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Estimation for fractal signals based on dyadic wavelet

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Abstract: Using Markov processes, the representation of fractal signals based on the dyadic wavelet transform (DYWT) is established. Then a new method of the estimation for fractal signals embedded a white noise is proposed. The numerical comparisons with previous method are shown.

Keywords: dyadic wavelet, fractal signal, $1/f$ processes, Markov processes

Classification: Science and engineering for electronics

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1 Introduction

Fractal signals are a class of nonstationary processes that have power spectrum, so also known as $1/f$ signals or $1/f$ processes. A $1/f$ process is used to describe many physical phenomena, such as variation in temperature, traffic flow, electronic device noises. A well known model for these processes is fractal Brownian motion (fBm) proposed by Mandelbrot and van Nees [1].

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Many research works focused on the generation of $1/f$ signals and the estimation for $1/f$ signals in presence of noise, where wavelet transform, as an adequate tool, was naturally used. Wornell [2] first presented an orthogonal wavelet basis expansion for $1/f$ processes in terms of a collection of uncorrelated random variables. Using such representation, algorithm for obtaining Bayesian minimum mean-square error signal estimation for $1/f$ processes is derived in [3] (WO algorithm). WO algorithm is based on the hypotheses that the orthogonal wavelet transformation is a whitening filter for $1/f$ signals and the approximation term of the wavelet expansion can be avoid when the number of scales in the mutiriresolution analysis is large enough. So the solution of the problem is simplified. To improve WO algorithm, Hirchoren considered the effect of the correlation of wavelet coefficients of fBm in each scales and design a bank of Wiener filters [4] and Kalman filters [5] using orthogonal wavelet transformation. However, the results obtained depend on the fBm model. In this paper, we consider the representation of $1/f$ signals based on DYWT that is of special interest in many applications such as transient detection. We model the sequences of dyadic wavelet coefficients in each scale as Markov processes, and give the approximated expressions of the autocorrelation function of dyadic wavelet coefficients that are not dependent on fBm model. This modeling is effective and very accurate. Numerical experimental results indicate that the signal generated by our method is more close to $1/f$ signals. On this basis, a new signal estimation algorithm for $1/f$-type signal embedded in white noise is presented.

2 Representation of $1/f$-type signals

A $1/f$ signal is a statistically-similar random process having power spectrum obeying a power law relationship of the form

$$S_f(\omega) \propto \frac{1}{|\omega|^\gamma} \tag{1}$$

where γ is called as spectrum parameter. Our aim is to establish the approximated representation of a $1/f$ signal by starting from a collection of zero-mean, second-order random processes. In the following discussions, we suppose that $w(m,b)(m \in Z)$ is a Markov process in b for any fixed m, and satisfies, for some $\gamma > 0$, $\sigma^2 > 0$ and $\rho > 0$

$$R_w(m,\tau) = E[w(m,b)w(m,b-\tau)] = \sigma^2 2^{\gamma m} e^{-2^{-m}\rho|\tau|}, \quad \forall m \in Z \tag{2}$$

$$E[w(m_1,b)w(m_2,b)] = 0, \quad m_1 \neq m_2, \quad \forall m_1, m_2 \in Z \tag{3}$$

Using $w(m,b)$, we define a new random process as

$$f(t) = \sum_{m=-\infty}^{\infty} \int_R 2^{-m} w(m,b)\psi_{m,b}(t)db, \quad t \in R \tag{4}$$

where $\psi_{m,b}(t) = 2^{-m/2}\psi(2^{-m}(t-b))$, $\psi(t)$ is a dyadic wavelet. Then we have the following results.

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Theorem 1 *Let $\psi(t)$ be a real dyadic wavelet with N-th order regularity for $N > \gamma/2 - 1$ and $\gamma > 0$. Then the power spectrum of $f(t)$ defined by Eq. (4) is*

$$S_f(\omega) = \sum_{m=-\infty}^{\infty} 2^{\gamma m} G(2^m\omega) | \hat{\psi} (2^m\omega)|^2 \tag{5}$$

and satisfies the following inequality

$$\frac{\sigma_L^2}{|\omega|^\gamma} \le S_f(\omega) \le \frac{\sigma_M^2}{|\omega|^\gamma} \tag{6}$$

for some $0 < \sigma_L^2 \le \sigma_M^2$, where $G(2^m\omega) = 2\sigma^2\rho/(\rho^2 + 2^{2m}\omega^2)$, and $\hat{\psi}(\omega)$ denotes the Fourier transform of the wavelet $\psi(t)$.

The proof of the Theorem 1 sees Appendix. The inequality (6) indicates that the expression (4) is a approximated representation of a $1/f$ signal with the parameter γ.

Table I gives the numerical experimental results corresponding to the spectral parameter $\gamma = 1.0, 1.5, 2.0, 2.5, 3$, and $\rho = 0.5$. The approximation error is denoted by the ratio σ_M^2/σ_L^2 where $\sigma_L^2 = \inf_\omega(|\omega|^\gamma S_f(\omega))$ and $\sigma_M^2 = \sup_\omega(|\omega|^\gamma S_f(\omega))$. E_{wor} and E_{our} denotes the approximation error resulted from the Wornell's method [2] and our method respectively. In our experiments, Daubechies 2th-order (Db2) wavelet and biorthogonal wavelet Bi9/7 are adopted. But we do not compute the error E_{wor} with Bi9/7 due to the supposition of the orthogonality for wavelet in Wornell's method. We can see that the approximation errors E_{our} are always lower than E_{wor} except the case $\gamma = 1$. When γ tends to 3, E_{our} is much less than E_{wor}. In addition, the approximation errors E_{our} very close to 1, which means that the modeling of wavelet coefficients by Markov processes is accurate.

Table I. Approximation errors

γ		1.0	1.5	2.0	2.5	3.0
Db2	E_{wor}	1.0500	1.1285	1.4125	2.3268	5.5885
Db2	E_{our}	1.0342	1.0364	1.0378	1.0387	1.0502
Bi9/7	E_{our}	1.1160	1.1001	1.0781	1.0490	1.0441

3 Signal estimation problem

Consider a received signal

$$r(t) = f(t) + v(t), \quad -\infty < t < \infty \tag{7}$$

where $f(t)$ is a zero-mean $1/f$ signal embedded in an additive stationary white Gaussian noise $v(t)$ with zero-mean and variance σ_v^2, and it is mutually independent with $v(t)$.

We define the DYWT of any $x(t) \in L^2(R)$ as

$$W_x(m, b) = 2^{-m} \int_R x(t)\psi_{m,b}(t)dt$$

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Then, applying the DYWT to Eq. (7), we can get the DYWT of $r(t)$

$$W_r(m,b) = W_f(m,b) + W_v(m,b) \tag{8}$$

where $W_f(m,b)$ and $W_v(m,b)$ are the DYWT of $f(t)$ and $v(t)$, respectively. For the fixed m, we model $W_v(m,b)$ as a white noise with variance σ_v^2, and $W_f(m,b)$ as a Markov process that satisfies Eq. (2) and Eq. (3). Based on this, It follows immediately using classical estimation theory that the estimation of $W_f(m,b)$ that minimizes the mean-square estimation error is given by

$$\hat{W}(m,b) = A_m W_r(m,b) + B_m W_r(m,b_1), \quad b_1 < b \tag{9}$$

$$A_m = \frac{K(m) - L(m)}{K^2(m) - L(m)}, \quad B_m = \frac{[K(m)-1](L(m))^{1/2}}{K^2(m) - L(m)}$$

where $K(m) = 1 + \sigma_v^2/(2^{\gamma m}\sigma^2)$ and $L(m) = e^{-2^{-m+1}\rho|b-b_1|}$. By this estimation, the optimal estimation of the $1/f$ signal $f(t)$ can be expressed as

$$\hat{f}(t) = \sum_{m=-\infty}^{\infty} 2^{-m} \int_R \hat{W}_f(m,b)\, \tilde{\psi}_{m,b}(t) db, \quad t \in R \tag{10}$$

where $\tilde{\psi}(t)$ is the reconstruction wavelet corresponding to $\psi(t)$.

In practice, we can only obtain a segment of observed data that is both time-limited and resolution-limited. In this case, we suppose the sampling interval in t is 1. Then Eq. (7) can be rewritten as

$$r(n) = f(n) + v(n), \quad n = 1, 2, 3, \cdots, 2^N \tag{11}$$

So, we can take $|b - b_1|$ to be 1. The dyadic wavelet representation of $r(n)$ is defined as the set of wavelet coefficients up to a scale $2^M (M \leq N)$ plus the remaining low-frequency information $a_M(n)$:

$$\{W_r(m,n), a_M(n) \quad |m \leq M, 1 \leq n \leq 2^N\}$$

that can be calculated by a filter bank algorithm [6] called the *algorithme á trous*.

4 Estimation of the parameters σ^2 and ρ

We consider the estimation of the parameters σ^2 and ρ with observed data when the parameters γ and σ_v^2 are known. The general problem of estimating the parameter of a Gaussian $1/f$ signal was discussed in [3] where the coefficients of DWT for $1/f$ signal was modeling as mutually independent zero-mean, Gaussian random variables.

Suppose that the DYWT $W_f(m,n)$ of $f(n)$ in Eq. (11) satisfies Eq. (2) and Eq. (3). We take τ to be 0 and 1 in Eq. (2), respectively. Then

$$var(W_f(m,n)) = \sigma^2 2^{\gamma m}, \quad R_{W_f}(m,1) = \sigma^2 2^{\gamma m} e^{-2^{-m}\rho}$$

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where $R_{W_f}(m,\tau)$ denotes the autocorrelation function of $W_f(m,n)$. By the above equations, we can derive the estimation of σ^2 and ρ with observed data $r(n)$,

$$\begin{cases} \hat{\sigma}^2 = \frac{1}{M2^N} \sum_{m=1}^{M} 2^{-\gamma m} \sum_{n=1}^{2^N} [W_r^2(m,n) - \sigma_v^2] \\ \hat{\rho} = \frac{1}{M} \sum_{m=1}^{M} 2^m [ln(\sigma^2 2^{\gamma m}) - ln(R_{W_r}(m,1))] \end{cases} \tag{12}$$

where R_{W_r} is the autocorrelation function of $W_r(m,n)$

5 Experiment

In this section, we will demonstrate the performance of the signal estimation. The $1/f$ signal $f(n)$ with a length of 2048 is generated from the Wavelab802 (http://WWW-stat.stanford.edu./ wavelab). To show the efficiency of our algorithm, we assume that γ is given *a priori* and $f(n)$ is embedded in additive Gaussian white noise with the variance $\sigma_v^2 = 4$.

Table II gives the mean-square errors by our estimation method with Daubechies 3-th wavelet (Db3) and Bi9/7, denoted by MSE_{our}, as well as those obtained by the WO algorithm with Db3, denoted by MSE_{wo}. We used a scale from 2^1 - 2^{11} ($M = 11$) in both cases. The estimation errors shown are the results of the averaged error from 64 realizations of noisy data. The parameters σ^2 and ρ are estimated by Eq. (12). As seen in Table II, the errors resulted from our method are much less than those from WO algorithm for $\gamma > 2$. In the other case of $\gamma \leq 2$, our method has slightly lower mean-square errors, which also means that the WO algorithm is indeed a simple and good method for fractal signal estimation. From these numerical results, we can conclude that Markov random field is a good modeling for wavelet coefficients. We also note that it is necessary to further improve our signal estimation method when $\gamma \leq 2$.

Table II. Approximation errors

γ		2.9	2.7	2.5	2.0	1.9
Db3	MSE_{wo}	0.7353	0.5736	1.0217	3.0833	12.0792
Db3	MSE_{our}	0.3089	0.3598	0.6205	2.4089	10.0670
Bi9/7	MSE_{our}	0.3123	0.3636	0.6256	2.4228	10.1097

6 Conclusion

The dyadic wavelet transform provides a translation-invariant wavelet representation of $1/f$ signals that is very useful in many applications such as physiological and computer vision studies. In this paper, we have studied the representation of these signals and signal estimation based on Markov random fields. Our next research works is going to develop the parameters estimation for $1/f$ signals with our wavelet model.

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Appendix: The proof of theorem 1

By the definition of $f(t)$, for any integer $p = 2^n, n \in Z$,

$$E[f(pt_1)f(pt_2)] = \sum_{m=-\infty}^{\infty} 2^{-3m} \int_{R^2} R_w(m, b-c)\psi_{m,b}(pt_1)\psi_{m,c}(pt_2)dbdc$$

Applying the changing of variables with $b = px$, $c = py$, we have $E[f(pt_1)$ $f(pt_2)] = p^{\gamma-1}E[f(t_1)f(t_2)]$. Therefore $f(t)$ is a statistically self-similar process.

By Eq. (2), the power spectrum of $f(t)$ is,

$$\begin{aligned}
S_f(\omega) &= \int_{R^3} R_f(t,\tau)e^{-j\omega\tau}d\tau \\
&= \sum_{m=-\infty}^{\infty} \int_{R^3} \sigma^2 2^{\gamma m-2m} e^{-2^{-m}\rho|b-c|}\psi_{m,b}(t)\psi_{m,c}(t-\tau)e^{-j\omega\tau}dbdcd\tau \\
&= \sum_{m=-\infty}^{\infty} \int_{R^2} \sigma^2 2^{\gamma m-3m/2} e^{-2^{-m}\rho|b-c|}\psi_{m,b}(t) \overline{\hat{\psi}} (2^m\omega)e^{-j\omega(t-c)}dbdc \\
&= \sum_{m=-\infty}^{\infty} \int_{R} \frac{2^{(\gamma-1/2)m+1}\sigma^2\rho}{\rho^2+2^{2m}\omega^2}\psi_{m,b}(t) \overline{\hat{\psi}} (2^m\omega)e^{-j\omega(t-b)}db \\
&= \sum_{m=-\infty}^{\infty} 2^{\gamma m}\frac{2\sigma^2\rho}{\rho^2+2^{2m}\omega^2}| \hat{\psi} (2^m\omega)|^2
\end{aligned}$$

So Eq. (5) holds.

Now we prove the inequality (6). Noting that there exist a integer k and real number $1 \leq \omega_0 \leq 2$ such that $\omega = 2^k\omega_0$, we conclude that

$$S_f(\omega) = \frac{\omega_0^\gamma}{(2^k\omega_0)^\gamma} \sum_{m=-\infty}^{\infty} 2^{\gamma m}G(2^m\omega_0)| \hat{\psi} (2^m\omega_0)|^2 \tag{13}$$

Denote the series on the right side of the above equation by $U(\omega_0)$. We need to show the convergence of this series. Since $\psi(t)$ has Nth-order regularity, $\hat{\psi} (\omega)$ decays at least as fast as $1/|\omega|^N$ as $|\omega| \longrightarrow \infty$. This implies that for any $\omega_0 \in [1,2]$, $| \hat{\psi} (2^m\omega_0)| \leq M/(1+|2^m\omega_0|^N)$, where $M > 1$. On the other hand, it is easily verified that $|G(2^m\omega_0)| \leq C2^{-2m}$ for some C when $m > 0$. Hence, we have the following estimation when $\gamma > 0$ and $N > \gamma/2 - 1$,

$$0 \leq U(\omega_0) \leq \sum_{m=-\infty}^{-1} 2^{\gamma m+1}\sigma^2M^2 + \sum_{m=0}^{\infty} 2^{\gamma m-2m(N+1)}CM^2\omega_0^{-2N} < \infty$$

This implies that $U(\omega_0)$ is uniformly convergent and continuous on $[1,2]$. Moreover, $\min\limits_{1\leq\omega_0\leq2} U(\omega_0) > 0$. To show this, it suffices to show $U(\omega_0) > 0$ for any $\omega_0 \in [1,2]$. Suppose that there exists $\tilde{\omega}_0 \in [1,2]$ such that $U(\tilde{\omega}_0) = 0$, i.e., $S_f(\tilde{\omega}_0) = 0$. Since $G(2^m \tilde{\omega}_0) > 0$, so we have that $\sum\limits_{m=-\infty}^{\infty} | \hat{\psi} (2^m \tilde{\omega}_0) | = 0$. However, this contradicts the supposition that ψ is a dyadic wavelet. Therefore, we must have that $\min\limits_{1\leq\omega_0\leq2} U(\omega_0) > 0$. Using Eq. (13), it is easy to show the inequality (6). The proof is completed.

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