The Conditional Information Leakage Given Eavesdropper's Received Signals in Wiretap Channels

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Abstract—Information leakage in Wyner's wiretap channel model is usually defined as the mutual information between the secret message and the eavesdropper's received signal. We define a new quantity called "conditional information leakage given the eavesdropper's received signals," which expresses the amount of information that eavesdropper gains from his/her received signal. A benefit of introducing this quantity is that we can develop a fast algorithm for computing the conditional information leakage, which has linear complexity in the code length n, while the complexity for computing the usual information leakage is exponential in n. Validity of such a conditional information leakage as a security criterion is confirmed by studying the cases of binary symmetric channels and binary erasure channels.

Index Terms—Wiretap Channel, Information-Theoritic Security, Information Leakage

I. INTRODUCTION

Information-theoretic security is a concept that we design a cryptosystem so that our private information must be kept hidden even if adversary's computational power is unlimited, as opposed to cryptography schemes whose secureness critically depends on computational hardness assumption [1]. Such an information theoretically secure systems include secret sharing [2], private information retrieval [3], and Wyner's wiretap channel [4].

Wiretap channel is a model of physical layer security in wireless communications, in which Alice wants to transmit a positive-rate message to Bob reliably and securely in the presence of the eavesdropper Eve, where the coding scheme is open to Eve as well as Bob. A discrete memoryless wiretap channel is described by a conditional probability p(y, z|x), where x, y, and z denote the symbols for input of the channel, output of the main channel (the channel from Alice to Bob), and the output of eavesdropper's channel (the channel from Alice to Bob), and the output of gystems in which Alice can securely transmit a message to Bob while keeping the information leakage to Eve arbitrarily small if the coding rate R is smaller than the secrecy capacity defined by

$$C_{S} = \max_{p_{X}} \{ I(X;Y) - I(X;Z) \},$$
(1)

where X, Y and Z are random variables for x, y and z, I(X;Y) is the mutual information between X and Y, and p_X is a probability mass function for X.

After Wyner's pioneering work, many research has been done. Wyner studied a degraded wiretap channel, where $X \to Y \to Z$ forms a Markov chain, while Csiszár and Korner extended the channel model to a broadcast channel with confidential messages [5]. The secrecy condition that Wyner posed was that $\frac{1}{n}I(S; Z^n) \to 0$ as $n \to \infty$, where S denotes the random variable of a secret message and $Z^n = Z_1, Z_2, \ldots, Z_n$ is a random vector of Eve's received symbols. Maurer and Wolf [6] pointed out that such a secrecy criterion is weak and they posed a stronger constraint that $I(S; Z^n) \to 0$ as $n \to \infty$. They showed that the secrecy capacity (1) can be achieved even under the strong secrecy criterion. Hayashi derived a secrecy exponent [7] which shows that there exists a sequence of encoders by which the information leakage goes to zero exponentially.

Construction of wiretap channel codes has been extensively studied. Coset-coding, also referred to as syndrome-coding, is generally used. A wiretap channel codes using low density parity check (LDPC) codes [8] and that using polar codes [9], [10] were studied. Codes in [8], [9] satisfy weak secrecy, while the wiretap codes based on polar codes [10] satisfy strong secrecy, i.e., codes in [10] is proved to satisfy that unnormalized information leakage goes to zero as n goes to infinity.

The purpose of our research is the evaluation of the secureness of a given encoder. Our fundamental question is that when *n* is finite, how much information is leaked to Eve for an explicitly designed wiretap code. This situation is the same as the evaluation of bit error rate, i.e., we must evaluate the bit error rate of an explicitly designed code, even if it is a capacity-achieving code. However, there is a difficulty in evaluation of information leakage; it takes exponential time in *n* for computing the mutual information $I(S^m; Z^n)$. This fact comes from the assumption that the eavesdropper is allowed to access to unlimited computational resources. Thus, previous studies for evaluating the information leakage focus on special cases where the eavesdropper's channel is a binary erasure channel (BEC) or a binary symmetric channel (BSC). Ozarow and Wyner [11] considered the case where the main channel is noiseless and Eve's channel is a BEC1. In this case, information leakage is obtained by computing the rank of submatrix consisting of all *i*-th rows of the parity check matrix, where *j*'s are the indices of the erased bits. Zhang et al. [12], [13]

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¹ To be precise, Eve is assumed to access μ symbols out of *n* transmitted symbols of her own choice [11]. Such a model is called a type II wiretap channel.

restricted their attention to the case when Eve's channel is a BSC and main channel is noiseless. Probability generating function is used to efficiently compute the information leakage. Unfortunately, Zhang et al.'s method cannot be applied for other DMCs to compute the information leakage. Mori and Ogawa evaluated an upper and lower bound of the information leakage for binary symmetric wiretap channel and reduced the complexity for computing the information leakage [14].

In this paper, we provide a method for evaluating the amount of information leakage to Eve when the Alice's encoding system is a coset coding and eavesdropper's channel is a general binary-input discrete memoryless channel (BI-DMC). The contribution of this paper is four-fold.

- 1) We introduce a new quantity called "conditional" information leakage given Eve's received signal, defined by $L(z^n) = \mathcal{H}(S) - \mathcal{H}(S|Z^n = z^n)$, where $\mathcal{H}(X)$ denotes the entropy of a random variable X. Introducing such a new quantity is the key of this paper. The standard definition of the information leakage $I(S; Z^n)$ is the expectation of $L(Z^n)$ over Z^n .
- We propose a method for computing L(zⁿ) when coset coding is employed and Eve's channel is a BI-DMC. The proposed method is a modified version of Zhang et al.'s method [12], [13]. Extension of the proposed method to *M*-ary input M ≥ 2 is straightforward.
- 3) We show that if Eve's channel is a BEC in addition to the condition stated in 2), then $L(z^n)$ is equal to the number of bits on *S* that Eve gains from z^n . This fact supports that the definition of $L(z^n)$ is reasonable. We also show that $L(z^n)$ is computed by the rank of a submatrix of the parity check matrix *A* for the coset coding.
- 4) We show that, under the conditions 2) and 3), the probability distribution of $L(z^n) = L(z^n|A)$ which depends on A can be well approximated by the ensemble average of the probability distribution over random A.

The rest part of the paper is organized as follows. In Section 2, we give definitions of the wiretap channel model and the coset coding. In Section 3, we define the "conditional" information leakage and give an efficient method for computing it when Eve's channel is a BI-DMC. Computation of the conditional information leakage when Eve's channel is a BEC is also discussed. In Section 4, the probability distribution of the conditional information leakage when Eve's channel is a BEC is discussed. The ensemble average of the probability distribution over a all possible coset coding is given. Section 5 concludes this paper.

II. WIRETAP CHANNELS WITH COSET CODES

Consider a wiretap channel with noiseless main channel and a coset coding as the encoder (See Fig.1). Alphabets of the input and Bob's received symbols are \mathbb{F}_2 , and that of Eve's received symbol is arbitrary, denoted by \mathbb{Z} . In Section 2 and 3, we consider BSC and BEC for Eve's channel.

Let $A \in \mathbb{F}_2^{m \times n}$ (m < n) be a parity check matrix, where \mathbb{F}_2 is a finite field of order 2. Assume that A is constructed as $A = [I_m \ A_2]$, where I_m is the identity matrix of *m*-th order. As in [12]–[14], we assume that the main channel is noiseless.



Fig. 1. System model: A wiretap channel with noiseless main channel. Coset coding with a parity check matrix $A \in \mathbb{F}_2^{m \times n}$ is employed as a encoder. S^m is Alice's secret message and U^{n-m} is a random bit sequence.

Let $S^m \in \mathbb{F}_2^m$ be a column vector of *m* random variables for the secret message. For a fixed $S^m = s^m$, a codeword is randomly chosen from $C(s^m) = \{x^n | Ax^n = s^m\}$, where $C(s^m)$ is the coset with a coset leader s^m . Then, the codeword for a secret message s^m is represented by a random variable X^n that follows a uniform distribution on $C(s^m)$.

Then, the codeword can be expressed by using $S^m \in \mathbb{F}_2^m$ and a column vector of n - m uniform random variables $U^{n-m} \in \mathbb{F}_2^{n-m}$ as

$$X^{n} = \begin{bmatrix} X_{1} \\ X_{2} \end{bmatrix} = \begin{bmatrix} S^{m} \\ 0 \end{bmatrix} + \begin{bmatrix} A_{2} \\ I_{n-m} \end{bmatrix} U^{n-m}$$
(2)

It is easy to check that $S^m = AX^n$ holds. The assumption of the noiseless main channel makes Bob's received signal Y^n equal to X^n , and therefore Bob can perfectly recover S^m by computing AY^n .

III. CONDITIONAL INFORMATION LEAKAGE GIVEN EVE'S RECEIVED SIGNAL

A. Definition

Wyner defined the information leakage by the mutual information between Eve's received signal Z^n and the secret message S^m , denoted by $I(S^m; Z^n)$. $\frac{1}{n}I(S^m; Z^n)$ or $I(S^m; Z^n)$ have been used as the security measure [5]–[10], [12]–[14]. Let us revisit this measure. Before receiving Z^n , Eve does not have any knowledge about the transmitted message S^m . Thus, her best guess for S^m is that S^m is uniformly distributed. After Eve receiving $Z^n = z^n$, where z^n is a realization of Z^n , her best guess for S^m is that S^m follows the a posteriori probability $p_{S^m|Z^n}(s^m|z^n)$. Equivocation for S^m is reduced from $\mathscr{H}(S^m)$ to $\mathscr{H}(S^m|Z_n = z_n)$. Thus, we can define the amount of information on S^m that Eve gained by receiving z^n is

$$L(z^n) = \mathscr{H}(S^m) - \mathscr{H}(S^m | Z^n = z^n).$$
(3)

We see that the mutual information $I(S^m; Z^n)$ is the expectation of Eq.(3) with respect to Z^n . The use of $I(S^m; Z^n)$ as the security criterion is reasonable since the code designer does not know the realization of Z^n beforehand. However, when we perform a computer simulation, z^n is available and therefore we can treat $L(z^n)$ as the information leakage. We refer to $I(S^m; Z^n)$ as the average information leakage and $L(z^n)$ as the conditional information leakage given z^n . Here we give a remark that Eq.(3) can take negative value in general.



Fig. 2. Illustration of the distribution of "conditional" information leakage given z^n .

However, if *a prior* probability distribution P_{S^m} is uniform, then $\mathscr{H}(S^m) = m$ and therefore Eq.(3) is always nonnegative.

It is considered that $L(Z^n)$ is distributed according to the probability distribution of Z^n . As the eavesdropper's received signals are random variables, it is natural to consider information leakage is also a random variable. An image of the distribution of $L(Z^n)$ is illustrated in Fig.2. The distribution of $L(Z^n)$ is more informative than the average information leakage $I(S^m; Z^n)$ which is a single scalar and is equal to the mean value of $L(Z^n)$.

B. Computation of the conditional information leakage

1) Probability Generating Function: In [12], [13], Zhang et al. proposed an efficient method for computing the information leakage when the encoder is a coset code with parity check matrix $A \in \mathbb{R}^{m \times n}$ and wiretapper's channel is a BSC. They used the probability generation function for computing the probability distribution of AV^n , where V^n denotes the random vector expressing bit flip in the BSC. Note that Zhang et al.'s method is only applicable when Eve's channel is a BSC.

For a random vector $X = (X_1, X_2, ..., X_m)^T \in \mathbb{F}_2^m$, where $(\cdot)^T$ denotes the transpose of a vector or a matrix, we define the probability generation function $G_X(t)$ as

$$G_X(t) = E[t^X] = \sum_{\boldsymbol{x} \in \mathbb{F}_m^2} P(\boldsymbol{X} = \boldsymbol{x}) t^{\boldsymbol{x}}.$$
(4)

The following property of the probability generating function is useful. Consider the "exclusive or" of two independent random vectors X and $Y \in \mathbb{F}_2^m$ denoted by $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_m \oplus Y_m)^T$. The probability generation function of $X \oplus Y$, $G_{X \oplus Y}(t) = E[t^{X \oplus Y}]$ is computed by

$$G_{\boldsymbol{X}\oplus\boldsymbol{Y}}(t) = G_{\boldsymbol{X}}(t)G_{\boldsymbol{Y}}(t).$$
⁽⁵⁾

The conditional probability mass function $P_{S^m|Z^n}(\cdot|z^n)$ can be computed using the property (5).

2) Computation of average information leakage for BSC: The conventional method: In this subsection, we explain Zhang et al.'s method [12], [13] for computing the information leakage of a wiretap channel under the assumption that main channel is noiseless, and the eavesdropper's channel is a BSC with crossover probability δ and the coset code with a parity check matrix A is employed. Zhang et al. [12], [13] proved that $\mathcal{H}(S^m|AZ^n) = \mathcal{H}(AV^n)$ holds and gave an efficient computation method for $\mathcal{H}(AV^n)$, by which mutual information $I(S^m; AZ^n) = \mathcal{H}(S^m) - \mathcal{H}(S^m|AZ^n)$ is computed. In [12], [13], relation between the information leakage $I(S^m; Z^n)$ and $I(S^m; AZ^n)$ was not explicitly given. In order to clarify the relation, we give the following theorem:

Theorem 1: Consider a wiretap channel with noiseless main channel. Assume that Eve's channel is a BSC and the encoder uses a coset code with a parity check matrix $A = [I_m|A_2] \in \mathbb{F}_2^{m \times n}$. Let V^n be a random vector expressing the noise of the BSC, i.e., $V_i = 1$ if *i*-th bit is flipped and $V_i = 0$ otherwise. Then we have,

$$I(S^m; Z^n) = m - \mathscr{H}(AV^n).$$
⁽⁶⁾

This theorem shows that $I(S^m; Z^n) = I(S^m; AZ^n)$ holds. We give a proof of this theorem in A.

A naive computation of the probability distribution of AV^n requires us to take a time proportional to 2^n . Zhang et al. showed that the computational complexity can be reduced by the use of probability generating function. Zhang et al. gave the following theorem [12], [13]:

Theorem 2 ([12], [13]): Consider a wiretap channel with noiseless main channel. Suppose that the eavesdropper's channel is a BSC with crossover probability δ and a coset code with a parity check matrix $A = [a_1 \ a_2 \cdots a_n]$ is used. The BSC is expressed by an additive noise vector $V^n = V_1 V_2 \cdots V_n$. Then, the probability generating function of AV^n is

$$G_{AV^{n}}(t) = \prod_{i=1}^{n} \left((1-\delta) + \delta t^{a_{i}} \right).$$
(7)

Although the proof was given in [12], to make this paper selfcontained, we give a proof here.

Proof: We have

$$Av^{n} = \boldsymbol{a}_{1}v_{1} \oplus \boldsymbol{a}_{2}v_{2} \oplus \cdots \oplus \boldsymbol{a}_{n}v_{n}.$$
(8)

Therefore, the following chain of equalities hold.

$$G_{AV^{n}}(t) = \sum_{\boldsymbol{x} \in \mathbb{F}_{2}^{m}} \Pr(AV^{n} = \boldsymbol{x})t^{\boldsymbol{x}}$$

$$= \sum_{v^{n} \in \mathbb{F}_{2}^{n}} p(v^{n})t^{Av^{n}}$$

$$\stackrel{(a)}{=} \prod_{i=1}^{n} \sum_{v_{i} \in \mathbb{F}_{2}} p(v_{i})t^{\boldsymbol{a}_{i}v_{i}}$$

$$= \prod_{i=1}^{n} \left((1 - \delta) + \delta t^{\boldsymbol{a}_{i}} \right).$$
(9)

Step (a) follows from the equality in (8) together with the assumption that $V_1, V_2, ..., V_n$ are independent.

By Theorem 2, the probability generation function of AV^n is expressed by the multiplication of *n* terms. Expansion of Eq.(9) as a polynomial of *t* is expressed as

$$G_{AV^n}(t) = \sum_{\boldsymbol{x} \in \mathbb{F}_2^m} \beta_{\boldsymbol{x}} t^{\boldsymbol{x}},\tag{10}$$

where β_x is the probability of the event $AV^n = x$. We can compute β_x recursively, as follows: For a given $A = [a_1 \ a_2 \ \dots \ a_n]$, we define for $r = 1, 2, \dots, n$,

$$G_{AV^n}^{(r)}(t) = \prod_{i=1}^r \left((1-\delta) + \delta t^{a_i} \right).$$
(11)

Let the expansion of $G_{AV^n}^{(r)}(t)$ be $G_{AV^n}^{(r)}(t) = \sum_{x \in \mathbb{F}_2^m} \beta_x^{(r)} t^x$. Then, we have

$$G_{AV^{n}}^{(r+1)}(t) = G_{AV^{n}}^{(r)}(t) \left((1-\delta) + \delta t^{a_{r+1}} \right)$$

= $(1-\delta)G_{AV^{n}}^{(r)}(t) + \delta \sum_{\boldsymbol{x} \in \mathbb{F}_{2}^{m}} \beta_{\boldsymbol{x}}^{(r)} t^{\boldsymbol{x} \oplus a_{r+1}}$
= $\sum_{\boldsymbol{x} \in \mathbb{F}_{2}^{m}} \left\{ (1-\delta)\beta_{\boldsymbol{x}}^{(r)} + \delta \beta_{\boldsymbol{x} \oplus a_{r+1}}^{(r)} \right\} t^{\boldsymbol{x}},$ (12)

and hence

$$\beta_{\mathbf{x}}^{(r+1)} = (1-\delta)\beta_{\mathbf{x}}^{(r)} + \delta\beta_{\mathbf{x}\oplus \mathbf{a}_{r+1}}^{(r)}.$$
(13)

Eq.(13) shows that we can compute $\beta_x = \beta_x^{(n)}$ recursively with initial value

$$\beta_{\boldsymbol{x}}^{(0)} = \begin{cases} 1, \, \boldsymbol{x} = (0, \dots 0)^T \\ 0, \, \text{otherwise.} \end{cases}$$

Hence, the computation time is linear in *n*, while it is still exponential in *m* because of the computation of $\beta_{x \oplus a_{r+1}}^{(r)}$ in (13). Further reduction of computational time, for example using Mori and Ogawa's method [14], is left to be studied in future.

3) Computation of the conditional information leakage for BI-DMCs: The proposed method: In this section, we give a method for computing the conditional information leakage $L(z^n)$. We have

$$L(z^{n}) = \mathscr{H}(S^{m}) - \mathscr{H}(S^{m}|Z^{n} = z^{n})$$

= $m + \sum_{s^{m} \in \mathbb{F}_{2}^{m}} p(s^{m}|z^{n}) \log p(s^{m}|z^{n}).$ (14)

Therefore, we compute the conditional probability distribution of $p(s^m|z^n)$ for a given z^n . We define the conditional probability of backward channel as

$$\Phi(x|z) \stackrel{\scriptscriptstyle \triangle}{=} \frac{P_X(x)W_E(z|x)}{\sum_{x' \in X} P_X(x')W_E(z|x')} = \frac{W_E(z|x)}{\sum_{x'} W_E(z|x)},$$

where $W_E(z|x)$ denotes the conditional probability for Eve's channel. The second equality follows from the assumption that s^m and u^{n-m} follow uniform distributions. We give the following theorem:

Theorem 3: Let β_s be the probability of the event $S^m = s$ given $Z^n = z^n$. Then, $\beta_s = \beta_s^{(n)}$ is obtained by computing

$$\beta_{s}^{(r+1)} = \Phi(0|z_{i})\beta_{s}^{(r)} + \Phi(1|z_{i})\beta_{s \oplus a_{r+1}}^{(r)}$$
(15)

for r = 0, 1, ..., n - 1 with initial values $\beta_0^{(0)} = 1$ and $\beta_j^{(0)} = 0$ for $j \neq 0$.

We give a proof of Theorem 3 in A

By Theorem 3, we can compute $p(s^m|z^n)$ by a modified Zhang et al's method, which is obtained by simply replacing $(1 - \delta)$ and δ in (13) with $\Phi(0|z_i)$ and $\Phi(1|z_i)$, respectively.

C. Computation of the conditional information leakage when *Eve's channel is a BEC*

In this section, binary erasure channel (BEC) is assumed for Eve's channel. Suppose that the main channel is noiseless and coset coding is used. We show that, unlike the BSC case, the conditional information leakage $L(Z^n)$ is distributed. We compute $L(z^n)$ for a BEC as follows:

Theorem 4: Suppose that Eve's channel is a BEC and that Eve receives $Z^n = z^n$. Let $A = [a_1 \ a_2 \ \dots \ a_n]$ be the parity check matrix for the coset code. Then the information leakage to Eve is

$$L(z^n) = m - \operatorname{rank}[\boldsymbol{a}_j : z_j = \mathbf{e}], \tag{16}$$

where $[a_j : z_j = e]$ denotes the submatrix of *A* consisting of all a_i 's for which $z_i = e$.

This is almost the same statement as [11, Lemma 4]. However, the authors in [11] assumed a combinatorial variation of an erasure channel in which Eve observes μ symbols out of the *n* transmitted symbols. Thus, in order to use the proof of Lemma 4 in [11] as the proof of Theorem 4, a translation is needed. Therefore, we give a direct proof of Theorem 4 in A.

It should be noted $L(z^n)$ is independent of the erasure probability δ . Theorem 4 shows that $L(z^n)$ is equal to the number of bit on S^m that Eve can recover from z^n . This is explained by the following example:

Example 1: Let m = 2, n = 3 and

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The secrete message is denoted by (S_1, S_2) . By (2), the codeword is expressed by $X_1 = S_1 \oplus U$, $X_2 = S_2 \oplus U$ and $X_3 = U$, where $U \in \mathbb{F}_2$ is a binary uniform random variable.

If no bit is erased so that $Z_i = X_i$ for i = 1, 2, 3, the secret message (S_1, S_2) is leaked to Eve. Suppose one of Z_i s is erased. If $Z_1 = e$, Eve does not know X_1 but she obtains X_2 from $Z_2 \oplus Z_3$. If $Z_2 = e$, Eve does not know X_2 but she obtains X_1 from $Z_1 \oplus Z_3$. If $Z_3 = e$, Eve does not know X_1 or X_2 but she obtains $X_1 \oplus X_2$ from $Z_1 \oplus Z_2$. Thus, the amount of information leakage is exactly 1 bit if one of Z_i s is erased. Suppose two of Z_i s are erased. If Z_2 and Z_3 are erased, Eve cannot extract any useful information from $Z_1 = S_1 \oplus U$.

We can also confirm that the probability distribution of S^m given $Z^n = z^n$ is obtained by using the probability generating function. For example, for the case of $(Z_1, Z_2, Z_3) = (1, 0, e)$ we have $G_{S^m|Z^n=z^n} = (0+1t^{a_1})(1+0t^{a_2})(\frac{1}{2}+\frac{1}{2}t^{a_3}) = \frac{1}{2}t^{a_1}+\frac{1}{2}t^{a_1\oplus a_3}$. Since $a_1 = (1, 0)^T$ and $a_1 \oplus a_3 = (0, 1)^T$, we have

$$P((S_1, S_2) = (1, 0)) = P((S_1, S_2) = (0, 1)) = \frac{1}{2}$$
$$P((S_1, S_2) = (0, 0)) = P((S_1, S_2) = (1, 1)) = 0.$$

Then, we have $\mathscr{H}(S^m|Z^n = z^n) = 1$, i.e., remaining ambiguity on S^m is exactly 1 bit. Therefore the conditional information leakage is $I(S^m; Z^n = z^n) = m - \mathscr{H}(S^m|Z^n = z^n) = 1$.

IV. DISTRIBUTION OF THE CONDITIONAL INFORMATION LEAKAGE

A. The case that Eve's channel is a BSC

In the coset coding, both s^m and u^{n-m} are uniformly distributed, which makes x^n also uniformly distributed. Thus, in the case of BSC, we have $\Phi(x|z) = 1 - \delta$ if x = z and $\Phi(x|z) = \delta$ if $x \neq z$. Then, the probability distribution obtained by (15) depends on z^n . However, we have the following:



Fig. 3. The histogram of $L(Z^n)$ for a fixed parity check matrix A, where the number of samples is 10^4

Theorem 5: If Eve's channel is a BSC, for any $z^n \in \mathbb{F}_2^n$, the conditional information leakage is independent of z^n , i.e., we have

$$L(z^n) = I(S^m; Z^n = z^n)$$

= $I(S^m; Z^n) = m - \mathscr{H}(AV^n)$

We give the proof in A

Theorem 5 shows that $L(Z^n)$ takes $I(S^m; Z^n)$ with probability one. BSC is a special case where $L(Z^n)$ is a fixed value. For a general DMC, $L(Z^n)$ is distributed, as shown in the next subsection.

B. The case that Eve's channel is a BEC

If Eve's channel is a BEC, the conditional information leakage $L(z^n)$ is given by (16). Thus it depends on the pattern of erased bits in z^n . To obtain the distribution of $L(z^n)$ exactly for a given parity check matrix A, we have to compute (16) for all z^n , taking a time proportional to 2^n . This subsection provides a numerical result by Monte Carlo simulation.

the simulation, we generate z^n s, In say $z^{n}(1), z^{n}(2), \ldots, z^{n}(N)$, compute $L(z^{n}(i))$ for $i = 1, \ldots, N$, and make a histogram of $L(z^n(i))$. We put (m, n) = (100, 200)and $N = 10^4$ and choose $\epsilon \in \{0.46, 0.5, 0.54, 0.58\}$. A parity check matrix A is generated once and fixed. Fig. 3 shows the histogram of $L(z^n)$. We observe that when $\epsilon = 0.54$ which is greater than 100/200 = 0.5, the average information leakage is small, but the conditional information leakage takes relatively large value, 6, 7 and 8 with probability 0.0081, 0.060 and 0.044. Thus, from a security perspective, the conditional information leakage is more meaningful than the averaged one.

C. Average distribution of $L(Z^n)$ over randomly generated parity check matrices

The distribution of $L(Z^n)$, denoted by $p_L(\ell)$, depends on the selection of A. However, if m is large enough, $p_L(\ell)$ can be well-approximated by the ensemble average of $p_L(\ell)$ over all possible A, as shown in this subsection.

Let us evaluate the ensemble average of the probability mass function of $L(Z^n)$ over randomly generated parity check matrices. Suppose Eve's channel is a BEC with erasure probability



Fig. 4. The ensemble average of the probability distribution, $\overline{p}_L(\ell)$.

 ϵ . To clarify the dependency of the parity check matrix A, denote the conditional information leakage $L(z^n)$ as $L(z^n|A)$. The probability mass function of $L(Z^n|A)$ is defined by

$$p_L(\ell) = \Pr(L(Z^n|A) = \ell).$$
(17)

Suppose *A* is generated equally randomly over $\mathbb{F}_2^{m \times n}$. Let us denote the random variable for *A* by *A*. Express the right hand side of (17) by $P_{L|A}(\ell|A)$ We define the average probability distribution of $L(Z^n|A)$ over *A* as

$$\overline{p}_L(\ell) = \mathcal{E}_A[P_{L|A}(\ell|A)]. \tag{18}$$

We can evaluate $\overline{p}_L(\ell)$ by counting the number of matrices in $\mathbb{F}_2^{m \times n}$ whose rank is *r*. The following function F(m, n) $(m \ge n)$ expresses the number of full-rank matrice in $\mathbb{F}_2^{m \times n}$:

$$F(m,n) = \prod_{i=0}^{n-1} (2^m - 2^i).$$

Then, the probability for $m \times n$ random binary matrix to have rank *r* is given by

$$Q(r|m,n) = \frac{1}{2^{nm}} \frac{F(m,r) \cdot F(n,r)}{F(r,r)}$$
(19)

See [15] for the derivation. We have

Theorem 6: Let K be a random variable following the binomial distribution with parameter n and ϵ . The average probability mass function $\overline{p}_L(\ell)$ of $L(Z^n|A)$ over the random matrix A is given by

$$\overline{p}_L(\ell) = \mathbb{E}_K[Q(m-\ell|m,K)]$$
$$= \sum_{k=m-\ell}^n \binom{n}{k} (1-\epsilon)^{n-k} \epsilon^k Q(m-\ell|m,k).$$
(20)

Proof: From (16), (17), and (18), we have

$$\overline{p}_{L}(\ell) = E_{A}[\Pr(L(Z^{n}|A) = \ell)] = E_{A}[\Pr(\operatorname{rank}[\{a_{j} : Z_{j} = e\}]) = m - \ell)] = \sum_{A \in \mathbb{R}_{2}^{m \times n}} \Pr(A = A)\Pr(Z^{n} = z^{n}) \cdot \mathbb{1}(\operatorname{rank}[\{a_{j} : z_{j} = e\}]) = m - \ell) = \Pr(Z^{n} = z^{n})Q(m - \ell \mid m, \#\{j : z_{j} = e\}) = E_{K}[Q(m - \ell \mid m, K)]$$
(21)

Since Q(r|m, n) = 0 if $r > \min\{m, n\}$, we have (20). This completes the proof.

Fig. 4 shows the graph of $\overline{p}_L(\ell)$ for (m, n) = (100, 200). We observe that the histogram in Fig. 3 is well approximated by the graph in Fig.4.

V. CONCLUSION

We have defined $L(z^n) = I(S^m; Z^n = z^n)$ as the conditional information leakage given Eve's received signal in Wyner's wiretap channel model and proposed to use it as a secrecy criterion. The standard definition of information leakage $I(S^m; Z^n)$ is the expectation of $L(Z^n)$ over Z^n . We have investigated the probability distribution of $L(Z^n)$.

We gave a method for computing $L(z^n)$ efficiently, which is a modified version of Zhang et al.'s method [12], [13]. Although Zhang et al.'s method is only applicable to BSCs, our method canbe applied to any BI-DMCs. Because of the space limitation, we only investigated the probability distribution of $L(Z^n)$ for the case of BSCs and BECs. Our proposed method will work better in other DMCs than these two examples. The case of other DMCs will be investigated in future.

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Appendix

In this section, we prove Theorem 1.

Proof of Theorem 1: We first expand $I(S^m; Z^n) = \mathscr{H}(Z^n) - \mathscr{H}(Z^n|S^m)$. The codeword X^n is given by (2). Although the additive noise in the BSC V^n is not uniform, $Z^n = X^n \oplus V^n$ follows a uniform distribution on \mathbb{F}_2^n since X^n is a uniform random vector because of its construction. Then, $\mathscr{H}(Z^n) = n$. Express the noise vector as $V^n = (V_1^T, V_2^T)^T$. Then the vector of Eve's received symbols is expressed by $Z^n = X^n + V^n$, where $Z^n = (Z_1^T, Z_2^T)^T$ is given by

$$Z_1 = S^m + A_2 U^{n-m} + V_1, (22)$$

$$\mathbf{Z}_2 = U^{n-m} + V_2. \tag{23}$$

We have the following chain of equalities:

$$\begin{split} \mathscr{H}(Z^{n}|S^{m}) &= \mathscr{H}(\mathbf{Z}_{1}, \mathbf{Z}_{2}|S^{m}) \\ &= \mathscr{H}(S^{m} + A_{2}U^{n-m} + \mathbf{V}_{1}, U^{n-m} + \mathbf{V}_{2}|S^{m}) \\ \stackrel{(a)}{=} \mathscr{H}(A_{2}U^{n-m} + \mathbf{V}_{1}, U^{n-m} + \mathbf{V}_{2}|S^{m}) \\ \stackrel{(b)}{=} \mathscr{H}(A_{2}U^{n-m} + \mathbf{V}_{1}, U^{n-m} + \mathbf{V}_{2}) \\ &= \mathscr{H}(A_{2}U^{n-m} + \mathbf{V}_{1}|U^{n-m} + \mathbf{V}_{2}) + \mathscr{H}(U^{n-m} + \mathbf{V}_{2}) \\ \stackrel{(c)}{=} \mathscr{H}(A_{2}U^{n-m} + \mathbf{V}_{1} + A_{2}(U^{n-m} + \mathbf{V}_{2})|U^{n-m} + \mathbf{V}_{2}) \\ &+ \mathscr{H}(U^{n-m} + \mathbf{V}_{2}) \\ &= \mathscr{H}(\mathbf{V}_{1} + A_{2}\mathbf{V}_{2}|U^{n-m} + \mathbf{V}_{2}) + \mathscr{H}(U^{n-m} + \mathbf{V}_{2}) \\ \stackrel{(d)}{=} \mathscr{H}(\mathbf{V}_{1} + A_{2}\mathbf{V}_{2}) + \mathscr{H}(U^{n-m} + \mathbf{V}_{2}) \\ \stackrel{(e)}{=} \mathscr{H}(\mathbf{V}_{1} + A_{2}\mathbf{V}_{2}) + (n-m). \end{split}$$

Step(a) holds because we can remove S_m from the random variable of the conditional entropy since S_m is a given random variable. Step(b) follows since U^n is generated independently of S^m . Step(c) holds since we can add a function of $U^{n-m} + V_2$ since it is given as the condition. Step(d) follows because U^{n-m} is uniform, we do not gain any information on V_2 by knowing $U^{n-m} + V_2$. This is confirmed more explicitly by the following inequality showing that $U^{n-m} + V_2$ and $V_1 + A_2V_2$ are independent:

$$\begin{split} &I(V_1 + A_2V_2; U^{n-m} + V_2) \\ &= \mathscr{H}(U^{n-m} + V_2) - \mathscr{H}(U^{n-m} + V_2|V_1 + A_2V_2) \\ &\leq \mathscr{H}(U^{n-m} + V_2) - \mathscr{H}(U^{n-m} + V_2|V_1, V_2) \\ &= \mathscr{H}(U^{n-m} + V_2) - \mathscr{H}(U^{n-m}|V_1, V_2) \\ &= \mathscr{H}(U^{n-m} + V_2) - \mathscr{H}(U^{n-m}) \\ &= (n-m) - (n-m) = 0. \end{split}$$

Lastly, Step(e) follows from that $U^{n-m} + V_2$ is uniformly distributed on \mathbb{F}_2^{n-m} .

Consequently, we have

$$I(S^m; Z^n) = \mathscr{H}(Z^n) - \mathscr{H}(Z^n|S^m)$$

= $n - \{\mathscr{H}(V_1 + A_2V_2) + (n - m)\}$
= $m - \mathscr{H}(V_1 + A_2V_2)$
= $m - \mathscr{H}(AV^n).$

This completes the proof.

In this section, a proof of Theorem 3 is given. *Proof of Theorem 3*: We have

$$p(s^{m}|z^{n}) = \sum_{x^{n}} p(s^{m}, x^{n}|z^{n})$$

= $\sum_{x^{n}} p(x^{n}|z^{n})p(s^{m}|x^{n}, z^{n})$
= $\sum_{x^{n}} \Phi^{n}(x^{n}|z^{n}) \mathbb{1}(s^{m} = Ax^{n}),$ (24)

where 1 denotes the indicator function. Because X^n follows uniform distribution and the eavesdropper's channel is memoryless, we have

$$\Phi^{n}(x^{n}|z^{n}) = \prod_{i=1}^{n} \Phi(x_{i}|z_{i}).$$
(25)

Then, the following chain of equalities holds for the probability generation function of $p(S^m|Z^n = z^n)$:

$$G_{S^{m}|Z^{n}=z^{n}}(t) = \sum_{s^{m} \in \mathbb{F}_{2}^{m}} p(s^{m}|z^{n})t^{s^{m}}$$

$$\stackrel{(a)}{=} \sum_{s^{m}} \sum_{x^{n}} \Phi(x^{n}|z^{n}) \mathbb{1}(s^{m} = Ax^{n})t^{s^{m}}$$

$$= \sum_{x^{n}} \Phi^{n}(x^{n}|z^{n})t^{Ax^{n}}$$
(26)

$$= \prod_{i=1}^{n} \left(\Phi(0|z_i) + \Phi(1|z_i) t^{a_i} \right), \tag{27}$$

where Step (a) follows from (24). \Box By expanding (27) in a way similar to Eqs.(10) to (12), we obtain (13), which completes the proof.

In this section, we prove Theorem 4. To this aim, we give two lemmas.

Lemma 1: Suppose that the encoder uses a coset code and Eve's channel is a BEC. Then, the information leakage to Eve only depends on the position of the erasure occurred and is independent of whether $z_i = 0$ or $z_i = 1$ is received.

Lemma 2: Suppose the encoder uses a coset code with parity check matrix $A = [a_1, \ldots, a_n]$ and Eve's channel is a BEC. Assume $x^n = 0^n$ is transmitted. Let z^n be Eve's received signal and let $\mathcal{J}_e(z^n) = \{j : z_j = e\}$. Let w be the rank of $[a_j|j \in \mathcal{J}_e(z^n)]$. Assume $a_{j_1}, a_{j_2} \ldots a_{j_w}$ are independent. Then, we have

$$G_{S^m|Z^n=z^n}(t) = \frac{1}{2^w} \prod_{i=1}^w (1+t^{a_{j_i}})$$
(28)

Proof of Lemma 1: Define the index set $\mathcal{J}_{e}(z^{n}) = \{j | z_{j} = e\}$ and $\mathcal{J}_{1}(z^{n}) = \{j | z_{j} = 1\}$. Then, we have

$$G_{S^{m}|Z^{n}=z^{n}}(t) = \prod_{i \in \mathcal{J}_{1}(z^{n})} t^{a_{i}} \prod_{j \in \mathcal{J}_{c}(z^{n})} \left(\frac{1}{2} + \frac{1}{2}t^{a_{j}}\right)$$
$$= t^{\sum_{i \in \mathcal{J}_{1}(z^{n})}a_{i}} \prod_{j \in \mathcal{J}_{c}(z^{n})} \left(\frac{1}{2} + \frac{1}{2}t^{a_{j}}\right)$$
(29)

Define

$$\hat{G}_{S^m | Z^n = z^n}(t) = \prod_{j \in \mathcal{J}_c(z^n)} \left(\frac{1}{2} + \frac{1}{2} t^{a_j} \right).$$
(30)

Then $\hat{G}_{S^m|Z^n=z^n}(t)$ is the probability generating function if all $z_i = 1$ in z^n is replaced by 0. Put its expansion as $\hat{G}_{S^m|Z^n=z^n}(t) = \sum_{s \in \mathbb{F}_2^m} \hat{\beta}_s t^s$. Then we have

$$G_{S^{m}|Z^{n}=z^{n}}(t) = \sum_{\boldsymbol{s}\in\mathbb{F}_{2}^{m}} \hat{\beta}_{\boldsymbol{s}} t^{\boldsymbol{s}\oplus\sum_{i\in\mathcal{J}_{1}}\boldsymbol{a}_{i}}$$
$$= \sum_{\boldsymbol{s}\in\mathbb{F}_{2}^{m}} \hat{\beta}_{\boldsymbol{s}\oplus\sum_{i\in\mathcal{J}_{1}(z^{n})}\boldsymbol{a}_{i}} t^{\boldsymbol{s}},$$
(31)

which implies $\beta_s = \hat{\beta}_{s \oplus \sum_{i \in \mathcal{J}_1(z^n)} a_i}$ and thus the distribution of S^m given z^n is a permutation of the distribution computed from $\hat{G}_{S^m|Z^n=z^n}(t)$. Since entropy does not change by a permutation of the probability distribution, we have Lemma 1.

Proof of Lemma 2 Since $a_{j_1}, a_{j_2}, ..., a_{j_w}$ are linearly independent, we can express a_{j_k} for $k \in \{w+1, w+2, ..., |\mathcal{J}_e(z^n)|\}$ as

$$a_{j_k} = \sum_{i=1}^{w} d_{k,i} a_{j_i}$$
 $(d_{k,i} \in \mathbb{F}_2 \text{ for } i = 1, 2, \dots, w)$

for some coefficient d_k , *i*. By Lemma 1, we can assume $x^n = 0^n$. Substituting the conditional probability of the backward

channel $\Phi(0|z_i)$ into (27) gives

$$\begin{split} G_{S^{m}|Z^{n}=z^{n}}(t) \\ &= \prod_{l=1}^{w} \left(\frac{1}{2} + \frac{1}{2}t^{a_{j_{l}}}\right) \prod_{k=w+1}^{|\mathcal{J}_{e}(z^{n})|} \left(\frac{1}{2} + \frac{1}{2}t^{a_{j_{k}}}\right) \\ &= \frac{1}{2^{|\mathcal{J}_{e}(z^{n})|}} \prod_{l=1}^{w} \left(1 + t^{a_{j_{l}}}\right) \prod_{k=w+1}^{|\mathcal{J}_{e}(z^{n})|} \left(1 + t^{\sum_{i=1}^{w} d_{k,i}a_{j_{i}}}\right) \\ &= \frac{1}{2^{|\mathcal{J}_{e}(z^{n})|}} \prod_{l=1}^{w} \left(1 + t^{a_{j_{l}}}\right) \left(1 + \prod_{i=1}^{w} t^{d_{w+1,i}a_{j_{i}}}\right) \\ &\prod_{k=w+2}^{|\mathcal{J}_{e}(z^{n})|} \left(1 + t^{\sum_{i=1}^{w} d_{k,i}a_{j_{i}}}\right) \\ &= \frac{1}{2^{|\mathcal{J}_{e}(z^{n})|}} \left\{\prod_{i=1}^{w} \left(t^{d_{w+1,i}a_{j_{i}}} + t^{(1\oplus d_{w+1,i})a_{j_{i}}}\right) \\ &+ \prod_{i=1}^{w} \left(1 + t^{a_{j_{i}}}\right)\right\} \prod_{k=w+2}^{|\mathcal{J}_{e}(z^{n})|} \left(1 + t^{\sum_{i=1}^{w} d_{k,i}a_{j_{i}}}\right). \end{split}$$

Because $t^{d_{w+1,i}a_{j_i}} + t^{(1 \oplus d_{w+1,i})a_{j_i}} = 1 + t^{a_{j_i}}$ holds for $i \in \{1, 2, ..., w\}$, we have

$$G_{S^{m}|Z^{n}=z^{n}}(t) = \frac{1}{2^{|\mathcal{J}_{e}(z^{n})|-1}} \prod_{i=1}^{w} (1 + t^{a_{j_{i}}})$$
$$\times \prod_{k=w+2}^{|\mathcal{J}_{e}(z^{n})|} \left(1 + t^{\sum_{i=1}^{w} d_{k,i}a_{j_{i}}}\right)$$

Continuing the same procedure for $k = w + 2, ..., |\mathcal{J}_{e}(z^{n})|$, we obtain Lemma 2

Proof of Theorem 4 As Lemma 2, let w be the rank of $[a_j|j \in \mathcal{J}_{e}(z^n)]$ and assume $a_{j_1}, a_{j_2}, ..., a_{j_w}$ are linearly independent. Then, for all $\sum_{i=1}^w b_i a_{j_i}, b_i \in \mathbb{F}_2$ are different. By expanding (28), we have $\beta_s = \Pr(S^m = s) = \frac{1}{2^w}$ if $s = \sum_{i=1}^w b_i a_{j_i}$ for some b_i s and $\beta_s = 0$ otherwise. This completes the proof.

This section gives a proof of Theorem 5.

Proof of Theorem 5: We have the following equality.

$$\mathcal{H}(S^m | Z^n = z^n) = \mathcal{H}(S^m \oplus HZ^n | Z^n = z^n)$$
$$= \mathcal{H}(HV^n | Z^n = z^n)$$

However, we have

$$p_{HV^{n}|Z^{n}}(y^{m}|z^{n})$$

$$= \frac{1}{p_{Z^{n}}(z^{n})} \sum_{x^{n}} \sum_{v^{n}} p_{X^{n}V^{n}HV^{n}Z^{n}}(x^{n},v^{n},y^{m},z^{n})$$

$$= \frac{1}{p_{Z^{n}}(z^{n})} \sum_{x^{n}} \sum_{v^{n}} p_{X^{n}}(x^{n})p_{V^{n}}(v^{n})\mathbb{1}(y^{m} = Hv^{n})$$

$$\cdot \mathbb{1}(z^{n} = x^{n} \oplus v^{n})$$

$$= \frac{1}{p_{Z^{n}}(z^{n})} \sum_{x^{n}} p_{X^{n}}(x^{n})p_{V^{n}}(x^{n} \oplus z^{n})$$

$$\cdot \mathbb{1}(y^{m} = H(x^{n} \oplus z^{n}))$$

$$= \frac{1}{p_{Z^{n}}(z^{n})} \sum_{x^{n}} p_{X^{n}}(x^{n} \oplus z^{n})p_{V^{n}}(x^{n})\mathbb{1}(y^{m} = H(x^{n}))$$
(32)

Since both X^n and Z^n follow the uniform distribution, Eq.(32) shows that $p_{HV^n|Z^n}(y^m|z^n)$ is independent of z^n . This completes the proof.

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