PAPER

Simple Proof of the Lower Bound on the Average Distance from the Fermat-Weber Center of a Convex Body

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SUMMARY We show that for any convex body Q in the plane, the average distance from the Fermat-Weber center of Q to the points in Q is at least $\Delta(Q)/6$, where $\Delta(Q)$ denotes the diameter of Q. Our proof is simple and straightforward, since it needs only elementary calculations. This simplifies a previously known proof that is based on Steiner symmetrizations.

key words: *computational geometry, Fermat-Weber center, average distance*

1. Introduction

The Fermat-Weber problem is fundamental and important in both convex geometry and facility location context [11]. Let Q be a measurable set with positive area in the plane. The Fermat-Weber center of Q is a point in the plane such that the average distance from it to the points in Q is minimum.

Let p and q be two points in the plane, and pq the line segment connecting p and q. Denote by $\|pq\|$ the Euclidean distance between p and q. For a point $y \in Q$, we denote by $\mu_Q(y)$ the average distance between y and the points x in Q, that is, $\mu_Q(y) = \int_{x \in Q} \|xy\| dx / area(Q)$, where $area(Q)$ is the area of Q. Let $\mathcal{FW}_Q$ be a point for which this average distance is minimum, namely, $\mu_Q(\mathcal{FW}_Q) = \min_y \mu_Q(y)$. We simply write $\mu^*_Q = \mu_Q(\mathcal{FW}_Q)$. The point $\mathcal{FW}_Q$ is a Fermat-Weber center of Q. Note that Q may be non-convex, or even consists of disjoint subregions.

In this paper, we focus our attention on compact convex bodies. It is well known that $\mathcal{FW}_Q \in Q$, if Q is convex [1]. Denote by c^* the infimum of $\mu^*_Q/\Delta(Q)$ over all convex bodies, where $\Delta(Q)$ denotes the diameter of Q. Carmi, Har-Peled and Katz were the first to show that $\frac{1}{7} \le c^* \le \frac{1}{6}$, and they conjectured that $c^* = \frac{1}{6}$, because a flat rhombus Q_ϵ can be constructed such that $\mu^*_{Q_\epsilon}$ tends to $\frac{\Delta(Q_\epsilon)}{6}$ [7]. The lower bound on c^* was later improved from $\frac{1}{7}$ to $\frac{4}{25}$ [1]. Dumitrescu et al. have eventually proved that $c^* = \frac{1}{6}$ [8], which confirms the conjecture due to Carmi, Har-Peled and Katz. Their work is based on two Steiner symmetrizations and a proof that the inequality $c^* \ge \frac{1}{6}$ holds for a convex body with two orthogonal symmetry axes. The Steiner symmetrizations adopt in [8] preserve the area and the diameter, and do not increase the average distance from the corresponding Fermat-Weber centers.

The goal of this paper is to provide a simple proof of $c^* = \frac{1}{6}$. As compared with the Steiner symmetrization method adopt in [8], the novelty of the method used in [1], [7] is its simplicity, because only elementary calculations (e.g., region partitions and recursive calculations) are needed. In this paper, we show that the simple method given in [1], [7] can be used to prove $c^* = \frac{1}{6}$, too. To this end, we make use of the whole body Q in the calculation of this lower bound, instead of only a portion of Q used in [1], [7].

The classical Fermat-Weber problem asks for a point in a set F of feasible facility locations, which minimizes the average distance to the points in a set D of (possibly weighted) demand locations. If D is a finite set of points and F is the entire plane, then the solution is algebraic [4]. Two polynomial-time approximation schemes have also been proposed [5], [6]. For a survey of the Fermat-Weber problem, see [11].

The Fermat-Weber center of a body Q, where the set of demand locations is continuous, is a very important point of Q. For instance, it is the ideal location for a fire or railroad station that serves the region Q. Although finding the Fermat-Weber center of Q is difficult, a simple, linear-time approximation scheme for the case where Q is a convex polygon is known [1], [8]. The result of $c^* = \frac{1}{6}$ helps give a better approximation ratio (Sect. 3).

2. Main Result

Our work is based on a refinement of the analysis of Abu-Affash and Katz [1]. Differing from the previous work in which only a portion of a convex body is used, our proof makes use of the whole body. We will first describe a method to divide a convex body into subbodies. Next, we establish a lower bound on the total distance between the Fermat-Weber center and the points in two particular subbodies, named T and T'. Although both T and T' are non-convex, all other subbodies used in our final analysis are convex. These ideas help give a simple proof of $c^* = \frac{1}{6}$.

Suppose that P is a convex body. Denote by $\mathcal{FW}_P$ a Fermat-Weber center of P. Let p and q be two points on the boundary of P such that the length of segment pq is $\Delta(P)$. Without loss of generality, assume that segment pq is horizontal, and p is its left endpoint, see Fig. 1. For ease of presentation, let $P[u, v]$ denote the clockwise closed boundary of P from a boundary point u to the other v.

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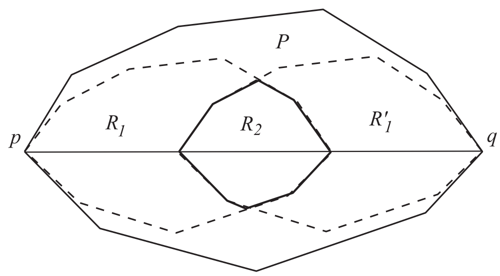

Fig. 1 The convex bodies P, R_1, R'_1 and R_2.

2.1 The Method for Dividing a Convex Body

Let P^α denote the body obtained from P by shrinking it by a factor α, that is, by applying the transformation $f(x, y) = (x/\alpha, y/\alpha)$ to the points (x, y) in P. We place two copies R_1, R'_1 of $P^{3/2}$ such that R_1 and R'_1 are contained in P and have a common tangent with P at p and q, respectively. See Fig. 1. Clearly, $area(R_1) = area(R'_1) = 4area(P)/9$.

Let $R_2 = R_1 \cap R'_1$. We place a copy R_3 (resp. R'_3) of R_2 such that R_3 (resp. R'_3) is contained in R_1 (resp. R'_1) and has a common tangent with P at the point p (resp. q) [1]. So, $\Delta(R_2) \geq \Delta(P)/3$, Fig. 2. Since P^3 is completely contained in R_2, we have $area(R_2) \geq area(P)/9$. Then, $area(P - (R_1 \cup R'_1)) = area(P) - area(R_1) - area(R'_1) + area(R_2) \geq 2area(P)/9$.

Let $R_4 = R_1 - (R_2 \cup R_3)$ and $R'_4 = R'_1 - (R_2 \cup R'_3)$. Clearly, $area(R_4) = area(R'_4)$. Let a and b be the topmost and bottommost points of P, respectively. (In the case that a or b is contained in a horizontal edge, we let a or/and b be the median point(s) of the edge(s).) Also, let c and d be the topmost and bottommost points of R_2, respectively. See Fig. 2(a). Let e (resp. f) be the intersection point of $P[p, a]$ (resp. $P[a, q]$) with the horizontal line through point c, and e' (resp. f') be the intersection point of $P[b, p]$ (resp. $P[q, b]$) with the horizontal line through point d. See Fig. 2(a).

The point set $P - (R_1 \cup R'_1)$ may consist of two disjoint regions; each of them is non-convex. So are the point sets R_4 and R'_4. Our first idea is to construct at most two convex subsets from $P - (R_1 \cup R'_1)$, R_4 and R'_4 such that the diameter of the regions formed by the point subsets is at least $\Delta(P)/3$.

Denote by R_5 (resp. R_6) the region formed by the points of $P - (R_1 \cup R'_1)$, which are between ef and $e'f'$, and to the left (resp. right) of the line through c and d. Let R'_5 and R'_6 be the copies of R_5 and R_6, inside R_4 and R'_4, respectively. See Fig. 2(a).

Denote by l a point on $P[e, a]$. Let s and t be the topmost points of R_3 and R'_3, respectively. Consider the following two regions: One is bounded by se, $P[e, l]$, lu and $R_4[u, s]$, and the other is bounded by $R_4[u, c]$ and cu, where u is the intersection point between segment cl and the boundary of R_4. The area of the former increases monotonically in the interval $P[e, a]$, starting from zero when $l = e$. And, the area of the latter decreases monotonically in $P[e, a]$, ending at zero when $l = a$. Hence, there exists the point l on $P[e, a]$ such that two considered regions are of equal area. Denote by R_7 and R'_7 the obtained regions (i.e., $area(R_7) = area(R'_7)$), see Fig. 2(b). Similarly, let r be the point on $P[a, f]$ such that the regions R_8 and R'_8 obtained by drawing the segment cr are of equal area. Region R_8 is bounded tf, $P[f, r]$, rv and $R'_4[v, t]$, and region R'_8 is bounded by $R'_4[c, v]$ and vc, where v is the intersection point between segment cr and the boundary of R'_4. Also, the same treatment is done for the portions of R_4 and R'_4 below segment pq. Denote by dl' and dr' two introduced segments, and (R_9, R'_9) and (R_{10}, R'_{10}) two pairs of obtained regions of equal area. See Fig. 2(b).

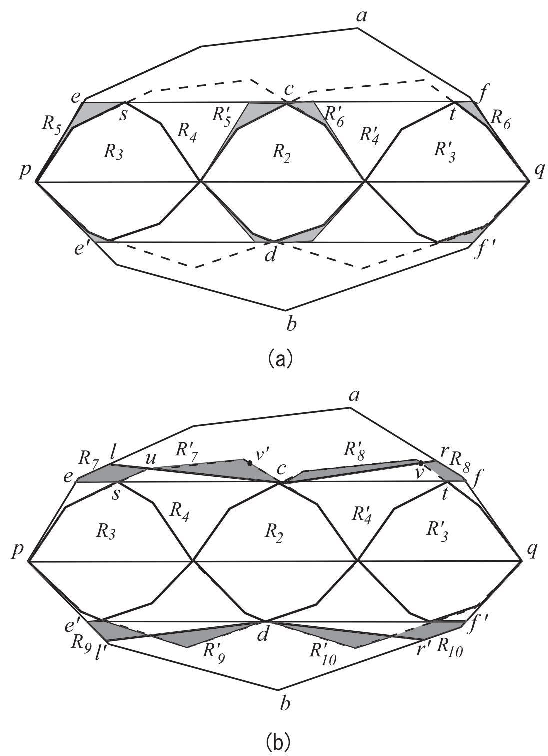

Fig. 2 Illustrating the construction of various regions.

Denote by R_{11} the region bounded by $P[l, r]$ and two segments cl and cr, and R_{12} the region bounded by $P[r', l']$ and two segments dl' and dr'. See Fig. 3. Note that if pq is a bounding segment of Q, then one of R_{11} and R_{12} is empty.

From the above construction of different regions, the following observation can be made.

Observation 1: Regions R_{11}, R_{12} and $R_2 \cup R'_5 \cup R'_6$ are all convex. Moreover, $\Delta(R_2 \cup R'_5 \cup R'_6) \geq \Delta(P)/3$, $area(R_4) = area((R_4 - R'_5 - R'_7 - R'_9) \cup R_5 \cup R_7 \cup R_9)$ and $area(R'_4) = area((R'_4 - R'_6 - R'_8 - R'_{10}) \cup R_6 \cup R_8 \cup R_{10})$.

Let us now give an important result, i.e., $\Delta(R_{11}) \geq \Delta(P)/3$.

Lemma 1: Suppose that region R_{11} is not empty. Then, $\Delta(R_{11}) \geq \Delta(P)/3$.

Proof. Without loss of generality, assume that the y-coordinate of point u is smaller than that of v. Recall that two regions R_4 and R'_4 are congruent. The copy of v inside

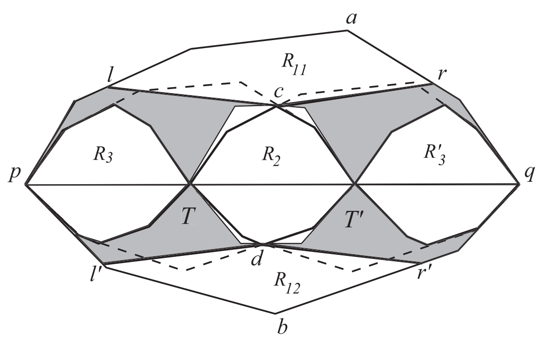

Fig. 3 Illustrating the regions R_{11}, R_{12}, T and T'.

R_4, denoted by v', is then on the boundary of R'_7, see also Fig. 2(b). Thus, $\|vv'\| = \Delta(P)/3$. Since both R'_7 (containing v') and R'_8 (containing v) are contained in R_{11} and since R_{11} is convex, the lemma follows. □

Analogously, we have $\Delta(R_{12}) \geq \Delta(P)/3$, provided that region R_{12} is not empty.

Let $T = (R_4 - R'_5 - (R'_7 \cup R'_9)) \cup (R_5 \cup R_7 \cup R_9)$, and $T' = (R'_4 - R'_6 - (R'_8 \cup R'_{10})) \cup (R_6 \cup R_8 \cup R_{10})$. See Fig. 3. In the subsequent section, we establish a lower bound on the total distance between $\mathcal{FW}_P$ and the points in T plus the total distance between $\mathcal{FW}_P$ and the points in T'.

2.2 Lower Bound on the Total Distance between $\mathcal{FW}_P$ and All Points in $T \cup T'$

As noted in [1], [7], regardless of the exact location of $\mathcal{FW}_P$, the distance between $\mathcal{FW}_P$ and any point x in R_3 plus the distance between $\mathcal{FW}_P$ and the corresponding point x' (i.e., the copy of x) in R'_3 is larger than $\frac{2\Delta(P)}{3}$. Also, the distance between $\mathcal{FW}_P$ and any point x in R_4 plus the distance between $\mathcal{FW}_P$ and the corresponding point x' in R'_4 is larger than $\frac{\Delta(P)}{3}$. Then, we have

$$\int_{x \in R_3} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R'_3} \|x\mathcal{FW}_P\|\, dx \geq \frac{2\Delta(P)}{3} area(R_3)$$

and

$$\int_{x \in R_4} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R'_4} \|x\mathcal{FW}_P\|\, dx \geq \frac{\Delta(P)}{3} area(R_4).$$

Analogously, the distance between $\mathcal{FW}_P$ and any point in $(R_4 - R'_5) \cup R_5$ plus the distance between $\mathcal{FW}_P$ and its corresponding point in $(R'_4 - R'_6) \cup R_6$ is larger than $\frac{\Delta(P)}{3}$. This is because we can establishes another one-to-one correspondence between any point $x \in R'_5$ (resp. R'_6) and its copy $x' \in R_5$ (resp. R_6). Since the distance between a point in R'_5 (resp. R'_6) and its copy in R_5 (resp. R_6) is $\Delta(P)/3$, we have

$$\int_{x \in (R_4 - R'_5) \cup R_5} \|x\mathcal{FW}_P\|\, dx + \int_{x \in (R'_4 - R'_6) \cup R_6} \|x\mathcal{FW}_P\|\, dx \geq \frac{\Delta(P)}{3} area(R_4).$$

From our construction of regions R_7, R'_7, R_8 and R'_8, all points of R_7 (resp. R'_7) are to the left (resp. right) of point u, and all points of R_8 (resp. R'_8) are to the right (resp. left) of point v. Let x be a point in R'_7 ($\subset R_4$), and x' its copy in R'_4. Then, the distance between x' and any point in R_7 is larger than $\Delta(P)/3$. Also, for any point y in R'_8 ($\subset R'_4$) and its copy y' in R_4, the distance between y' and any point in R_8 is larger than $\Delta(P)/3$. Moreover, the distance between a point in R_7 and a point in R_8 is larger than $\Delta(P)/3$. The same analysis works for regions R_9, R'_9, R_{10} and R'_{10}, too. Again, by establishing another one-to-one correspondence between the points in R_7 (resp. R_9) and those in R'_7 (resp. R'_9) and between the points in R_8 (resp. R_{10}) and those in R'_8 (resp. R'_{10}), we have

$$\int_{x \in (R_4 - (R'_7 \cup R'_9)) \cup (R_7 \cup R_9)} \|x\mathcal{FW}_P\|\, dx + \int_{x \in (R'_4 - (R'_8 \cup R'_{10})) \cup (R_8 \cup R_{10})} \|x\mathcal{FW}_P\|\, dx \geq \frac{\Delta(P)}{3} area(R_4).$$

From the discussion made above, the total distance between $\mathcal{FW}_P$ and the points in T plus the total distance between $\mathcal{FW}_P$ and the points in T' is larger than $\frac{\Delta(P)}{3} area(R_4)$. After the one-to-one correspondence between points of R_4 and points R'_4 is set, six other one-to-one correspondences between R_i and R'_i, for all $5 \leq i \leq 10$, are further established; they assure the correctness of the inequality. More precisely,

$$\int_{x \in T} \|x\mathcal{FW}_P\|\, dx + \int_{x \in T'} \|x\mathcal{FW}_P\|\, dx \geq \frac{\Delta(P)}{3} area(R_4).$$

2.3 An Alternative Proof of $c^* = 1/6$

In the following, we prove $c^* \geq 1/6$. Since $\mu^*_{Q_\epsilon}$ tends to $\frac{\Delta(Q_\epsilon)}{6}$ for a flat rhombus Q_ϵ [7], we then obtain $c^* = 1/6$.

Theorem 1: Let P be a convex body. Then $\mu^*_P \geq \Delta(P)/6$.

Proof. As in [1], [7], the proof proceeds in two stages. In the first stage, since $area(R_4) = area(R_1) - (area(R_2 \cup R_3) = \frac{4}{9} area(P) - 2area(R_3)$, we can obtain the following intermediate result (which is the same as that in [1] but the used set P' is different)

$$\int_{x \in P} \|x\mathcal{FW}_P\|\, dx \geq \int_{x \in R_3} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R'_3} \|x\mathcal{FW}_P\|\, dx$$

$$+\int_{x\in T}\|x\mathcal{FW}_P\|\,dx+\int_{x\in T'}\|x\mathcal{FW}_P\|\,dx$$
$$\geq\frac{2\Delta(P)}{3}area(R_3)+\frac{\Delta(P)}{3}(\frac{4}{9}area(P)-2area(R_3))$$
$$=\frac{4\Delta(P)}{27}area(P).$$

This implies that for any convex body Q, $\mu_Q^*\geq 4\Delta(Q)/27$. In the second stage, we apply this intermediate result to the *whole* collection of all convex subsets of $P-P'$, which are pairwise disjoint, to obtain the final result, i.e., $\mu_P^*\geq\Delta(P)/6$.

From Observation 1, we can apply the intermediate result to the remaining regions R_{11}, R_{12} and $R_2\cup R_5'\cup R_6'$. Recall that $area(P)-area(R_1)-area(R_1')+area(R_2)\geq 2area(P)/9$, and $area(R_2)\geq area(P)/9$. So, $area(R_{11}\cup R_{12})=area(P)-area(R_1)-area(R_1')+area(R_2)-area(R_5)-area(R_6)\geq 2area(P)/9-area(R_5)-area(R_6)\,(=2area(P)/9-area(R_5')-area(R_6'))$. From Lemma 1 and Observation 1, we then have

$$\int_{x\in R_{11}}\|x\mathcal{FW}_P\|\,dx+\int_{x\in R_{12}}\|x\mathcal{FW}_P\|\,dx$$
$$+\int_{x\in(R_2\cup R_5'\cup R_6')}\|x\mathcal{FW}_P\|\,dx$$
$$\geq\frac{4\Delta(R_{11})}{27}area(R_{11})+\frac{4\Delta(R_{12})}{27}area(R_{12})$$
$$+\frac{4\Delta(R_2\cup R_5'\cup R_6')}{27}(area(R_2\cup R_5'\cup R_6'))$$
$$\geq\frac{4\Delta(P)}{81}(area(R_{11}\cup R_{12})+area(R_2\cup R_5'\cup R_6'))$$
$$\geq\frac{4\Delta(P)}{81}(\frac{2}{9}area(P)+area(R_2))$$
$$\geq\frac{4\Delta(P)}{243}area(P).$$

Therefore,

$$\int_{x\in P}\|x\mathcal{FW}_P\|\,dx$$
$$=\int_{x\in R_3}\|x\mathcal{FW}_P\|\,dx+\int_{x\in R_3'}\|x\mathcal{FW}_P\|\,dx$$
$$+\int_{x\in T}\|x\mathcal{FW}_P\|dx+\int_{x\in T'}\|x\mathcal{FW}_P\|\,dx$$
$$+\int_{x\in R_{11}}\|x\mathcal{FW}_P\|\,dx+\int_{x\in R_{12}}\|x\mathcal{FW}_P\|\,dx$$
$$+\int_{x\in R_2\cup R_5'\cup R_6'}\|x\mathcal{FW}_P\|\,dx$$
$$\geq\frac{4\Delta(P)}{27}area(P)+\frac{4\Delta(P)}{243}area(P)$$
$$=\frac{40\Delta(P)}{243}area(P).$$

At this point we obtain that for any convex body Q, $\mu_Q^*\geq 40\Delta(Q)/243$. Then, we repeat the above calculation using this slightly stronger result (i.e., $\mu_Q^*\geq 40\Delta(Q)/243$) for the regions R_{11}, R_{12} and $R_2\cup R_5'\cup R_6'$ (instead of the previous result $\mu_Q^*\geq 4\Delta(Q)/27$). This calculation will yield a more stronger result, etc. By noticing that $\frac{4}{27}\times\frac{1}{9}=\frac{4}{243}$, the result after the kth iteration is $\mu_Q^*\geq c_k\Delta(Q)$, where $c_k=4/27+c_{k-1}/9$ and $c_0=4/27$. More precisely, we obtain after the kth iteration that $\mu_Q^*\geq\frac{4}{27}(1+\frac{1}{9}+\cdots+\frac{1}{9^k})\Delta(Q)$. This sequence of results eventually converges to $\mu_Q^*\geq\Delta(Q)/6$. □

3. Applications

Note first that there exists another constant c_2^* such that the average distance between a Fermat-Weber center and the points in Q is at most $c_2^*\Delta(Q)$. Abu-Affash and Katz were the first to show that c_2^* is between $\frac{2}{3\sqrt{3}}$ and $\frac{1}{3}$ [1]. The upper bound on c_2^* was later improved to $2(4-\sqrt{3})/13$ in [8], and further to $(99-5\sqrt{3})/36$ (<0.3444) in [10]. Since the average distance between the points in a disk D and the Fermat-Weber center (i.e., the center) of D is $\Delta(D)/3$, one may conjecture that $c_2^*=\frac{1}{3}$ [7]. Proving this conjecture is an interesting open problem.

For a convex polygon P, the center o of the smallest disk enclosing P can be computed in linear-time [9]. Abu-Affash and Katz [1] showed that point o approximates the Fermat-Weber center of P, with $\frac{\mu_P(o)}{\mu_P^*}$. Combining Theorem 1 with Tan and Jiang's result on $\mu_P(o)$ [10] gives us the approximation ratio $(99-5\sqrt{3})/6$ (<2.07). This ratio may be further improved to 2 if $\mu_P(o)\leq\Delta(P)/3$ could be proved.

As noted in [1], [8], the result of $c^*=\frac{1}{6}$ can also be applied to the problem of balancing the load among several service-providing facilities, while keeping the total cost low. Let D the demand region, and let $p_1,p_2,\ldots,p_m$ be the points representing m facilities. Aronov, Carmi and Katz [3] have considered the following *load balancing problem*: Subdivide D into m equal-area regions $R_1,R_2,\ldots,R_m$, so that region R_i is served by facility p_i, and the total cost of the subdivision is minimized. Given a subdivision, the cost $\mu_{R_i}(p_i)$ associated with facility p_i is the average distance between p_i and the points in R_i, and the total cost of the subdivision is $\sum_i\mu_{R_i}(p_i)$. One of the main results in [3] is an $(8+\sqrt{2\pi})$-approximation algorithm, under the assumption that D is a rectangle and all regions R_i are convex. The approximation ratio was later improved to about 9.0344 [8].

The study of the load balancing problem might be related to crowdsourced package delivery that has gained great interest from the logistics industry and academe, due to its significant economic and environmental impact. Suppose that the package delivery demands and drivers (moving facilities) are dynamically given in a social region. One may often want to figure out the (low) cost that represents a state of balance between the package demand and the driver supply. Here, a demand is usually associated with a specific location and time. The cost may be defined to minimize time cost or maximize the number of transshipments (e.g., [2]). How the approximation scheme of Aronov, Carmi and Katz [3] or its variant can be applied is of course an interesting work. We are working in this direction.

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