Extended *k-Q* product formulas for capacitive- and inductive-coupling wireless power transfer schemes

Takashi Ohira[a]

Toyohashi University of Technology,

1–1 Hibarigaoka, Tempaku, Toyohashi 441–8580, Japan

a) *ohira@tut.jp*

Abstract: This paper presents versatile formulas that characterize wireless power transfer schemes in arbitrary coupling topologies. Maximum available coupling efficiency is formulated in two-port immittance matrix domain invariant to source and load conditions. Key performance index k-Q product is extended to general cases even where resonance or filtering model is inapplicable. Typical capacitive- and inductive-coupling schemes are shown with their k-Q product instances.

Keywords: coupling coefficient, Q factor, port parameter, immittance matrix

Classification: Microwave and millimeter wave devices, circuits, and systems

References

[1] I. Awai: IEICE Electron. Express **10** (2013) 20132008. DOI:10.1587/elex.10.20132008

[2] T. Komaru, M. Koizumi, K. Komurasaki, T. Shibata and K. Kano: IEEE International Conference Industrial Technology (2010) 789. DOI:10.1109/ICIT.2010.5472632

[3] M. Zargham and P. G. Gulak: IEEE Trans. Biomed. Circuits Systems **6** (2012) 228. DOI:10.1109/TBCAS.2011.2174794

[4] D. M. Pozar: *Microwave Engineering* (John Wiley & Sons, 2005) 3rd ed. 536.

1   Introduction

Capacitive- or inductive-coupling structures for wireless power transfer are usually characterized by assuming a simple circuit model that consists of a few discrete capacitors or inductors. They conclude that power transfer efficiency is primarily dominated by coupling coefficient k multiplied by Q factor or so-called resonant k-Q product. It may be straightforward to define k-Q product if its equivalent circuit is represented by single-mode resonators. Practical coupler systems however exhibit labyrinthine structures involving multiple capacitors, mutual inductors, and distributed-constant effects. Recent progress found out about existence of electric coupling components even between

pure coils made of just wound metal wire [1]. They considerably affect overall coupling in its efficiency. It is also often discussed that the efficiency complicatedly depends on source and load impedance conditions, and a clear pilotage is anticipated for lucid design of effective power transfer systems [2, 3]. This paper gives an extended definition of k-Q product as an elegant index by which we can predict the maximum efficiency of arbitrary two-port power transfer schemes.

2 Maximum available coupling efficiency

Consider a system having two RF ports which are internally coupled via electric and/or magnetic field between them. We do not assume any particular circuit topology but regard the system as a black box in general. The only what we need is its RF two-port parameters, i.e. $\mathbf{S}$, $\mathbf{Y}$, or $\mathbf{Z}$ matrix at the point frequency assigned for power transfer. Those parameters are observable by measurement without engineering its internal structure. Another way to get the port parameters is electromagnetic field simulation in case that the physical structure inside is known in detail. Anyway, there is no need to even imply its equivalent circuit model in the following theory.

We start formulation from a sophisticated criterion called *maximum transducer power gain*

$$G_{\mathrm{Tmax}} = \frac{|S_{21}|}{|S_{12}|}\left(K - \sqrt{K^2 - 1}\right) \tag{1}$$

which often appears in microwave amplifier design process. It is in-situ invariant against source or load impedance. Forward power gain $|S_{21}|^2$ can never exceed but achieves G_{Tmax} only when ports #1 and #2 are simultaneously conjugate-matched to its source and load respectively [4]. Associatively K is known as Rollett's stability factor

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |S_{11}S_{22} - S_{12}S_{21}|^2}{2|S_{12}S_{21}|}. \tag{2}$$

What we do here beside usual amplifiers is forcing $S_{21} = S_{12}$ since the coupler consists of only reciprocal components. We then introduce positive parameter α defined as

$$\alpha = \sqrt{\frac{2}{K-1}} \quad \text{or} \quad K = 1 + \frac{2}{\alpha^2}. \tag{3}$$

This may look somewhat heuristic, but we will soon find how α plays an essential roll in formulation. Employing α, we can rewrite Eq. (1) as

$$\eta_{mac} = K - \sqrt{K^2 - 1} = 1 - \frac{2}{1 + \sqrt{1 + \alpha^2}}. \tag{4}$$

The left-hand side is denoted as η_{mac} because it means no longer *gain* as the system involves no active device. This is what we call *maximum available coupling efficiency* in this paper. Since α appears only once on the right-hand side, we can notice at a single glance that η_{mac} is uniquely determined as a monotonously increasing scalar function of α. See Table I for easy look-up. It

is also worth notifying that the right-hand side agrees with peak efficiency formula derived for coupled series LCR resonators in Ref. [2] if we just algebraically replace α^2 by $k^2 Q_S Q_D$. This implies that α carries out the same mission as conventional k-Q product and suggests its meta concept.

Table I. Factor α to achieve typical goals of η_{mac}

α^2	0	8	15	80	360	9800	∞
η_{mac} [%]	0	50	60	80	90	98	100

3   Immittance-domain expression

To mathematically show how α works indeed for coupling systems, formulating the two-port network in the immittance domain is a persuasive way for circuit engineers as shown in Fig. 1(a). Any two-port coupling structure except for trivial topologies can be fully characterized by its impedance matrix $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$. Each component of the matrix can be decomposed into its real and imaginary parts e.g. $z_{11} = r_{11} + jx_{11}$. Thanks to the network reciprocity again, we read $z_{12} = z_{21}$ throughout the formulation below. With a help of matrix conversion

$$\mathbf{S} = (\mathbf{Z} - z_0\mathbf{I})(\mathbf{Z} + z_0\mathbf{I})^{-1}$$

or its constituent form

$$\begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} (z_{11} - z_0)(z_{22} + z_0) - z_{21}^2 & 2z_{21}z_0 \\ 2z_{21}z_0 & (z_{11} + z_0)(z_{22} - z_0) - z_{21}^2 \end{bmatrix}$$
$$\Delta = (z_{11} + z_0)(z_{22} + z_0) - z_{21}^2$$

we can translate Rollett's stability factor from S- into Z-parameter domain as

$$K = \frac{2r_{11}r_{22} - r_{21}^2 + x_{21}^2}{|z_{21}|^2} \tag{5}$$

where z_0 denotes the reference impedance used in S parameter measurement but does not remain in the impedance-domain stability factor. Substituting Eq. (5) into Eq. (3), we obtain a simple and elegant formula

$$\alpha = \frac{|z_{21}|}{\sqrt{|\mathbf{R}|}} = \sqrt{\frac{r_{21}^2 + x_{21}^2}{r_{11}r_{22} - r_{21}^2}}. \tag{6}$$

This is the impedance-domain expression of α. The result reveals that α monotonously increases with mutual reactance x_{21} in square, and decreases with diagonal resistance product $r_{11}r_{22}$. This gives us a general insight on how to improve η_{mac} since it is simply monotonous to α as was shown in Eq. (4). On the contrary, diagonal reactance x_{11} or x_{22} does not contribute to improving or degrading η_{mac}. This stems from the physics in which any reactance element can be neutralized by putting its conjugate reactor in series to the input or output port without affecting additional power profit or loss. Note again that z_0 does not appear in the above result. This is also physically self-evident because the maximum available coupling efficiency of any network is something that must keep constant against any external environment change.

Replacing r_{ij} and x_{ij} by conductance g_{ij} and susceptance b_{ij} with notation $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$, duality theorem leads us to alternative expression

$$\alpha = \frac{|y_{21}|}{\sqrt{|\mathbf{G}|}} = \sqrt{\frac{g_{21}^2 + b_{21}^2}{g_{11}g_{22} - g_{21}^2}} \tag{7}$$

in the admittance domain. The formula in terms of Y-parameters is convenient to characterize coupling topologies mainly having parallel connections of circuit elements.
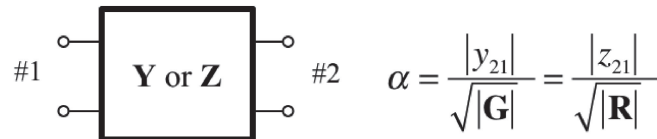
4   Capacitive coupling

To open up a clearer vista on the physical meaning of factor α, we apply Eq. (7) to electric coupling phenomena in capacitive power transfer schemes. Consider a π-shape topology consisting of three passive elements: $y_1 = g_1 + jb_1$, $y_2 = g_2 + jb_2$, and $y_3 = g_3 + jb_3$ as shown in Fig. 1(b). It is regarded as a two-port network having admittance matrix
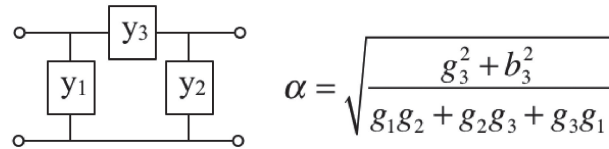
$$\begin{aligned}
\mathbf{Y} &= y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + y_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} g_1 + g_3 + j(b_1 + b_3) & -g_3 - jb_3 \\ -g_3 - jb_3 & g_2 + g_3 + j(b_2 + b_3) \end{bmatrix}.
\end{aligned} \tag{8}$$

By decomposition of each component into its real and imaginary components for usual dyadic y_{ij} parameters, we substituted
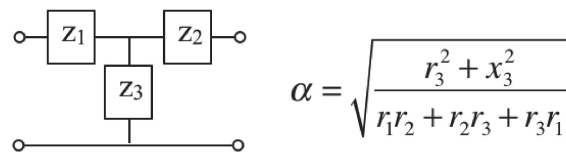
$$g_{11} = g_1 + g_3, \quad g_{22} = g_2 + g_3, \quad g_{21} = -g_3, \quad b_{21} = -b_3. \tag{9}$$

$$\#1 \quad \boxed{\mathbf{Y} \text{ or } \mathbf{Z}} \quad \#2 \qquad \alpha = \frac{|y_{21}|}{\sqrt{|\mathbf{G}|}} = \frac{|z_{21}|}{\sqrt{|\mathbf{R}|}}$$

(a) black box

$$\alpha = \sqrt{\frac{g_3^2 + b_3^2}{g_1 g_2 + g_2 g_3 + g_3 g_1}}$$

(b) π-shape

$$\alpha = \sqrt{\frac{r_3^2 + x_3^2}{r_1 r_2 + r_2 r_3 + r_3 r_1}}$$

(c) T-shape

Fig. 1.   Two-port reciprocal passive network.

Note that diagonal susceptance b_{11} or b_{22} does not contribute to improving or degrading α for the same physical reason as described right after Eq. (9) into Eq. (7), which results in

$$\alpha = \sqrt{\frac{g_3^2 + b_3^2}{g_1 g_2 + g_2 g_3 + g_3 g_1}}. \tag{10}$$

If each of the passive elements is represented not directly by its conductance and susceptance but normalized coefficients $k_1 = g_3/g_1$, $k_2 = g_3/g_2$, and $Q = b_3/g_3$, Eq. (10) is rewritten as

$$\alpha = \sqrt{\frac{k_1 k_2 (1 + Q^2)}{1 + k_1 + k_2}}. \tag{11}$$

This is the k-Q product expression in the admittance domain extended for arbitrary π-shape coupling topologies. Especially in well-designed coupling regimes, we can suppose $g_3 \ll g_1, g_2, b_3$ which lets $Q^2 \gg 1$ and the above formula approximately falls into $\alpha^2 \approx k_1 k_2 Q^2$.

The general expression Eq. (7) is also applicable to pragmatic configurations such as a wireless power transfer scheme featuring two pairs of parallel facing planar electrodes, which is equivalent to the two-port network shown in Fig. 2(a). The two pairs have common coupling capacitance C_3 along with stray capacity C_1 and C_2 hanging on input and output ports. As a linear combination of these terms, we get overall admittance matrix

$$\mathbf{Y} = \left(\frac{1}{r_1} + j\omega C_1 \right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \left(\frac{1}{r_2} + j\omega C_2 \right) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \left(\frac{1}{r_3} + j\omega C_3 \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Resistors r_1, r_2, and r_3 represent parasitic loss assumed in parallel to C_1, C_2, and C_3, respectively (hidden from circuit schematics to avoid confusion). To estimate k-Q product for this scheme, we do not have to renew the formulation but just substitute the above components into Eq. (7). We thus get

$$\alpha = \sqrt{\frac{r_1 r_2}{2 r_3} \cdot \frac{1 + \omega^2 C_3^2 r_3^2}{r_1 + r_2 + 2 r_3}}. \tag{12}$$

The same result also stems from Eq. (11) by defining $k_1 = r_1/2r_3$, $k_2 = r_2/2r_3$, and $Q = \omega C_3 r_3$. Beside their parasitic loss, neither C_1 nor C_2 itself affects α because they can be cancelled by adding external lossless inductors. If both C_1 and C_2 come to lossless ($r_1 = r_2 = \infty$), α goes to infinite resulting in K and η_{mac} reaching unity. This is true even if C_3 has finite loss ($r_3 \neq \infty$). In this case, power transfer efficiency $|S_{21}|^2$ possibly approaches one hundred percent since we can increase source and load impedance as high as necessary at least theoretically. If C_3 comes to lossless ($r_3 = \infty$) on the contrary, Eq. (12) converges into

$$\lim_{r_3 \to \infty} \alpha = \frac{1}{2} \omega C_3 \sqrt{r_1 r_2}$$

as shown in Fig. 2(a). This is quite analogous to k-Q product for inductive coupling that will appear next section.

5 Inductive coupling

In the same way as described in the previous section, we consider a T-shape topology. It consists of three passive elements: $z_1 = r_1 + jx_1$, $z_2 = r_2 + jx_2$, and $z_3 = r_3 + jx_3$ as shown in Fig. 1(c). Two-port impedance matrix Z of the network is formulated as

$$\mathbf{Z} = z_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{13}$$

Expressing each component in terms the dyadic z parameters, we substitute

$$r_{11} = r_1 + r_3, \quad r_{22} = r_2 + r_3, \quad r_{21} = r_3, \quad x_{21} = x_3. \tag{14}$$

Diagonal reactance neither x_{11} nor x_{22} contributes to improving or degrading α for the same reason as mentioned in Eq. (9). Then we put Eq. (14) back into Eq. (6), which results in $k\text{-}Q$ product

$$\alpha = \sqrt{\frac{2}{K-1}} = \sqrt{\frac{r_{21}^2 + x_{21}^2}{r_{11}r_{22} - r_{21}^2}} = \sqrt{\frac{r_3^2 + x_3^2}{r_1 r_2 + r_2 r_3 + r_3 r_1}} \tag{15}$$

extended for any two-port scheme equivalent to the T-shape topology.

The most frequently referred example that falls into T-shape is mutual coupling of lossy coils shown in Fig. 2(b) represented by impedance matrix
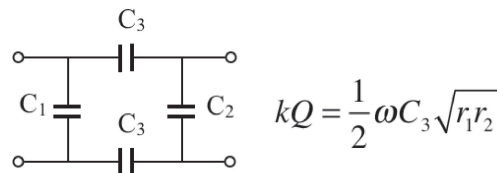
$$\mathbf{Z} = \begin{bmatrix} r_1 + j\omega L_1 & j\omega M \\ j\omega M & r_2 + j\omega L_2 \end{bmatrix}$$

assuming r_1 and r_2 for inductor loss in series, and mutual inductance M free of loss r_3. We can decompose the matrix into three terms
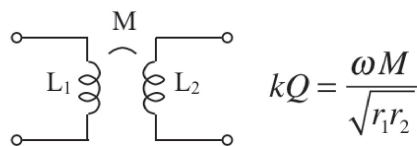
$$\{r_1 + j\omega(L_1 - M)\} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \{r_2 + j\omega(L_2 - M)\} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + j\omega M \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which has the same fashion as Eq. (13). We can therefore apply Eq. (15) and yield

$$\lim_{r_3 \to 0} \alpha = \frac{\omega M}{\sqrt{r_1 r_2}}$$

$$kQ = \frac{1}{2}\omega C_3 \sqrt{r_1 r_2}$$

(a) two pairs of parallel facing electrodes

$$kQ = \frac{\omega M}{\sqrt{r_1 r_2}}$$

(b) mutually coupled solenoids

Fig. 2. Typical lumped-constant schemes.

which exactly agrees with a familiar expression of k-Q product. The formula finds α not containing L_1 or L_2 directly, which means that η_{mac} does not suffer from increasing or decreasing the self inductance by appending lossless inductors or capacitors at either port. This is theoretically true no matter whether they are in resonance or not. Remembering the role of α in η_{mac} by Eq. (4), we get physical insight from the above formula. Especially if either r_1 or r_2 vanishes, η_{mac} approaches unity even in a very loose (long distance between coils) coupling structure.

6   Conclusion

Indeed wireless power transfer schemes feature electric and magnetic coupling or more complicated factor, we have bird-eye-viewed them as just an RF two-port network. Introduced α plays an essential roll and should be called *extended k-Q product*. The formulas presented in this paper enable us to calculate α of any topology of circuits. This can be done even without extracting the circuit's internal topology or affecting their electromagnetic properties if we just have measured or simulated two-port parameters **S**, **Y**, or **Z**. We no longer need the concept of resonance to estimate the k-Q product. It will contribute to development of high-efficiency wireless power transfer systems as a useful design criterion.

Acknowledgment