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Converting matrices between ternary helix and Reed-Muller transforms over Galois Fields

Bogdan J. Falkowski[1a)] **and Cheng Fu**[1]

[1] *School of Electrical and Electronic Engineering*
Nanyang Technological University
50 Nanyang Avenue, Block S1, Singapore 639798
a) *efalkowski@ntu.edu.sg*

Abstract: In this article, new relations between ternary helix and Reed-Muller transforms have been analyzed. In addition, the new converting matrices and corresponding hardware implementations in Galois Field (3) that can convert directly between ternary helix and Reed-Muller spectra have also been presented.

Keywords: Galois Field (3), helix transform, Reed-Muller transform
Classification: Science and engineering for electronics

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1 Introduction

Most of the digital VLSI/ULSI circuits that are used presently are based on binary logic which has been found to be reliable and compact. However, lately there is an increasing interest in circuits based on multiple-valued logic. This interest is fueled by their potential advantages over the binary ones, such as increased data processing capability per unit area, reduced number

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of complexity of interconnections, as well as smaller number of active devices inside a chip. Multiple-valued logic allows circuits to have simpler, more flexible and more compact implementation with higher speed and reduced power dissipation. Among the papers written on multiple-valued logic, quite a large number of them are on the design and implementation techniques as well as applications of 3-valued logic or ternary logic [1, 2]. The ternary logic has some inherent advantages in an environment where a 'middle' state between two outer ones can be found in which the outer devices are either both on or both off, as well as in the environment where two binary elements are combined at an upper and lower signal levels. Proposed applications of ternary logic include fail-safe logic and detection of hazard in binary logic circuits [1] as well as evaluation of logic functions in the presence of unknown inputs [2]. The ternary logic in Galois Field (GF) (3) is also used for error-correcting codes in CDMA systems [3].

In this article, properties and mutual relations of two ternary transforms over GF(3) are considered. The first one is well-known Ternary Reed-Muller (TRM) transform [4] while the second one, introduced by these authors in [5], is named helix transform due to the symmetrical structure along the diagonal or reverse-diagonal in the transform matrices. The presented relations and properties show that both ternary transforms are directly related through converting matrices that can be easily implemented in hardware.

2 Basic definitions

Definition 1. Let M_n be a $N \times N(N = 3^n)$ matrix with columns corresponding to some ternary functions of n variables. If the set of columns is linearly independent with respect to ternary Galois Field, then M_n has one unique inverse M_n^{-1} over GF(3) and is said to be linearly independent, i.e.

$$M_n \cdot M_n^{-1} = I_n, \tag{1}$$

where I_n is a identity matrix with order N and all the operations are performed over GF(3).

The linearly independent transform based on Definition 1 can be described by the following general formulae,

$$M_n \cdot \overrightarrow{A} = \overrightarrow{F} \tag{2}$$

and

$$M_n^{-1} \cdot \overrightarrow{F} = \overrightarrow{A} \tag{3}$$

where $\overrightarrow{F} = \left[\begin{array}{cccc} f_0, & f_1, & ..., & f_{3^n-1} \end{array} \right]^T$ is a column vector defining the truth vector of a ternary function $f(x_n)$ in natural ternary ordering, M_n is a matrix of order N defined by any linearly independent set of n-variable ternary functions and $\overrightarrow{A} = \left[\begin{array}{cccc} a_0, & a_1, & ..., & a_{3^n-1} \end{array} \right]^T$ is the spectra coefficient column vector for the particular transform matrix M_n while T is the matrix transpose operator.

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Formula (2) can be also expressed by

$$f(x_n) = \sum_{i=0}^{3^n-1} a_i g_i \tag{4}$$

where g_i denotes a ternary discrete function, which truth vector constitutes the number $i+1$ column of the matrix M_n, $0 \le i \le 3^n - 1$ and all operations are over GF(3).

3 Properties and relations

The Ternary Reed-Muller ($\boldsymbol{TRM}$) transform is defined by the following equation from [4].
For order N,

$$TRM_n = \overset{n-1}{\otimes} TRM_1 = \overset{n-1}{\otimes} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \tag{5}$$

where "$\overset{n-1}{\otimes}$" represents Kronecker product [4] applied $n-1$ times to the matrix with additions and multiplications over GF(3) and n is the number of ternary variables.

In Table I, the ternary functions of $\boldsymbol{TRM}$ transform for $n = 2$ are presented.

Table I. Functions of $\boldsymbol{TRM}$ transform for $n = 2$

x_2	x_1	$a_{\langle TRM \rangle}$
0	0	f_0
0	1	$2f_1 + f_2$
0	2	$2f_0 + 2f_1 + 2f_2$
1	0	$2f_3 + f_6$
1	1	$f_4 + 2f_5 + 2f_7 + f_8$
1	2	$f_3 + f_4 + f_5 + 2f_6 + 2f_7 + 2f_8$
2	0	$2f_0 + 2f_3 + 2f_6$
2	1	$f_1 + 2f_2 + f_4 + 2f_5 + f_7 + 2f_8$
2	2	$f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8$

The basic 3×3 matrix of the Left-Positive-Helix ($\boldsymbol{LPH}$) transform are given by,

$$LPH_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

For order N, $\boldsymbol{LPH}$ transform matrix is derived from the basic matrix by Kronecker product as shown in the following equation,

$$LPH_n = \begin{bmatrix} LPH_{n-1} & O_{n-1} & O_{n-1} \\ LPH_{n-1} & LPH_{n-1} & LPH_{n-1} \\ O_{n-1} & O_{n-1} & LPH_{n-1} \end{bmatrix} = \overset{n-1}{\otimes} LPH_1 \ . \tag{7}$$

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The inverse transform matrix of $\boldsymbol{LPH}$ for order N can be derived by calculating their Kronecker product from the inverse matrix of (6).

$$LPH_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

For order N,

$$LPH_n^{-1} = \begin{bmatrix} LPH_{n-1}^{-1} & O_{n-1} & O_{n-1} \\ 2\cdot LPH_{n-1}^{-1} & LPH_{n-1}^{-1} & 2\cdot LPH_{n-1}^{-1} \\ O_{n-1} & O_{n-1} & LPH_{n-1}^{-1} \end{bmatrix} = \overset{n-1}{\otimes} LPH_1^{-1} \ . \tag{9}$$

The ternary functions of $\boldsymbol{LPH}$ transform for $n = 2$ are given in Table II.

Table II. Functions of $\boldsymbol{LPH}$ transform for $n = 2$

x_2	x_1	$a_{\langle LPH \rangle}$
0	0	f_0
0	1	$2f_0 + f_1 + 2f_2$
0	2	f_2
1	0	$2f_0 + f_3 + 2f_6$
1	1	$f_0 + 2f_1 + f_2 + 2f_3 + f_4 + 2f_5 + f_6 + 2f_7 + f_8$
1	2	$2f_2 + f_5 + 2f_8$
2	0	f_6
2	1	$2f_6 + f_7 + 2f_8$
2	2	f_8

Table II shows that some spectral coefficients can be obtained directly from the truth vector and require no computational cost. There are some connections between the ternary representation of the row number of the coefficients' vector and the truth vector, as shown in the following properties.

Property 1. The row i in the spectral coefficients' vector of $\boldsymbol{LPH}$ transform can be obtained without any computational cost, where the ternary representation of the row number i does not contain any '1's.

Four coefficients in the spectra of $\boldsymbol{LPH}$, $a_{(00)}$, $a_{(02)}$, $a_{(20)}$ and $a_{(22)}$, are obtained directly from f_0, f_2, f_6 and f_8, all of which have the same subscripts, respectively.

For $n = 2$, the spectra of $\boldsymbol{TRM}$ can be derived from $\boldsymbol{LPH}$ transform by the following equation,

$$\overrightarrow{A_{TRM}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix} \times \overrightarrow{A_{LPH}}$$

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The transform matrix of size $3^n \times 3^n$ between ***LPH*** and ***TRM*** is denoted by Q_n. The following equations give the definition of Q_n. For order N,

$$Q_n = \overset{n-1}{\otimes} Q_1 = \overset{n-1}{\otimes} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}. \tag{10}$$

The transform Q_n is a self-inverse transform matrix and it is also extended from the basic matrix Q_1 as shown below:

$$Q_n^{-1} = \overset{n-1}{\otimes} Q_1^{-1} = \overset{n-1}{\otimes} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}. \tag{11}$$

Due to this property, the spectra $\overrightarrow{A_{LPH}}$ can be presented by the spectra $\overrightarrow{A_{TRM}}$ by the equation below,

$$\overrightarrow{A_{LPH}} = Q_n \times \overrightarrow{A_{TRM}}. \tag{12}$$

Similar relations also exist between ***RPH*** and ***TRM*** transforms. The general transform matrix $\overset{\frown}{Q}_n$ connecting ***RPH*** and ***TRM*** transforms is given by

$$\overset{\frown}{Q}_n = \overset{n-1}{\otimes} \overset{\frown}{Q}_1 = \overset{n-1}{\otimes} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}. \tag{13}$$

The general inverse transform matrix of $\overset{\frown}{Q}_n$ is given by

$$\overset{\frown}{Q}_n^{-1} = \overset{n-1}{\otimes} \overset{\frown}{Q}_1^{-1} = \overset{n-1}{\otimes} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix}. \tag{14}$$

$\overrightarrow{A_{RPH}}$ spectra can be expressed by $\overrightarrow{A_{TRM}}$ spectra as shown below:

$$\overrightarrow{A_{RPH}} = \overset{\frown}{Q}_n^{-1} \times \overrightarrow{A_{TRM}}. \tag{15}$$

In our previous article [5], the permutation properties of the four types of helix transforms are presented. These properties also exist in the relations between helix transform and ***TRM*** transform. Due to their properties, the transform matrix $\breve{Q}_n$ connecting ***LNH*** and ***TRM*** transforms can be derived by horizontal permutation of all the elements inside the matrix Q_n. For order N,

$$\breve{Q}_n = \overset{n-1}{\otimes} \breve{Q}_1 = \overset{n-1}{\otimes} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \tag{16}$$

The inverse transform matrix $\breve{Q}_n$ is obtained by vertical permutation on all the elements inside the inverse transform matrix Q_n^{-1}.

$$\breve{Q}_n^{-1} = \overset{n-1}{\otimes} \breve{Q}_1^{-1} = \overset{n-1}{\otimes} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{17}$$

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The connecting transform matrix $\tilde{Q}_n$ between ***RNH*** and ***TRM*** transforms is derived from $\widehat{Q}_n$ by horizontal permutation on all the elements inside the matrix.

$$\tilde{Q}_n = \overset{n-1}{\otimes}\, \tilde{Q}_1 = \overset{n-1}{\otimes} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}. \tag{18}$$

Similarly, the inverse transform matrix $\tilde{Q}_n^{-1}$ is derived from $\widehat{Q}_n^{-1}$ by horizontal permutation on all the elements inside the matrix.

$$\tilde{Q}_n^{-1} = \overset{n-1}{\otimes}\, \tilde{Q}_1^{-1} = \overset{n-1}{\otimes} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}. \tag{19}$$

Fig. 1 presents the circuit realizations of the transform matrix Q_n for $n = 1$ and $n = 2$ using GF(3) operations.

4 Conclusion

The detailed analysis of relations between ternary helix and Reed-Muller transforms is shown in this article. The presented properties and relationships give an efficient method to calculate the corresponding spectral polynomial expansions of any ternary logic function directly between ternary helix and Reed-Muller transforms. The whole transfer between both transforms and their spectra can be implemented in hardware by using basic operators over GF (3). The presented results will be very useful, especially for ternary functions with big number of variables. In addition, all the properties presented here have simple procedures and can be implemented not only in hardware but also in software by using parallel programming.

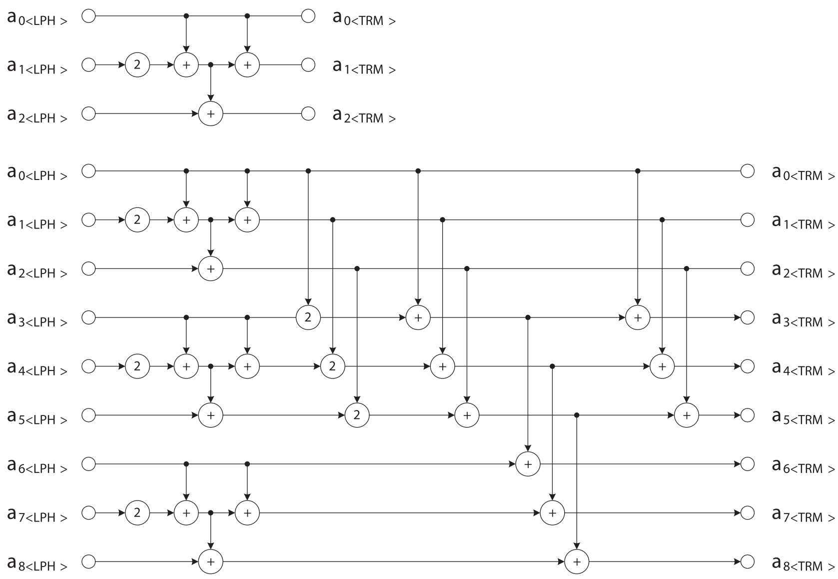

Fig. 1. Circuit realizations of Q_n for $n = 1$ and $n = 2$.

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