle$, and $|P|$ denotes the length of P. A cycle C of length ℓ is similarly denoted by $C = \langle v_0, v_1, \cdots, v_{\ell-1}, v_0 \rangle$, and $|C|$ denotes the length of C. A *chord* of a path (cycle) $\langle v_0, \cdots, v_\ell \rangle$ ($\langle v_0, \cdots, v_{\ell-1}, v_0 \rangle$)

is an edge between two vertices of the path (cycle) that is not an edge of the path (cycle). A path (cycle) is *chordless* if it contains no chords, i.e., an induced cycle must be chordless. Let $G_1, G_2, \cdots, G_\ell$ be ℓ graphs and also let v_i and v'_i be two vertices in G_i for $1 \le i \le \ell$. Then, $\langle G_1, G_2, \cdots, G_\ell \rangle$ denotes a graph $G = (V(G_1) \cup V(G_2) \cup \cdots \cup V(G_\ell), E(G_1) \cup E(G_2) \cup \cdots \cup E(G_\ell) \cup \{(v'_1, v_2), (v'_2, v_3), \cdots, (v'_{\ell-1}, v_\ell)\})$. That is, two adjacent graphs G_{i-1} and G_i are connected by only one edge (v'_{i-1}, v_i) and G roughly forms a path, which will be called *path-like structure*. Similarly, $\langle G_1, G_2, \cdots, G_\ell, G_1 \rangle$ roughly forms a cycle, which will be called *cycle-like structure*.

Let MaxP_1 and MaxP_2 be maximization problems. A *gap-preserving reduction*, say, Γ, from MaxP_1 to MaxP_2 comes with four parameter functions, g_1, α, g_2 and β. Given an instance x of MaxP_1, the reduction Γ computes an instance y of MaxP_2 in polynomial time such that if $OPT_{\mathsf{MaxP}_1}(x) \ge g_1(x)$, then $OPT_{\mathsf{MaxP}_2}(y) \ge g_2(y)$, and if $OPT_{\mathsf{MaxP}_1}(x) < g_1(x)/\alpha(|x|)$, then $OPT_{\mathsf{MaxP}_2}(y) < g_2(y)/\beta(|y|)$, where $OPT_{\mathsf{MaxP}_1}(x)$ and $OPT_{\mathsf{MaxP}_2}(y)$ denote the objective function values of optimal solutions to the instances x and y, respectively. Note that $\alpha(|x|)$ is the approximation gap, i.e., the hardness factor of approximation for MaxP_1 and the gap-preserving reduction Γ shows that there is no $\beta(|y|)$ factor approximation algorithm for MaxP_2 unless $\mathcal{P} = \mathcal{NP}$ (see, e.g., Chapter 29 in [12]).

3. Hardness of Approximating r-MaxRICS

In this section we give the proofs of Theorem 3 and Corollary 1. The hardness of approximating r-MaxRICS for $r \ge 3$ is shown via a gap-preserving reduction from LONGEST INDUCED CYCLE problem, i.e., 2-MaxRICS. Consider an input graph $G = (V(G), E(G))$ of 2-MaxRICS with n vertices and m edges. Then, we construct a graph $H = (V(H), E(H))$ of r-MaxRICS. First we show the $n^{1/6-\varepsilon}$ inapproximability of 3-MaxRICS and then the same $n^{1/6-\varepsilon}$ inapproximability of the general r-MaxRICS problem for $r \ge 4$.

Let $OPT_2(G)$ (and $OPT_r(H)$, respectively) denote the number of vertices of an optimal solution for G of 2-MaxRICS (and H of r-MaxRICS, respectively). Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ of n vertices and $E(G) = \{e_1, e_2, \cdots, e_m\}$ of m edges. Let $g(n)$ be a parameter function of the instance G. Then we provide the gap preserving reduction such that (C1) if $OPT_2(G) \ge g(n)$, then $OPT_r(H) \ge 4(n^3+1) \times g(n)$, and (C2) if $OPT_2(G) < \frac{g(n)}{n^{1-\varepsilon}}$ for a positive constant ε, then $OPT_r(H) < 4(n^3+1) \times \frac{g(n)}{n^{1-\varepsilon}}$. As we will explain it, the number of vertices in the reduced graph H is $O(n^6)$. Hence the approximation gap is $n^{1-\varepsilon} = \Theta(|V(H)|^{1/6-\varepsilon})$ for any constant $\varepsilon > 0$. By redefining $|V(H)| = n$, we obtain the $n^{1/6-\varepsilon}$ inapproximability of r-MaxRICS.

3.1 Reduction for $r = 3$

Without loss of generality, we can assume that there is no vertex whose degree is one in the input graph G of 2-MaxRICS. The reason is that such a vertex does not contribute to any feasible solution, i.e., a cycle, of 2-MaxRICS and can be removed from G.

The constructed graph H consists of (i) n subgraphs, H_1 through H_n, which are associated with n vertices, v_1 through v_n, respectively, and (ii) m edge sets, E_1 through E_m, which are associated with m edges, e_1 through e_m, respectively. Now we only give a rough outline of the construction and explain the details later. See Fig. 3. If an input instance G of 2-MaxRICS is the left graph, then the reduced graph H of 3-MaxRICS is illustrated in the right graph, where some details are omitted due to the space. Since the graph G has five vertices, v_1 through v_5, the graph H has five subgraphs, H_1 through H_5, each of which is illustrated by a dotted oval. One can see that each H_i roughly consists of $\binom{deg(v_i)}{2} = deg(v_i)(deg(v_i)-1)/2$ path-like structures. For example, since two vertices v_1 and v_2 are connected via the edge e_1 in G, $u_{1,2}$ in H_1 is connected to $u_{2,1}$ in H_2. Similarly to e_2 through e_6, there are five edges, $(u_{1,3}, u_{3,1})$, $(u_{3,4}, u_{4,3})$, $(u_{2,4}, u_{4,2})$, $(u_{2,5}, u_{5,2})$, and $(u_{4,5}, u_{5,4})$ in H. The edge (γ_1, γ_2) between path-like structures labeled by $P_{1,2,5}$ in H_2 and by $P_{3,4,5}$ in H_4 plays an important role as described later.

(i) Here we describe the construction of the ith subgraph H_i in detail for every i $(1 \le i \le n)$. See Fig. 4, which illustrates H_i. Suppose that the set of vertices adjacent to v_i is $N(v_i) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{deg(v_i)}}\}$, where $i_j \in \{1, 2, \cdots n\} \setminus \{i\}$ for $1 \le j \le deg(v_i)$. The subgraph $H_i = (V(H_i), E(H_i))$ includes $deg(v_i)$ vertices, u_{i,i_1} through $u_{i,i_{deg(v_i)}}$ that correspond to the vertices adjacent to v_i, and $deg(v_i)(deg(v_i)-1)/2$ *path gadgets*, P_{i_1,i,i_2}, P_{i_1,i,i_3}, $\cdots$, $P_{i_1,i,i_{deg(v_i)}}$, P_{i_2,i,i_3}, $\cdots$, $P_{i_{deg(v_i)-1},i,i_{deg(v_i)}}$, where two vertices u_{i,i_j} and u_{i,i_k} are connected via the path gadget P_{i_j,i,i_k} for $v_{i_j}, v_{i_k} \in N(v_i)$. As an example, in Fig. 4, the top vertex u_{i,i_1} and the bottom u_{i,i_4} are connected via P_{i_1,i,i_4}. Each path gadget P_{i_j,i,i_k} includes n^3 subgraphs, $P^1_{i_j,i,i_k}$ through $P^{n^3}_{i_j,i,i_k}$, where, for each $1 \le p \le n^3$,

$$
\begin{aligned}
V(P^p_{i_j,i,i_k}) &= \{w^{p,1}_{i_j,i,i_k}, w^{p,2}_{i_j,i,i_k}, w^{p,3}_{i_j,i,i_k}, \gamma^p_{i_j,i,i_k}\},\\
E(P^p_{i_j,i,i_k}) &= (\gamma^p_{i_j,i,i_k}, \{w^{p,1}_{i_j,i,i_k}, w^{p,2}_{i_j,i,i_k}, w^{p,3}_{i_j,i,i_k}\})\\
&\qquad \cup \{(w^{p,1}_{i_j,i,i_k}, w^{p,2}_{i_j,i,i_k}), (w^{p,2}_{i_j,i,i_k}, w^{p,3}_{i_j,i,i_k})\}.
\end{aligned}
$$

Note that the above number "n^3" of the subgraphs $P^p_{i_j,i,i_k}$'s comes from the upper bound of the total number of path gadgets: Each H_i contains $deg(v_i)(deg(v_i)-1)/2$ path gadgets and thus, in total, $deg(v_i)(deg(v_i)-1)/2 \times n$ path gadgets in H_1 through H_n, which is bounded above by n^3. Thus, we want to prepare n^3 subgraphs $P^p_{i_j,i,i_k}$'s (or, more precisely, we want to prepare n^3 γ-vertices which are defined later).

In the path gadget P_{i_j,i,i_k}, two vertices $w^{1,1}_{i_j,i,i_k}$ and $w^{n^3,3}_{i_j,i,i_k}$ are respectively identical to the vertices u_{i,i_j} and u_{i,i_k} prepared in the above. For $2 \le p \le n^3$, contiguous two subgraphs $P^{p-1}_{i_j,i,i_k}$ and $P^p_{i_j,i,i_k}$ are connected by one edge $(w^{p-1,3}_{i_j,i,i_k}, w^{p,1}_{i_j,i,i_k})$ except for a pair $P^{q-1}_{i_j,i,i_k}$ and $P^q_{i_j,i,i_k}$ for some q: the two subgraphs $P^{q-1}_{i_j,i,i_k}$ and $P^q_{i_j,i,i_k}$ are connected by a path of length four $\langle w^{q-1,3}_{i_j,i,i_k}, \beta^1_{i_j,i,i_k}, \beta^2_{i_j,i,i_k}, \beta^3_{i_j,i,i_k}, w^{q,1}_{i_j,i,i_k} \rangle$. This

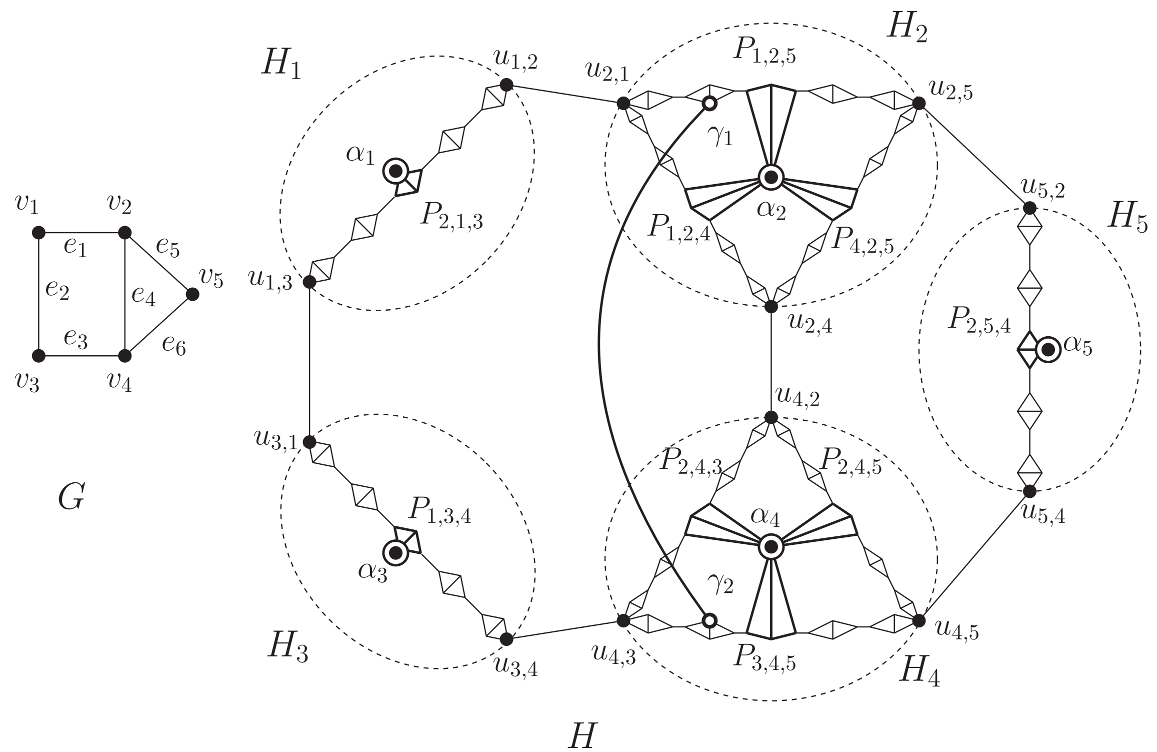

Fig. 3 Input graph G (left) and reduced graph H (right).

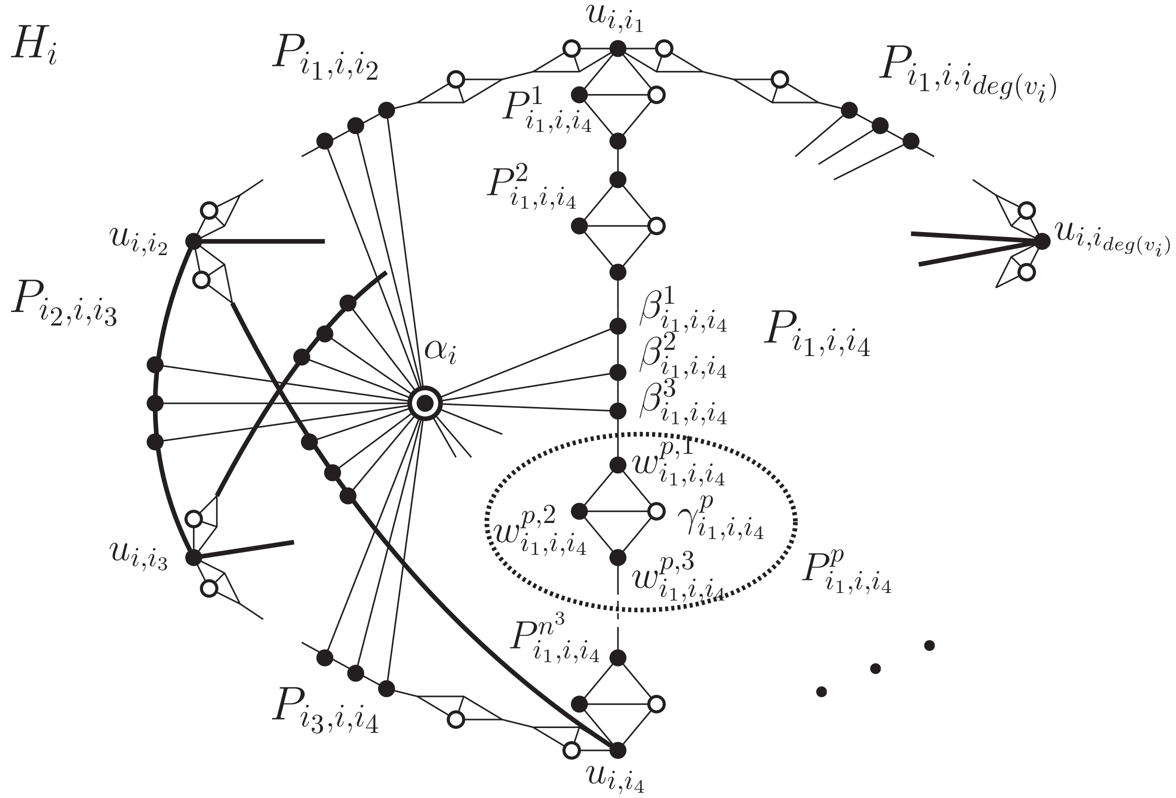

Fig. 4 Subgraph H_i.

q can be arbitrary since we just want to insert the path of length four into the path gadget, and as an example, $q = 3$ in the path gadget P_{i_1,i,i_4} in Fig. 4. Finally, we prepare a special vertex α_i, and α_i is connected to all $\{\beta^1_{i_i,i,i_k}, \beta^2_{i_i,i,i_k}, \beta^3_{i_i,i,i_k}\}$'s. In the following, α_1, α_2, $\cdots$, α_n are called *α-vertices*. Similarly, β-vertices and γ-vertices mean the vertices labeled by

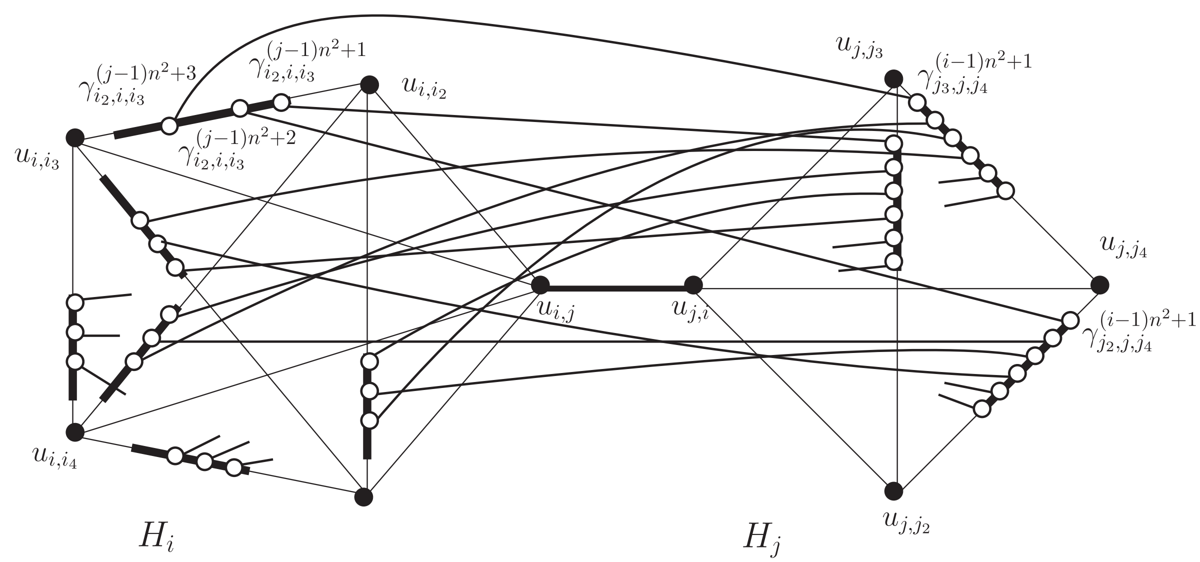

Fig. 5 E_k connecting H_i and H_j.

β and γ, respectively. Since each path gadget has $4n^3 + 3$ vertices (two of which are shared with other path gadgets), the total number of vertices in H_i is

$$|V(H_i)| = \frac{deg(v_i)(deg(v_i) - 1)(4n^3 + 1)}{2} + deg(v_i) + 1,$$

i.e., there are $O(n^5)$ vertices in H_i.

(ii) Next we explain construction of the edge sets E_1 through E_m. Now suppose that e_k connects v_i with v_j for $i \neq j$. Also suppose that the sets of vertices adjacent to v_i and v_j are $N(v_i) = \{j, i_2, \cdots, i_{deg(v_i)}\}$ and $N(v_j) = \{i, j_2, \cdots, j_{deg(v_j)}\}$, respectively. Then, $(u_{i,j}, u_{j,i}) \in E_k$ where $u_{i,j} \in V(H_i)$ in the ith subgraph H_i and $u_{j,i} \in V(H_j)$ in the jth subgraph H_j. Furthermore, by the following rules, γ-vertices in the path gadgets are connected: See Fig. 5. No vertex other than $u_{i,j}$ in the path gadget $P_{x,i,y}$ for $x = j$ or $y = j$ in H_i is connected to any vertex in H_j. Similarly, no vertex other than $u_{j,i}$ in the path gadget $P_{s,j,t}$ for $s = i$ or $t = i$ in H_j is connected to any vertex in H_i. For a path gadgets $P_{x,i,y}$ in H_i, where $j \notin \{x, y\}$ we prepare a set of edges as follows. Let $D = \min_{k\in\{i,j\}}\{deg(v_k)(deg(v_k) - 1)/2 - (deg(v_k) - 1)\}$.

- In $P_{x,i,y}$, there are n^3 γ-vertices, $\gamma_{x,i,y}^1$ through $\gamma_{x,i,y}^{n^3}$. Consider D γ-vertices among those n^3 γ-vertices, the $((j-1)n^2+1)$th vertex $\gamma_{x,i,y}^{(j-1)n^2+1}$ through the $((j-1)n^2+D)$th vertex $\gamma_{x,i,y}^{(j-1)n^2+D}$.
- Next take a look at the jth subgraph H_j and the path gadgets $P_{s,j,t}$'s for $i \notin \{s, t\}$. Note that the number of such gadgets is $deg(v_j)(deg(v_j) - 1)/2 - (deg(v_j) - 1)$ and hence at least D. Then, consider the $((i-1)n^2+1)$th vertex $\gamma_{s,j,t}^{(i-1)n^2+1}$ in each $P_{s,j,t}$. Here, the term "+1" in the superscript of γ comes from the assumption that $j_1 = i$; if $j_k = i$, we consider the $((i-1)n^2 + k)$th γ-vertex.
- Then, we can choose any function f which assigns each element in $\{1, \ldots, D\}$ to a string s, j, t such that $i \notin \{s, t\}$ and it holds $f(b) \neq f(c)$ if $b \neq c$. Finally, we connect $\gamma_{x,i,y}^{(j-1)n^2+k}$ with $\gamma_{f(k)}^{(i-1)n^2+1}$ for $1 \leq k \leq D$. It is important that the path gadget $P_{x,i,y}$ is connected to $P_{s,j,t}$ via only one edge.

Each subgraph H_i has $O(n^5)$ vertices and thus the total number of vertices $|V(H)| = O(n^6)$. Clearly, this reduction can be done in polynomial time. In the next two subsections, we show that both conditions (C1) and (C2) are satisfied by the above reduction.

3.2 Proof of Condition (C1)

Without loss of generality, suppose that a longest induced cycle in G is $C^* = \langle v_1, v_2, \cdots, v_\ell, v_1\rangle$ of length ℓ, and thus $OPT_2(G) = |C^*| = \ell \geq g(n)$. Then we select the following subset S of $4(n^3 + 1) \times \ell$ vertices and the induced subgraph $G[S]$:

$$\begin{aligned} S = &\ V(P_{\ell,1,2}) \cup \{\alpha_1\} \cup V(P_{1,2,3}) \cup \{\alpha_2\} \\ &\cup \cdots \cup V(P_{\ell-1,\ell,1}) \cup \{\alpha_\ell\}. \end{aligned}$$

For example, take a look at the graph G illustrated in Fig. 3 again. One can see that the longest induced cycle in G is $\langle v_1, v_3, v_4, v_2, v_1\rangle$. Then, we select the connected subgraph induced on the following set of vertices:

$$\begin{aligned} &V(P_{2,1,3}) \cup \{\alpha_1\} \cup V(P_{1,3,4}) \cup \{\alpha_3\} \\ &\cup V(P_{2,4,3}) \cup \{\alpha_4\} \cup V(P_{1,2,4}) \cup \{\alpha_2\} \end{aligned}$$

It is easy to see that the induced subgraph is 3-regular and connected. Hence, the reduction satisfies the condition (C1).

3.3 Proof of Condition (C2)

We show that the reduction satisfies the condition (C2) by showing its contraposition. Suppose that $OPT_3(H) \geq 4(n^3 + 1) \cdot \frac{g(n)}{n^{1-\varepsilon}}$ holds for a positive constant ε, and S^* is an optimal

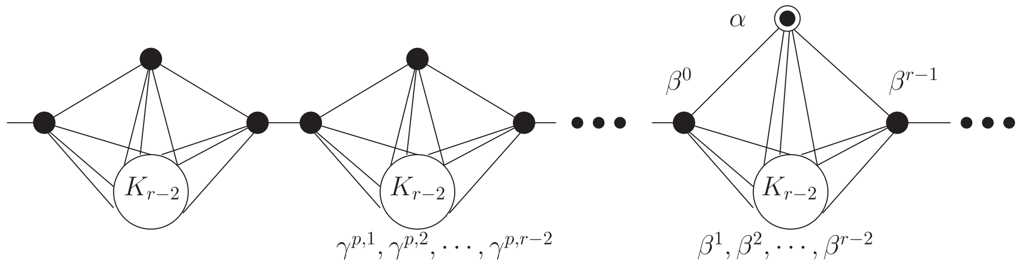

Fig. 6 Modified path gadget in the proof of Corollary 1.

set of vertices such that the subgraph $H[S^*]$ induced on S^* is connected and 3-regular. In the following, one of the crucial observations is that we can select at most one path gadget from each subgraph H_i into the optimal set S^* of vertices, and if a portion of the path gadget is only selected, then the induced subgraph cannot be 3-regular.

(I) See Fig. 4 again. Suppose for example that two path gadgets P_{i_1,i,i_4} and P_{i_2,i,i_3} are selected, and put their vertices into S^*. In order to make the degree of β-vertices three, we need to also select α_i. However, the degree of α_1 becomes six. This implies that we can select at most three β-vertices from each subgraph H_i.

(II) From the above observation (I), we consider the case that at most two of $\beta^1_{j,i,k}$, $\beta^2_{j,i,k}$, and $\beta^3_{j,i,k}$ are selected for some i, j, k. Let us assume that we select $\beta^1_{j,i,k}$ and $\beta^2_{j,i,k}$ ($\beta^1_{j,i,k}$ and $\beta^3_{j,i,k}$, resp.) are put into S^*, but $\beta^3_{j,i,k}$ ($\beta^2_{j,i,k}$, resp.) is not selected. Then, the degree of $\beta^2_{j,i,k}$ ($\beta^1_{j,i,k}$ and $\beta^3_{j,i,k}$, resp.) is at most 2 even if we select α_i, i.e., the induced subgraph cannot be 3-regular. By a similar reason, we cannot select only one of the β-vertices. Hence, if we select β-vertices, all of the three β-vertices in one path gadget must be selected.

As for w-vertices, a similar discussion can be done: For example, if we select $w^{p,1}_{j,i,k}$ and $w^{p,3}_{j,i,k}$ for some i, j, k, p, but $w^{p,2}_{j,i,k}$ ($\gamma^p_{j,i,k}$, resp.) is not selected, then the degree of $\gamma^p_{j,i,k}$ ($w^{p,2}_{j,i,k}$, resp.) is only 2. Thus, we need to select all the vertices of the part $P^p_{k,i,j}$ if we select some vertices from it.

Combining two observations above, one can see that the edges connecting $P^{p-1}_{k,i,j}$ and $P^p_{k,i,j}$, or w-vertices and β-vertices are necessary to make the degrees of the vertices three. As a result, we can conclude that if only a part of one path gadget is chosen, then the induced subgraph obtained cannot be 3-regular.

(III) From (I) and (II), we can assume that if some vertices of a path gadget are selected into S^*, it means that all vertices of the path gadget are selected. For example, suppose that P_{i_1,i,i_4} is selected. Since the degree of the endpoint u_{i,i_1} (u_{i,i_4}) of P_{i_1,i,i_4} is only 2, we have to put at least one vertex into S^* from another subgraph adjacent to H_i, say, a vertex $u_{j,i}$ in H_j. This implies that the induced subgraph $H[S^*]$ forms a cycle-like structure $\langle H_{i_1}, H_{i_2}, \cdots, H_{i_j}, H_{i_1}\rangle$ connecting $H_{i_1}, H_{i_2}, \cdots, H_{i_j}, H_{i_1}$ in order, where $\{i_1, i_2, \cdots, i_j\} \subseteq \{1, 2, \cdots, n\}$.

We mention that such an induced subgraph $H[S^*]$ is 3-regular if and only if the corresponding subgraph in the original graph G is an induced cycle. The if-part is clear by the discussion of the previous section. Let us look at the induced subgraph $H[V(P_{2,1,3}) \cup V(P_{1,3,4}) \cup V(P_{3,4,5}) \cup V(P_{2,5,4}) \cup V(P_{1,2,5})]$ in the right graph H shown in Fig. 3. Then, the induced subgraph includes the edge (γ_1, γ_2) and thus the degrees of γ_1 and γ_4 are 4. The reason why the induced subgraph cannot be 3-regular comes from the fact that the cycle $\langle v_1, v_3, v_4, v_5, v_2, v_1\rangle$ includes the chord edge (v_1, v_4) in the original graph G. The edges between γ-vertices are placed because there is an edge between their corresponding vertices in G. As a result, the assumption that $H[S^*]$ is an optimal solution, i.e., 3-regular, implies that the corresponding induced subgraph in the original graph G forms a cycle $\langle v_{i_1}, v_{i_2}, \cdots, v_{i_j}, v_{i_1}\rangle$.

Since the number of vertices in each path gadget is $4(n^3 + 1)$, $OPT_2(G) \geq \frac{g(n)}{n^{1-\varepsilon}}$ holds by the assumption $OPT_3(H) \geq 4(n^3 + 1) \cdot \frac{g(n)}{n^{1-\varepsilon}}$. Therefore, the condition (C2) is also satisfied.

3.4 Reduction for $r \geq 4$

In this section, we give a brief sketch of the ideas to prove Corollary 1, i.e., the $O(n^{1/6-\varepsilon})$ inapproximability for r-MaxRICS for any fixed integer $r \geq 4$.

The proof is very similar to that of Theorem 3. The main difference between those proofs is the structure of each path gadget. See Fig. 6, which shows the modified path gadget. (i) We replace each of γ-vertices in Fig. 4 with the complete graph K_{r-2} of $r - 2$ vertices, and then connect one γ-vertex in H_i and one γ-vertex in H_j for $i \neq j$ by a similar manner to the reduction for the case $r = 3$. (ii) As for β-vertices, we prepare K_{r-2} of $r - 2$ vertices, say, $\beta^1, \cdots, \beta^{r-2}$, and two vertices, say, β^0 and β^{r-1}, such that each of the two vertices β^0 and β^{r-2} is adjacent to all the vertices in K_{r-2}. Then, all of the β-vertices are connected to the α-vertex similar to the reduction for $r = 3$. Since the reduction requires n^3 γ-vertices to connect all the pairs of H_i's, which is independent of the value of r, the path gadget consists of $\lceil \frac{n^3}{r-2} \rceil$ subgraphs, say, $P^1_{j,i,k}$ through $P^{\lceil n^3/(r-2)\rceil}_{j,i,k}$. As a result, the total number of vertices in the constructed graph remains $O(n^6)$. This completes the proof and thus we can obtain the $n^{1/6-\varepsilon}$ inapproximability of the general r-MaxRICS problem for $r \geq 4$.

4. Conclusion

In this paper, we have shown that r-MaxRICS is $\mathcal{NP}$-hard for any fixed integer $r \geq 4$. Furthermore, we have shown that r-MaxRICS for any fixed integer $r \geq 3$ cannot be approximated within a factor of $n^{1/6-\varepsilon}$ in polynomial time for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$. An apparent future work is to prove a stronger hardness ratio for r-MaxRICS. Also, it is an interesting topic for further researches to investigate the (in) tractability and the (in) approximability of r-MaxRICS on subclasses of graphs such as planar graphs and degree-bounded graphs.

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