LETTER
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A note on CRLB formulation for underdetermined DOA estimation in circularly configured planar arrays

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Abstract: This letter presents the Cramer–Rao Lower Bound (CRLB) for circularly configured planar arrays. CRLB sets a lower bound on the variance of unbiased estimators. It has been extensively studied in the field of array signal processing especially for direction-of-arrival (DOA) estimation using uniform or non-uniform linear arrays. We consider an underdetermined signal model for circularly configured planar arrays and investigate the conditions under which CRLB exist. A new closed-form expression for the CRLB is derived. We numerically compare the CRLB of uniform circular array (UCA) and nested sparse circular array (NSCA) to confirm whether the proposed formulation is effective for both uniform and non-uniform circular planar arrays.

Keywords: Cramer–Rao lower bound, DOA estimation, underdetermined, Fisher information matrix
Classification: Microwave and millimeter-wave devices, circuits, and modules

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1 Introduction

Target localization with less sensors than sources, i.e., underdetermined direction of arrival (DOA) estimation, has been receiving considerable interest in recent years [1, 2, 3, 4, 5]. The Cramer–Rao Lower bound (CRLB) provides a fundamental lower bound on the estimation error of any unbiased DOA estimator [1, 6]. The CRLB can therefore be used as a tool to assess the performance of DOA estimation algorithms. In literature, there has been a lot of research work on CRLB for DOA estimation problems. Although this is the case, most of the existing derivations assume an overdetermined signal model, where the number of sources (D) is smaller than the number of sensors (M) [1, 2, 3, 7].

Recently, there has been an upsurge of attention for the derivation of CRLB for underdetermined DOA estimation [6, 7, 8, 9, 10, 11]. These derivations consider non-uniform or sparse linear array geometries such as nested arrays, co-prime arrays, and minimum redundancy arrays (MRAs). In terms of underdetermined DOA estimation, many estimation methods use the Khatri–Rao subspace method. In array geometry consideration, there exists some 2-D non-uniform planar array configurations such as nested sparse circular arrays (NSCA) [12] which are capable of overdetermined and underdetermined DOA estimation. Thus, there is an existing gap for the derivation of the CRLB for planar arrays especially circularly configured arrays such as uniform circular array (UCA) and nested sparse circular array which are two examples of circularly configured arrays. NSCA is a circularly configured array in which two sub-circular arrays are nested together. One part of the array is dense and the other is sparse [12].

In this letter, we consider the case of stochastic CRLB for circularly configured planar arrays starting from the Fisher information matrix (FIM) [1]. We derive a new closed-form expression for the CRLB. The new CRLB expressions are valid if and only if the FIM is non-singular. In this letter, we observe that for an underdetermined DOA estimation case, as the SNR tends to infinity, the CRLB stagnates to a constant value which is not the case for overdetermined DOA estimation case.

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2 Preliminaries

We consider an M element circularly configured uniform and non-uniform planar array as shown in Fig. 1, where r is the array's radius, d_1 and d_2 are the element's separation distance in the dense and sparse parts respectively. The detailed consideration of Fig. 1(b) can be found in [12]. We assume that D narrowband sources with the wavenumber $k = 2\pi/\lambda$ are impinging on this array from the directions $\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_D]$. The received signal $\boldsymbol{x}(t) = [x_1(t), x_2(t), \cdots, x_M(t)]$ is therefore given by

$$\boldsymbol{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{s}(t) = [s_1(t), s_2(t), \cdots, s_D(t)]$ is a signal vector which is assumed to be uncorrelated, and $\mathbf{n}(t) = [n_1(t), n_2(t), \cdots, n_M(t)]$ is noise vector. The array manifold matrix is therefore given by;

$$\mathbf{A} = [\mathbf{a}(\phi_1), \mathbf{a}(\phi_2), \ldots, \mathbf{a}(\phi_D)] \tag{2}$$

$\mathbf{a}(\phi)$ is the steering vector given by $\mathbf{a}(\phi) = [e^{-jkr\cos(\phi-\gamma_m)}]^T$ where γ_m is the angular position of the m-th element. The source autocorrelation matrix of $\mathbf{s}(t)$ is diagonal. Thus,

$$\mathbf{R}_{xx} = E[\boldsymbol{x}\boldsymbol{x}^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma\mathbf{I} \tag{3}$$

where $\mathbf{R}_{ss}$ is the signal covariance matrix given by the diagonal of signal powers i.e $\mathbf{R}_{ss} = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_D)$, ρ_d is the power of the d-th source, σ is noise variance and $\mathbf{I}$ is an identity matrix. From (3), we vectorize the covariance matrix such that

$$\begin{aligned}\mathbf{y} &= \mathrm{vec}(\mathbf{R}_{xx}) = \mathrm{vec}(\mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H) + \mathrm{vec}(\sigma\mathbf{I}) \\ &= (\mathbf{A}^* \odot \mathbf{A})\hat{\mathbf{R}}_{ss} + \mathrm{vec}(\sigma\mathbf{I})\end{aligned} \tag{4}$$

where $\hat{\mathbf{R}}_{ss} = [\rho_1, \rho_2, \ldots, \rho_D]^T$ is the equivalent source signal vector and $(\mathbf{A}^* \odot \mathbf{A})$ is the manifold of longer array after vectorization.

3 Cramer–Rao lower bound

The expressions for CRLB comes from the inversion of the Fisher information matrix (FIM), which contains information about all the unknown parameters [6, 8, 11]. In this letter, we are interested in the stochastic CRLB of circularly configured planar array for underdetermined DOA estimation in which $D \geq M$. Let $\boldsymbol{\alpha} = [\boldsymbol{\phi}^T, \rho_d^T, \sigma]^T$ denote unknown parameter vector where $d = 1, 2, \ldots, D$. The (η, ℓ)-th entry of the Fisher information matrix (FIM) $\mathfrak{F}(\boldsymbol{\alpha})$ is defined as

$$\mathfrak{F}(\boldsymbol{\alpha}) = N \bullet \mathrm{Tr}\left\{\mathbf{R}_{xx}^{-1}\frac{\partial\mathbf{R}_{xx}}{\partial[\boldsymbol{\alpha}]_\eta}\mathbf{R}_{xx}^{-1}\frac{\partial\mathbf{R}_{xx}}{\partial[\boldsymbol{\alpha}]_\ell}\right\} \tag{5}$$

$\mathrm{Tr}\{\}$ is the trace which can be defined as;

$$\mathrm{Tr}\{\mathbf{WXYZ}\} = \mathrm{vec}(\mathbf{X}^H)^H(\mathbf{W}^T \otimes \mathbf{Y})\mathrm{vec}(\mathbf{Z})$$

$$(\mathbf{W} \otimes \mathbf{X})^{-1} = (\mathbf{W})^{-1} \otimes (\mathbf{X})^{-1}$$

for non-singular $\mathbf{W}$ and $\mathbf{X}$. Therefore (5) can be written as

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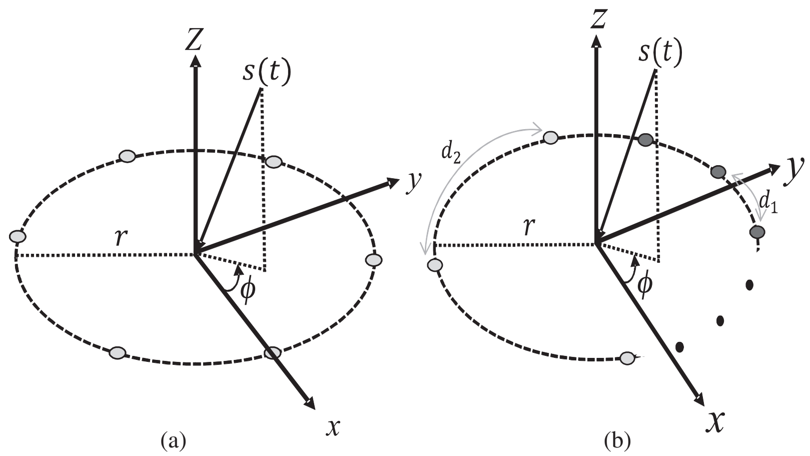

Fig. 1. Array configuration (a) uniform circular array and (b) nested sparse circular antenna array

$$\begin{aligned}\mathfrak{I}(\boldsymbol{\alpha}) &= N\left[\mathrm{vec}\left(\frac{\partial \mathbf{R}_{xx}}{\partial[\boldsymbol{\alpha}]_\eta}\right)\right]^H (\mathbf{R}_{xx}^{-T} \otimes \mathbf{R}_{xx}^{-1})\mathrm{vec}\left(\frac{\partial \mathbf{R}_{xx}}{\partial[\boldsymbol{\alpha}]_\ell}\right) \\ &= N\left[(\mathbf{R}_{xx}^T \otimes \mathbf{R}_{xx})^{-\frac{1}{2}}\left(\frac{\partial \mathbf{y}}{\partial[\boldsymbol{\alpha}]_\eta}\right)\right]^H (\mathbf{R}_{xx}^T \otimes \mathbf{R}_{xx})^{-\frac{1}{2}}\left(\frac{\partial \mathbf{y}}{\partial[\boldsymbol{\alpha}]_\ell}\right) \end{aligned} \tag{6}$$

The derivatives of $\mathbf{y}$ with respect to $\boldsymbol{\alpha}$ is given by

$$\frac{\partial \mathbf{y}}{\partial[\boldsymbol{\alpha}]} = \left[\frac{\partial \mathbf{y}}{\partial \phi_1}, \frac{\partial \mathbf{y}}{\partial \phi_2}, \cdots, \frac{\partial \mathbf{y}}{\partial \phi_D}, \frac{\partial \mathbf{y}}{\partial \rho_1}, \frac{\partial \mathbf{y}}{\partial \rho_2}, \cdots, \frac{\partial \mathbf{y}}{\partial \rho_D}, \frac{\partial \mathbf{y}}{\partial \sigma^2}\right] \tag{7}$$

Let $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ denote:

$$\boldsymbol{\Lambda} = (\mathbf{R}_{xx}^T \otimes \mathbf{R}_{xx})^{-\frac{1}{2}}\left[\frac{\partial \mathbf{y}}{\partial \phi_1}, \frac{\partial \mathbf{y}}{\partial \phi_2}, \cdots, \frac{\partial \mathbf{y}}{\partial \phi_D}\right]$$

$$\boldsymbol{\Gamma} = (\mathbf{R}_{xx}^T \otimes \mathbf{R}_{xx})^{-\frac{1}{2}}\left[\frac{\partial \mathbf{y}}{\partial \rho_1}, \frac{\partial \mathbf{y}}{\partial \rho_2}, \cdots, \frac{\partial \mathbf{y}}{\partial \rho_D}, \frac{\partial \mathbf{y}}{\partial \sigma^2}\right]$$

The FIM therefore becomes;

$$\mathfrak{I}(\boldsymbol{\alpha}) = N\begin{bmatrix}\boldsymbol{\Lambda}^H \\ \boldsymbol{\Gamma}^H\end{bmatrix}\begin{bmatrix}\boldsymbol{\Lambda} & \boldsymbol{\Gamma}\end{bmatrix} = N\begin{bmatrix}\boldsymbol{\Lambda}^H\boldsymbol{\Lambda} & \boldsymbol{\Lambda}^H\boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}\boldsymbol{\Lambda}^H & \boldsymbol{\Gamma}^H\boldsymbol{\Gamma}\end{bmatrix} \tag{8}$$

If the FIM is non-singular, then the CRLB for the DOAs $\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_D]^T$ can be expressed as the inverse of the Schur complement of the block $\boldsymbol{\Lambda}^H\boldsymbol{\Lambda}$ of $\mathfrak{I}(\boldsymbol{\alpha})$. Therefore the CRLB will be given by

$$\mathrm{CRLB}(\phi) = \mathfrak{I}(\boldsymbol{\alpha})^{-1} = \frac{1}{N}(\boldsymbol{\Lambda}^H \boldsymbol{\Pi}_{\boldsymbol{\Gamma}}^{\perp}\boldsymbol{\Lambda})^{-1} \tag{9}$$

where $\boldsymbol{\Pi}_{\boldsymbol{\Gamma}}^{\perp} = \mathbf{I} - \boldsymbol{\Gamma}(\boldsymbol{\Gamma}^H\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}^H$. The non-singularity of the FIM is equivalent to non-singularity of $\boldsymbol{\Lambda}^H\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^H\boldsymbol{\Pi}_{\boldsymbol{\Gamma}}^{\perp}\boldsymbol{\Lambda}$.

4 Numerical examples

In order to validate our claims, we now conduct simulations and perform a CRLB performance comparison for uniform circular array and nested sparse circular array [12] as shown in Fig. 1. We consider a six element NSCA (three dense and three sparse elements [12]), and a six, nine and twelve element UCA all with $r = \lambda$. The

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eight uncorrelated sources are located at $\boldsymbol{\phi} = [15°, 36°, 70°, 90°, 112°, 130°, 145°, 162°]$, all with the same amount of power. In this letter, the CRLB is evaluated by estimating all the DOAs and find the average CRLB for all estimated angles. The signal-to-noise ratio (SNR) used in this analysis is defined as;

$$\mathrm{SNR} = 10\log_{10}\frac{\min_d \rho_d}{\sigma^2} \tag{10}$$

where $d = 1, 2, \ldots, D$.

In Fig. 2, we compare the CRLB of a six element NSCA to the CRLB of UCA with twelve elements. We then reduce the number of elements of the UCA to nine and then six. In terms of a six element UCA, the number of elements is the same to that of NSCA but the element interval is different. In the case of NSCA, the elements in the sparse part have a larger element interval as compared to UCA whose element interval is the same for all elements. In this case, the CRLB performance of the NSCA and UCA becomes very close to each other and same for higher SNR cases as shown in blue curve and black dots in Fig. 2. The result in Fig. 2 for the two circularly configured planar arrays is comparable which proves the CRLB derivation works for an arbitrary circularly configured planar array for underdetermined DOA estimation.

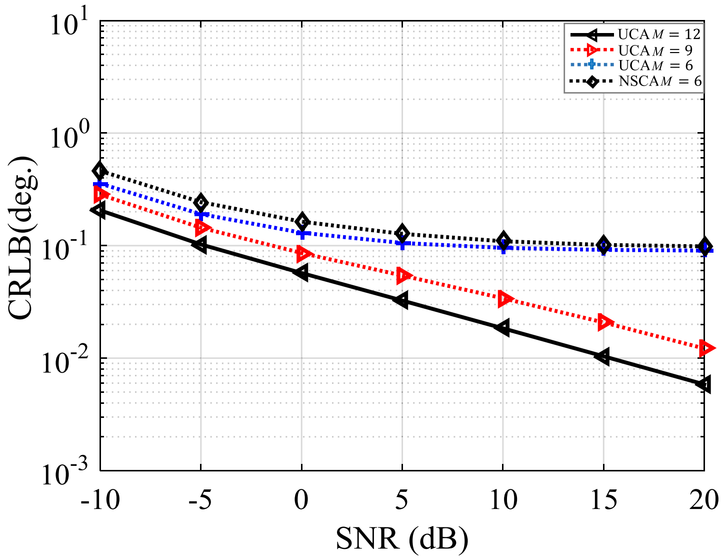

Fig. 2. A comparison of CRLB versus SNR for UCA and NSCA with different number of elements M for 7 sources ($D = 7$).

In Fig. 3, a plot of the proposed CRLB expression for different number of sources as a function of SNR, with 5000 snapshots is shown using NSCA. In this figure, we observe that the CRLB in the case of an overdetermined DOA estimation is inversely proportional to the SNR. In an underdetermined DOA estimation scenario, the value of the CRLB becomes stagnant as the SNR increases. In this case, as SNR converges to infinity, the CRLB converges to a positive constant which is as a result of asymptotic behavior of the MSE error of the CRLB for $D \geq M$ which is quite different in the case when $D \leq M$. This kind of behavior is often observed in the case of underdetermined DOA estimation for linear arrays. Such behavior is also observed in the case of circularly configured planar arrays.

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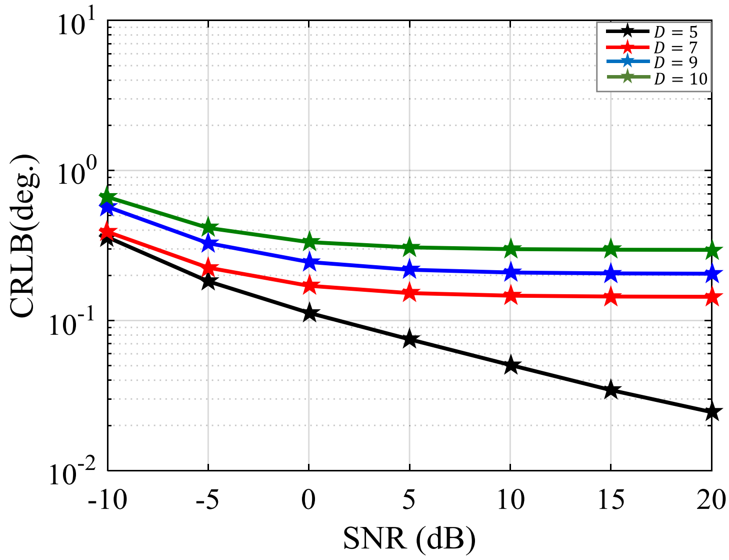

Fig. 3. CRB versus SNR for NSCA with $M = 6$, snapshots = 5000, for different number of sources D

5 Conclusion

In this paper, we derived a closed-form expression for the CRLB of circularly configured planar arrays for underdetermined DOA estimation. We performed a numerical comparison of the CRLB for circular array geometries; NSCA and UCA. In the underdetermined DOA estimation case, as the SNR tends to infinity, the CRB stagnates to a constant value. From the results of two circularly configured planar arrays, we obtained a comparable result which proves that the CRLB derivation works for a circularly configured planar array for underdetermined DOA estimation. These results can be a good starting point to benchmark DOA estimation algorithms for circular planar arrays.

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