

Γ -point group velocity of lossy Dirac cone composite right/left-handed metamaterials

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Abstract: The Γ -point group velocity, $v_{g\Gamma}$, of lossy Dirac cone composite right/left-handed (CRLH) metamaterials is derived theoretically for the first time based on the equivalent circuit analysis. With the balanced CRLH condition, it is theoretically shown that $v_{g\Gamma}$ takes the maximum value of a half of the speed of light in the background medium under the condition where the quality factors of the resonators in the series and shunt branches in the unit cell, Q_{se} and Q_{sh} , are equal, whereas under the condition of $Q_{se} \neq Q_{sh}$, $v_{g\Gamma}$ becomes smaller with the slow wave factor of $\kappa = (k^{1/2} + k^{-1/2})^{-1}$ where k is the quality factor ratio $k = Q_{sh}/Q_{se}$.

Keywords: Dirac cone, composite right/left-handed transmission lines, metamaterials

Classification: Microwave and millimeter-wave devices, circuits, and modules

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1 Introduction

The Dirac cone [1, 2] is the dispersion characteristic with a linear ω - β relation at high symmetric points in the Brillouin zone of periodic electromagnetic systems that can be found in artificial composite right/left-handed (CRLH) metamaterial systems [3, 4] and photonic crystals [1, 2] as well as natural electron systems such as graphene [5]. For a lossless CRLH system, it has been found that the Dirac cone can be realized and the group velocities take non-zero values at the Γ -point (where $\beta = 0$) when a specific condition called the *balanced condition* [3, 4] (recalled later) is satisfied, whereas a bandgap appears and group velocities become always zero at the Γ -point without the balanced condition (see Fig. 1). By exploiting the property of the non-zero group velocity of CRLH metamaterials, epoch-making applications such as leaky-wave antennas with broadside radiation have been proposed [6, 7, 8, 9].

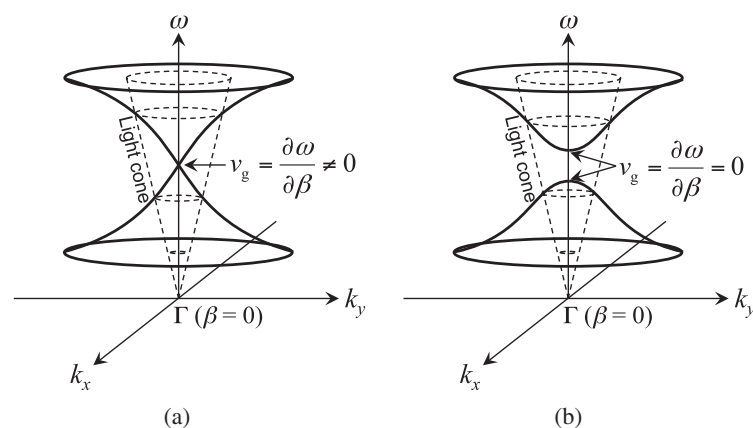


Fig. 1. Dispersion characteristics of 2-D CRLH metamaterials. (a) Dirac cone with the balanced condition. (b) Without the balanced condition.

On the other hand, for lossy CRLH systems with inevitable conductor, dielectric, magnetic, and radiation losses, it has been empirically or numerically shown that the group velocities at the Γ -point still takes non-zero value with the balanced condition [10], however, rigorous values of the Γ -point group velocities are not given to the authors' best knowledge. It is practically important to know the values

of the Γ -point group velocities in realistic lossy systems for future wireless applications of Dirac cone CRLH metamaterials. In this letter, an explicit form of Γ -point group velocities is theoretically given based on an equivalent circuit model, and properties of lossy Dirac cone CRLH metamaterials at the Γ -point are discussed.

2 Γ -point group velocity of lossy Dirac cone CRLH systems

Let us consider a periodic lossy CRLH metamaterial whose unit cell is shown by the equivalent circuit of Fig. 2. Possible realistic losses, R and G , are introduced in the series and shunt branches, respectively. R represents a total loss proportional to the magnetic field, such as the conductor loss and the magnetic material loss, for instance. G also represents one proportional to the electric field such as dielectric losses. R and G even include radiation losses. Based on the periodic analysis of the circuit [11], the dispersion characteristics can be readily obtained as

$$\begin{aligned}\cosh \gamma p &= \cosh \alpha p \cos \beta p + j \sinh \alpha p \sin \beta p \\ &= 1 + \frac{1}{2} \{RG - XB + j(RB + GX)\},\end{aligned}\quad (1)$$

where $\gamma = \alpha + j\beta$ is the propagation constant, and X and B are the reactance and susceptance of the series and shunt branches, respectively, given by

$$X = \omega L_R - \frac{1}{\omega C_L}, \quad B = \omega C_R - \frac{1}{\omega L_L}. \quad (2)$$

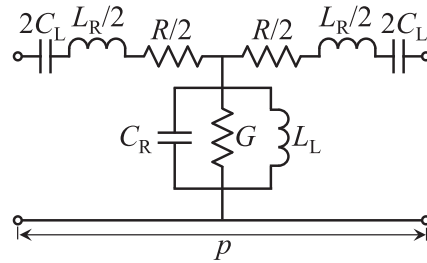


Fig. 2. Equivalent circuit of the unit cell of lossy CRLH TLs.

Under the *balanced condition* [3, 4]:

$$\frac{1}{\sqrt{L_R C_L}} = \frac{1}{\sqrt{L_L C_R}} \equiv \omega_0, \quad (3)$$

where the resonant frequencies of the series and shunt resonators are degenerated at ω_0 , Eq. (1) near the Γ -point can be expressed with the quality factors of the series and shunt resonators, Q_{se} and Q_{sh} , as

$$\cosh \gamma p = 1 + \frac{\omega^2}{2Q_{se}Q_{sh}} \frac{p^2}{v_R^2} (1 + 2jQ_{se}\delta)(1 + 2jQ_{sh}\delta), \quad (4)$$

where Q_{se} and Q_{sh} are given by

$$Q_{se} \equiv \frac{\omega_0 L_R}{R} \quad \text{and} \quad Q_{sh} \equiv \frac{\omega_0 C_R}{G}, \quad (5)$$

δ is the detuning degree near ω_0 given by

$$\delta = \frac{1}{2} \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \approx \frac{\omega - \omega_0}{\omega_0} \equiv \frac{\Delta\omega}{\omega_0}, \quad (6)$$

and v_R is the speed of light in the background medium defined as

$$v_R \equiv \frac{1}{\sqrt{L'_R C'_R}}. \quad (7)$$

Here, $L'_R \equiv L_R/p$ and $C'_R \equiv C_R/p$ are defined as per-unit-length quantities. For instance, if the background medium is vacuum, $L'_R = \mu_0$ and $C'_R = \varepsilon_0$, then v_R becomes the speed of light $c_0 = (\varepsilon_0 \mu_0)^{-1/2}$.

Assuming the Γ -point ($\omega \sim \omega_0$ and $\beta \sim 0$), and also assuming that losses are not very large ($Q_{se}\delta \ll 1$ and $Q_{sh}\delta \ll 1$), we obtain the propagation constant γ near the Γ -point from Eq. (4) as

$$\gamma \cong \frac{\omega_0}{v_R} \left\{ \frac{1}{\sqrt{Q_{se}Q_{sh}}} + j \left(\sqrt{\frac{Q_{sh}}{Q_{se}}} + \sqrt{\frac{Q_{se}}{Q_{sh}}} \right) \delta \right\}. \quad (8)$$

Therefore, the phase constant β can be obtained as

$$\beta = \frac{\omega_0}{v_R} \left(\sqrt{\frac{Q_{sh}}{Q_{se}}} + \sqrt{\frac{Q_{se}}{Q_{sh}}} \right) \delta. \quad (9)$$

From Eq. (9), the Γ -point group velocity is readily obtained as

$$v_g|_{\Gamma} = \frac{\partial\omega}{\partial\beta} = \pm \frac{1}{\sqrt{\frac{Q_{sh}}{Q_{se}}} + \sqrt{\frac{Q_{se}}{Q_{sh}}}} v_R \equiv \pm \kappa v_R, \quad (10)$$

where κ is defined as

$$\kappa \equiv \frac{1}{\sqrt{\frac{Q_{sh}}{Q_{se}}} + \sqrt{\frac{Q_{se}}{Q_{sh}}}}. \quad (11)$$

It is noted from Eq. (10) that the Γ -point group velocity takes non-zero values that are proportional to v_R by the factor of κ , leading to Dirac cone dispersion characteristics in lossy CRLH systems. In the following, the parameter κ is referred to as the *slow wave factor*.

Incidentally, for the lossless case, the dispersion characteristics can be given by letting $R = 0$ and $G = 0$ in Eq. (1) as

$$\beta = \frac{1}{p} \cos^{-1} \left(1 - \frac{1}{2} XB \right), \quad (12)$$

and the Γ -point group velocity can be derived as [3, 4]

$$v_{g, \text{lossless}}|_{\Gamma} = \frac{\partial\omega}{\partial\beta} = \pm \frac{1}{2} v_R. \quad (13)$$

Therefore, the slow wave factor κ is 1/2 in the lossless case.

3 Properties of the Γ -point group velocity

Let us discuss the Γ -point group velocity of Eq. (10) in detail. Since the slow wave factor κ of Eq. (11) is a function of the ratio of the series and shunt resonators Q_{se} and Q_{sh} , we introduce the quantity of the ratio of Q_{se} and Q_{sh} as

$$k \equiv \frac{Q_{sh}}{Q_{se}}. \quad (14)$$

In this case, the slow wave factor κ is given as

$$\kappa = \frac{1}{\sqrt{k} + \sqrt{\frac{1}{k}}}. \quad (15)$$

Fig. 3 shows the slow wave factor κ versus the ratio k . As seen in the figure, κ is a symmetric function with $k = 1$ and takes the same value in the cases of $k = k'$ and $1/k'$. The value of κ takes the maximum value of $1/2$ when $k = 1$, i.e.,

$$Q_{se} = Q_{sh} \quad \text{or} \quad \frac{L_R}{R} = \frac{C_R}{G}. \quad (16)$$

It is worth noticing that, with this condition of Eq. (16), the Γ -point group velocity becomes maximum:

$$|v_g|_{\Gamma_{\max}} = \frac{1}{2} v_R, \quad (17)$$

which is the same value as that in the lossless case. Incidentally, it is interesting to note that the condition Eq. (16) is the same form as the Heviside condition providing distortionless transmission for conventional transmission lines [11]. If the values of Q_{se} and Q_{sh} differ ($k \neq 1$), $|v_g|_{\Gamma_{\max}}$ becomes smaller and waves slow down with the factor of κ at the Γ -point.

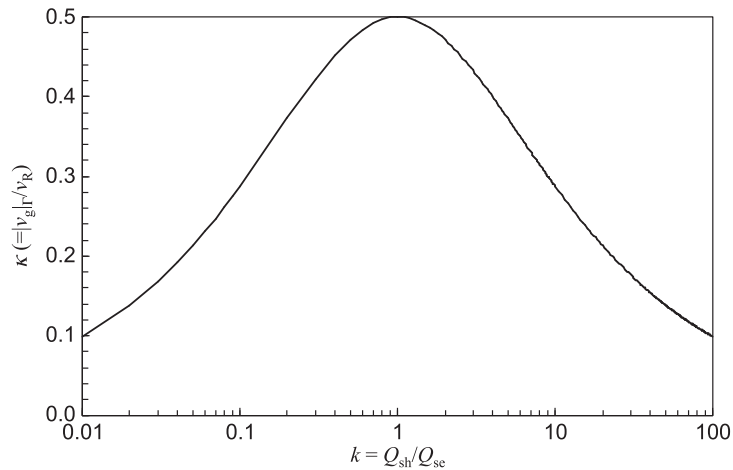


Fig. 3. Slow wave factor κ versus k .

In the following, let us show examples how dispersion diagrams near the Γ -point change with the parameter κ . Fig. 4 shows rigorous dispersion characteristics calculated from Eq. (1) with the slow wave factor κ as a parameter. Note that the parameter κ is a unique parameter in this case, and the parameter $k(= Q_{sh}/Q_{se})$ is given from the inverse function of Eq. (15). It can be seen from Fig. 4 that the absolute value of the slope of the dispersion curve at the Γ -point ($\beta = 0$), $|\partial\omega/\partial\beta|_{\Gamma}$ coincides with the value of κ by definition of the group velocity. The maximum value of the slope is $1/2$ when $k = Q_{sh}/Q_{se} = 1$, while the slope becomes smaller

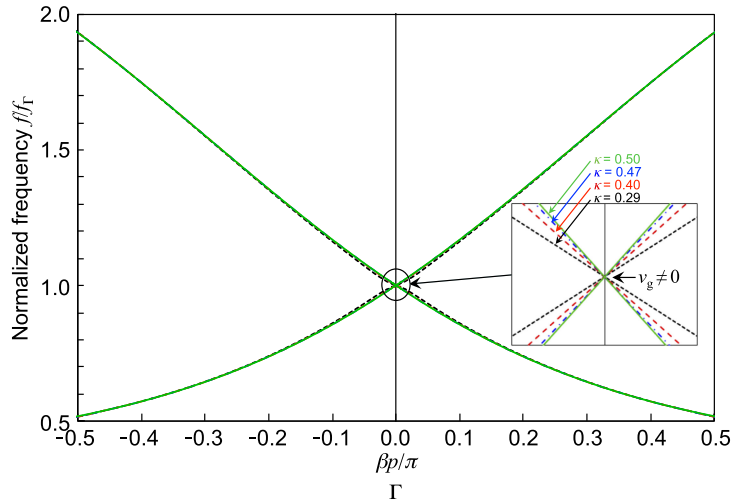


Fig. 4. Dispersion characteristics of lossy CRLH systems with the parameter of κ . The horizontal axis is the normalized phase constant. The vertical axis is the normalized frequency by the Γ -point frequency.

than $1/2$ when $k = Q_{\text{sh}}/Q_{\text{se}} \neq 1$. Apart from the Γ -point, the dispersion curves coincide with each other even with different values of κ as seen in Fig. 4. In summary, even in lossy CRLH systems, κ never becomes zero with the balanced condition of Eq. (3), leading to the Dirac cone dispersion characteristics.

Incidentally, the attenuation constant α at the Γ -point is given from Eqs. (5) and (8) as

$$\alpha = \frac{\omega_0}{v_R} \frac{1}{\sqrt{Q_{\text{se}}Q_{\text{sh}}}} = \sqrt{R'G'}, \quad (18)$$

where $R' = R/p$ and $G' = G/p$ are the resistance and conductance in the series and shunt resonators per unit length. Therefore, α is inversely proportional to the geometric mean of Q_{se} and Q_{sh} and is proportional to the geometric mean of R' and G' at the Γ -point. This leads to the conclusion that to reduce the attenuation constant α at the Γ -point of Dirac cone CRLH metamaterials, it is important to reduce both R' and G' simultaneously.

4 Conclusions

We have derived the Γ -point group velocity in the lossy Dirac cone CRLH metamaterials theoretically in terms of quality factors of the resonators in the series and shunt branches, Q_{se} and Q_{sh} . It has been shown that the Γ -point group velocity takes the maximum value of a half of the speed of light in the background medium under the condition of $Q_{\text{se}} = Q_{\text{sh}}$, whereas the Γ -point group velocity becomes small under the condition of $Q_{\text{se}} \neq Q_{\text{sh}}$ with the slow wave factor of $\kappa = (k^{1/2} + k^{-1/2})^{-1}$ where k is the quality factor ratio $k = Q_{\text{sh}}/Q_{\text{se}}$.

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