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Newton-type method in spectrum estimaion-based AOA estimation

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Abstract: For numerical optimization of the cost function of the maximum likelihood (ML) algorithm for angle-of-arrival (AOA) estimation, Newton-type method has been widely employed. In this paper, we apply the Newton-type method to the optimization of cost function for spectrum estimation-based AOA estimation algorithm of the conventional beamforming, the Capon beamfomring and the MUSIC algorithm. Explicit expressions of the first derivatives and the second derivatives of the cost functions are presented. The expressions are used for the Newton iteration to improve the accuracy of the initial estimates. The performance improvement in terms of estimation accuracy and computational burden is demonstrated using the Monte-Carlo simulations.

Keywords: conventional beamforming algorithm, Capon beamforming algorithm, MUSIC algorithm, Newton method, AOA (angle-of-arrival)

Classification: Electromagnetic theory

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1 Introduction

Determination of the AOA (angle-of-arrival) of signal has been of interest to the signal processing community [1].

The spectrum estimation methods like the conventional beamforming, the Capon beamforming and the MUSIC algorithm have p-dimensional cost function where p is the number of parameters to be estimated per signal [2]. Therefore, to implement the optimization of the p-dimensional cost function, we have to perform p-dimensional search. Note that, for spectrum estimation-based algorithm, the search dimension is independent of the number of incident signals, N. It is only dependent on the number of parameters to be estimated for each incident signal, p.

For the maximum likelihood (ML) algorithm the dimension of the search space is equal to Np, where N is the number of the incident signals [2]. Therefore, the computational burden of the ML algorithm is much larger than that of the spectrum estimation-based algorithm.

The search can be exhaustive grid search or Newton-type search. For the ML algorithm, it is very impractical to apply the exhaustive grid search for optimization of Np-dimensional cost function, which is why there have been many studies on the application of the Newton-type search to the nonlinear optimization of the ML cost function [2, 3, 4]. The compact expressions for the gradient, the Hessian and the approximate Hessian corresponding to each ML algorithm for use with the Newton-type search have been derived [2, 3, 4].

We consider the case of $p = 1$, where the cost function for the beamformng and the MUSIC is one-dimensional. We can easily extend the formulation presented in this paper to the case of $p = 2$ or $p = 3$.

In this paper, we consider the application of the Newton-search for the optimization of the beamforming algorithm and the MUSIC algorithm. We call the estimates from the exhaustive grid search the initial estimates. The final estimates refer to the estimates obtained by applying the Newton-search to the initial estimates. In this paper, to get the initial estimates, we use the exhaustive search, but there are many other ways to get the initial estimates like the alternating projection [5] or iterative quadratic maximum likelihood [3].

2 Proposed algorithm

Assume that there are N incident signals and that the array antenna consists of M antennas. The cost function of the spectrum-estimation-based algorithms like the conventional beamforming, the Capon beamforming and the MUSIC algorithm for azimuth estimation and the simultaneous estimation of azimuth/elevation can be written as

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$$P(\theta)=\sum_{k=1}^{M}\sum_{i=1}^{M}a_i^*(\theta)C_{ik}a_k(\theta) \qquad \text{azimuth} \tag{1}$$

$$P(\theta,\phi)=\sum_{k=1}^{M}\sum_{i=1}^{M}a_i^*(\theta,\phi)C_{ik}a_k(\theta,\phi) \qquad \text{azimuth and elevation} \tag{2}$$

where array manifolds can be explicitly writen as [1]

$$\mathbf{a}(\theta)=[\exp(j\psi_1(\theta))\ \exp(j\psi_2(\theta))\ \cdots\ \exp(j\psi_M(\theta))]^{\mathrm{T}} \tag{3}$$

azimuth

$$\mathbf{a}(\theta,\phi)=[\exp(j\psi_1(\theta,\phi))\ \exp(j\psi_2(\theta,\phi))\ \cdots\ \exp(j\psi_M(\theta,\phi))]^{\mathrm{T}}. \tag{4}$$

azimuth and elevation

$\psi_m(\theta)$ and $\psi_m(\theta,\phi)$ is a phase response of the m-th antenna element for azimuth estimation and azimuth/elevation estimation, respectively. C_{ik} in (1) or (2) for each algorithm is [1]

$$C_{ik}=\hat{R}_{ik} \qquad \text{Conventional beamforming} \tag{5}$$

$$C_{ik}=\hat{R}_{ik}^{-1} \qquad \text{Capon beamforming} \tag{6}$$

$$C_{ik}=\mathbf{U}_N\mathbf{U}_N^H \qquad \text{MUSIC} \tag{7}$$

where $\hat{R}_{ik}$ is an entry of the estimate of the covariance matrix, $\hat{\mathbf{R}}$, and $\mathbf{U}_N$ is a matrix whose columns consist of the noise eigenvectors of $\hat{\mathbf{R}}$. Note that, we have to maximize (1) or (2) in the conventional beamforming algorithm, and have to minimize (1) or (2) in the Capon beamforming algorithm and the MUSIC algorithm.

2.1 Estimation of azimuth

The Newton-iteration for estimation of the azimuth angle using the spectrum estimation-based algorithm is, for $i=0,1,\cdots$, and $n=1,2,\cdots,N$,

$$\hat{\theta}_n^{(i+1)}=\hat{\theta}_n^{(i)}-\frac{\frac{dP(\hat{\theta}_n^{(i)})}{d\theta}}{\frac{d}{d\theta}\left(\frac{dP(\hat{\theta}_n^{(i)})}{d\theta}\right)}. \tag{8}$$

The first differentiation and the second differentiation of the cost function in (8) with respect to the azimuth angle are

$$\frac{\partial P(\theta)}{\partial\theta}=\sum_{k=1}^{M}\sum_{i=1}^{M}\left(\frac{\partial a_k(\theta)}{\partial\theta}\frac{\partial P(\theta)}{\partial a_k(\theta)}+\frac{\partial a_i^*(\theta)}{\partial\theta}\frac{\partial P(\theta)}{\partial a_i^*(\theta)}\right) \tag{9}$$

$$=\sum_{k=1}^{M}\sum_{i=1}^{M}\left(ja_i^*(\theta)C_{ik}a_k(\theta)\left(\frac{\partial\psi_k(\theta)}{\partial\theta}-\frac{\partial\psi_i(\theta)}{\partial\theta}\right)\right) \tag{10}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial P(\theta)}{\partial\theta}\right)=\sum_{k=1}^{M}\sum_{i=1}^{M}\frac{\partial}{\partial\theta}\left(\frac{\partial a_k(\theta)}{\partial\theta}\frac{\partial P(\theta)}{\partial a_k(\theta)}+\frac{\partial a_i^*(\theta)}{\partial\theta}\frac{\partial P(\theta)}{\partial a_i^*(\theta)}\right) \tag{11}$$

$$=\sum_{k=1}^{M}\sum_{i=1}^{M}a_i^*C_{ik}a_k\left(2\frac{\partial\psi_k(\theta)}{\partial\theta}\frac{\partial\psi_i(\theta)}{\partial\theta}\right.$$
$$\left.+j(\psi_i(\theta)-\psi_k(\theta))-\left(\frac{\partial\psi_k(\theta)}{\partial\theta}\right)^2-\left(\frac{\partial\psi_i(\theta)}{\partial\theta}\right)^2\right) \tag{12}$$

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where we used the chain-rule, and $\frac{\partial a_i(\theta)}{\partial \psi_i(\theta)} = j a_i(\theta)$, $\frac{\partial a_i^*(\theta)}{\partial \psi_i(\theta)} = -j a_i^*(\theta)$ and $\frac{\partial^2 \psi_i(\theta)}{\partial \theta^2} = -\psi_i(\theta)$.

In the conventional beamforming algorithm, since the final estimates of the Newton iteration should correspond to the local maxima of the cost function in (1), the second derivatives evaluated at the final estimates should be negative. On the contrary, in the Capon beamforming algorithm and the MUSIC algorithm, since the final estimates should correspond to the local minima of the cost function in (1) or (2), the second derivatives evaluated at the final estimates should be positive.

2.2 Estimation of azimuth and elevation

The Newton iteration for simultaneous estimation of the azimuth angle and the elevation angle is, for $i = 0, 1, \cdots$, and $n = 1, 2, \cdots, N$,

$$\begin{bmatrix} \hat{\theta}_n^{(i+1)} \\ \hat{\phi}_n^{(i+1)} \end{bmatrix} = \begin{bmatrix} \hat{\theta}_n^{(i)} \\ \hat{\phi}_n^{(i)} \end{bmatrix} - \left[\begin{matrix} \frac{\partial}{\partial \theta}\left(\frac{\partial P(\theta,\phi)}{\partial \theta}\right) \frac{\partial}{\partial \phi}\left(\frac{\partial P(\theta,\phi)}{\partial \theta}\right) \\ \frac{\partial}{\partial \theta}\left(\frac{\partial P(\theta,\phi)}{\partial \phi}\right) \frac{\partial}{\partial \phi}\left(\frac{\partial P(\theta,\phi)}{\partial \phi}\right) \end{matrix} \right]^{-1} \Bigg|_{\substack{\theta=\hat{\theta}_n^{(i)} \\ \phi=\hat{\phi}_n^{(i)}}} \begin{bmatrix} \frac{\partial P(\theta,\phi)}{\partial \theta} \\ \frac{\partial P(\theta,\phi)}{\partial \phi} \end{bmatrix} \Bigg|_{\substack{\theta=\hat{\theta}_n^{(i)} \\ \phi=\hat{\phi}_n^{(i)}}} . \tag{13}$$

Note that, for notational brevity, in the case of the estimation of the azimuth and the elevation, we suppress the argument in (15)–(24) as follows:

$$P(\theta, \phi) \equiv P, a_i(\theta, \phi) \equiv a_i, \psi_i(\theta, \phi) \equiv \psi_i. \tag{14}$$

The first differentiations are

$$\frac{\partial P}{\partial \theta} = \sum_{k=1}^{M} \sum_{i=1}^{M} \left(\frac{\partial a_k}{\partial \theta} \frac{\partial P}{\partial a_k} + \frac{\partial a_i^*}{\partial \theta} \frac{\partial P}{\partial a_i^*} \right) \tag{15}$$

$$= \sum_{k=1}^{M} \sum_{i=1}^{M} \left(j a_i^* C_{ik} a_k \left(\frac{\partial \psi_k}{\partial \theta} - \frac{\partial \psi_i}{\partial \theta} \right) \right) \tag{16}$$

$$\frac{\partial P}{\partial \phi} = \sum_{k=1}^{M} \sum_{i=1}^{M} \left(\frac{\partial a_k}{\partial \phi} \frac{\partial P}{\partial a_k} + \frac{\partial a_i^*}{\partial \phi} \frac{\partial P}{\partial a_i^*} \right) \tag{17}$$

$$= \sum_{k=1}^{M} \sum_{i=1}^{M} \left(j a_i^* C_{ik} a_k \left(\frac{\partial \psi_k}{\partial \phi} - \frac{\partial \psi_i}{\partial \phi} \right) \right) \tag{18}$$

The second differentiations are

$$\frac{\partial}{\partial \theta}\left(\frac{\partial P}{\partial \theta}\right) = \sum_{k=1}^{M} \sum_{i=1}^{M} \frac{\partial}{\partial \theta} \left(\frac{\partial a_k}{\partial \theta} \frac{\partial P}{\partial a_k} + \frac{\partial a_i^*}{\partial \theta} \frac{\partial P}{\partial a_i^*} \right) \tag{19}$$

$$= \sum_{k=1}^{M} \sum_{i=1}^{M} a_i^* C_{ik} a_k \left(2 \frac{\partial \psi_k}{\partial \theta} \frac{\partial \psi_i}{\partial \theta} + j(\psi_i - \psi_k) - \left(\frac{\partial \psi_k}{\partial \theta}\right)^2 - \left(\frac{\partial \psi_i}{\partial \theta}\right)^2 \right) \tag{20}$$

$$\frac{\partial}{\partial \phi}\left(\frac{\partial P}{\partial \theta}\right) = \sum_{k=1}^{M} \sum_{i=1}^{M} \frac{\partial}{\partial \phi} \left(\frac{\partial a_k}{\partial \theta} \frac{\partial P}{\partial a_k} + \frac{\partial a_i^*}{\partial \theta} \frac{\partial P}{\partial a_i^*} \right) \tag{21}$$

$$= \sum_{k=1}^{M} \sum_{i=1}^{M} a_i^* C_{ik} a_k \left(\frac{\partial \psi_i}{\partial \phi} \frac{\partial \psi_k}{\partial \theta} + \frac{\partial \psi_k}{\partial \phi} \frac{\partial \psi_i}{\partial \theta} \right.$$

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$$+j\left(\frac{\partial}{\partial\phi}\left(\frac{\partial\psi_k}{\partial\theta}\right)-\frac{\partial}{\partial\phi}\left(\frac{\partial\psi_i}{\partial\theta}\right)\right)-\frac{\partial\psi_k}{\partial\phi}\frac{\partial\psi_k}{\partial\theta}-\frac{\partial\psi_i}{\partial\phi}\frac{\partial\psi_i}{\partial\theta}\Bigg) \tag{22}$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial P}{\partial\phi}\right)=\sum_{k=1}^{M}\sum_{i=1}^{M}\frac{\partial}{\partial\phi}\left(\frac{\partial a_k}{\partial\phi}\frac{\partial P}{\partial a_k}+\frac{\partial a_i^*}{\partial\phi}\frac{\partial P}{\partial a_i^*}\right) \tag{23}$$

$$=\sum_{k=1}^{M}\sum_{i=1}^{M}a_i^*C_{ik}a_k\Bigg(2\frac{\partial\psi_k}{\partial\phi}\frac{\partial\psi_i}{\partial\phi}$$
$$+j\left(\psi_i-\psi_k\right)-\left(\frac{\partial\psi_k}{\partial\phi}\right)^2-\left(\frac{\partial\psi_i}{\partial\phi}\right)^2\Bigg). \tag{24}$$

Note that $\frac{\partial}{\partial\theta}\left(\frac{\partial P}{\partial\phi}\right)$ in (13) can be obtained from $\frac{\partial}{\partial\theta}\left(\frac{\partial P}{\partial\phi}\right)=\frac{\partial}{\partial\phi}\left(\frac{\partial P}{\partial\theta}\right)$. In the conventional beamforming algorithm, since the final estimates of the Newton iteration should correspond to the local maxima of the cost function in (2), the Jacobian matrix evaluated at the final estimates should be negative definite. On the contrary, in the Capon beamforming algorithm and the MUSIC algorithm, the final estimates should correspond to the local minima of the cost function in (2), the Jacobian matrix evaluated at the final estimates should be positive definite.

3 Numerical results

3.1 Estimation of azimuth

We compute the cost function of each algorithm at

$$\left\{\theta_{\text{start}},\theta_{\text{start}}+\Delta\theta,\theta_{\text{start}}+2\Delta\theta,\cdots\theta_{\text{start}}+\left\lfloor\frac{\theta_{\text{stop}}-\theta_{\text{start}}}{\Delta\theta}\right\rfloor\Delta\theta\right\} \tag{25}$$

where $\lfloor\rfloor$ rounds the argument toward zero. θ_{start} and θ_{stop} specify a search range of the angle, $\Delta\theta$ is a search step. This corresponds to the exhaustive grid search with search step $\Delta\theta$.

The array manifold for the ULA is, for $m=1,\cdots,M$,

$$\psi_m(\theta)=(m-1)\,\pi\sin\theta. \tag{26}$$

The ULA is used and the number of the antenna element, M, is chosen to be five. $\theta^{(\text{start})}$ and $\theta^{(\text{stop})}$ are selected to be $\theta^{(\text{start})}=-80°$ and $\theta^{(\text{start})}=80°$.

The iterations in (8) is terminated when the criterion is met:

$$|\hat{\theta}^{(i+1)}-\hat{\theta}^{(i)}|<10^{-5}. \tag{27}$$

The root mean square error (RMSE) and operation time in Fig. 1 are obtained from the 1000 repetitions. The search steps, $\Delta\theta$, in (25) are chosen to be 4.7°. We investigate the RMSE and the execution time. We consider the case that there are two incident signals, which implies that N is equal to two.

The results with legend $'\mathrm{CA}'$ and $'\mathrm{MU}'$ refer to the initial estimates for the Capon beamforming and the MUSIC algorithm. The results with legend $'\mathrm{CA}+\mathrm{NT}'$, and $'\mathrm{MU}+\mathrm{NT}'$ refer to the final estimates for Capon beamforming algorithm and the MUSIC algorithm, respectively.

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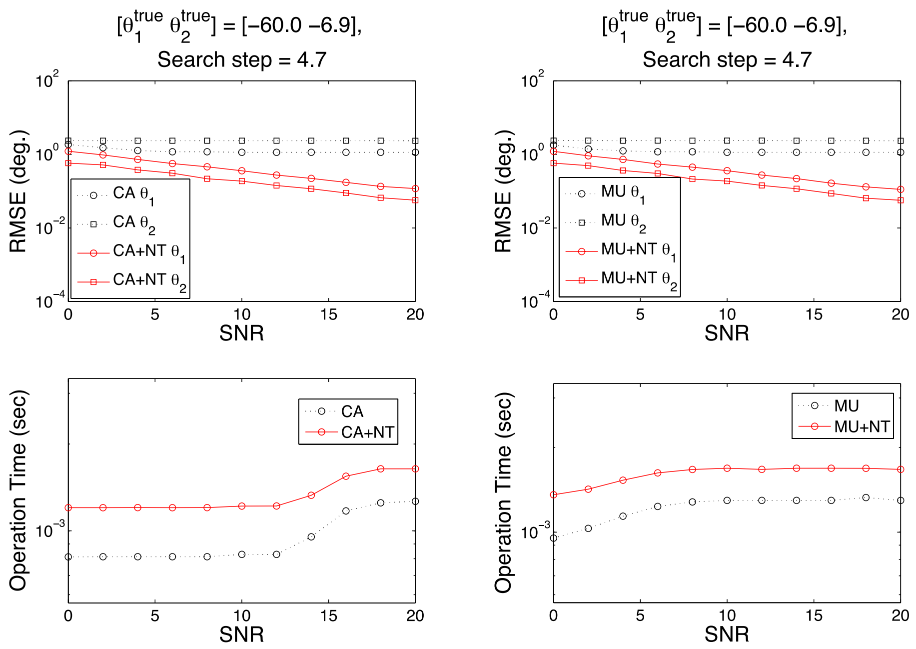

(a) Capon beamforming for ULA (b) MUSIC for ULA

Fig. 1. The initial estimates with $\Delta\theta = 4.7°$ and the final estimates ($[\theta_1^{\text{true}}\ \theta_2^{\text{true}}] = [-60°\ -6.9°]$)

Fig. 1 shows how the Newton-type search improves the accuracy of the initial estimates. The results with $[\theta_1^{\text{true}}\ \ \theta_2^{\text{true}}] = [-60°\ \ -6.9°]$ and with the search step of $\Delta\theta = 4.7°$ for various SNR's are shown in Fig. 1.

The final estimates are superior to the initial estimates in terms of the RMSE, but getting the final estimates takes more time than getting the initial estimates, which can be seen in the second figure of each case because the final estimates are obtained by applying the Newton -type search to the initial estimates.

3.2 Estimation of azimuth and elevation

Two-dimensional exhaustive grid search with search steps of $\Delta\theta$ and $\Delta\phi$ is performed.

The iteration for the n-th incident signal stops if both of the following criteria are met:

$$\begin{aligned} \left|\hat{\theta}_n^{(i+1)} - \hat{\theta}_n^{(i)}\right| &< \text{tolerance} \\ \left|\hat{\phi}_n^{(i+1)} - \hat{\phi}_n^{(i)}\right| &< \text{tolerance}. \end{aligned} \tag{28}$$

The array manifold for the UCA is, for $m = 1, \cdots, M$,

$$\psi_m(\theta,\phi) = 2\pi\frac{r}{\lambda}\cos\phi\cos\left(\theta - \frac{2\pi\,(m-1)}{M}\right). \tag{29}$$

We will show the performance of the proposed scheme in this section. The uniform circular array (UCA) with five antenna elements is considered. $[\theta^{\text{start}}\ \ \theta^{\text{stop}}] = [-180°\ \ 180°]$ and $[\theta^{\text{start}}\ \ \theta^{\text{stop}}] = [0°\ \ 90°]$ are employed for the numerical simulation to obtain the initial estimates of the Capon beamforming algorithm and the MUSIC algorithm. In this simulation, $\Delta\theta = \Delta\phi = 8.8°$ is adopted. The number of snapshots used for the calculation of the estimate

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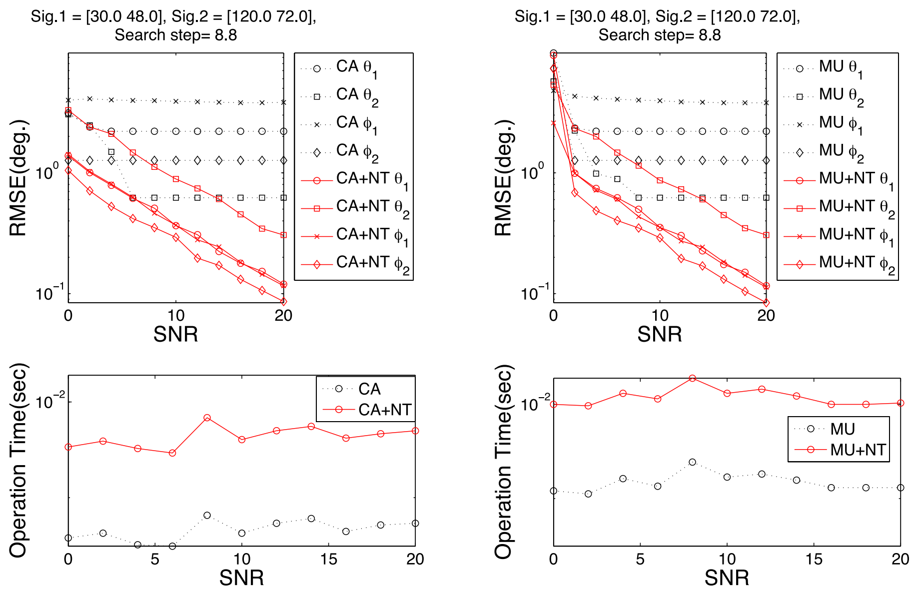
of the covariance matrix is 64 and tolerance used for the termination criterion which is the termination criterion for the iteration in (28) is 10^{-3}. The RMSE and operation time in Fig. 2 are obtained from the 1000 repetitions. The simulation results in Fig. 2.(a) show how much improvement we can get if we apply the Newton scheme to the initial estimates obtained from the Capon beamforming algorithm. Fig. 2.(b) shows the corresponding results for the MUSIC algorithm. The results with legend $'\mathrm{CA}'$ and $'\mathrm{CA}+\mathrm{NT}'$ correspond to the initial estimates and the final estimates of the Capon algorithm, respectively. In Fig. 2, we can see that the computation complexity required for the final estimates is nearly equal to that required for the initial estimates, which can be seen in the lower figures. In addition, in the upper figures, we can see that the final estimates are more accurate than the initial estimates.



(a) Capon beamforming for UCA (b) MUSIC for UCA

Fig. 2. The initial estimates with $\Delta\theta = \Delta\phi = 8.8°$ and the final estimates $([\theta_1^{\mathrm{true}}\ \phi_1^{\mathrm{true}}] = [30°\ 48°])$, $([\theta_2^{\mathrm{true}}\ \phi_2^{\mathrm{true}}] = [120°\ 72°])$

4 Conclusions

In this paper, explicit expressions for the Newton-type search to improve the accuracy of the initial estimates of two beamforming algorithms and the MUSIC algorithm are presented. It is quite straightforward to extend the proposed scheme to the other spectrum estimation-based AOA estimation algorithm such as min-norm algorithm, maximum entropy algorithm and Pisarenko algorithm by modifying the cost functions in (1) and (2) consistently.

We showed the results for the case that there are two incident signals. It is also possible to apply the proposed scheme when there are more than

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two incident signals because the Newton search is applied to each incident signal independently. Similarly, the scheme can also be applied to any other array structure by modifying the array vectors in (3), (4), (26) and (29) consistently.

When we want to estimate the elevation as well as the azimuth using the MUSIC or the beamforming algorithm, we have to optimize two-dimensional cost function. Two dimensional exhaustive grid search can be computationally intensive if the search step is small. It is more useful to apply the Newton-search for optimization of two-dimensional cost function than for optimization of one-dimensional cost function. For $p = 2$ or $p = 3$, the gradient and the Hessian of the cost function need to be derived for the implementation.

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