PAPER

On the Construction of Variable Strength Orthogonal Arrays*

Qingjuan ZHANG†, ***Nonmember***, **Shanqi PANG**††a), ***Member***, ***and*** **Yuan LI**†, ***Nonmember***

SUMMARY Variable strength orthogonal array, as a special form of variable strength covering array, plays an important role in computer software testing and cryptography. In this paper, we study the construction of variable strength orthogonal arrays with strength two containing strength greater than two by Galois field and construct some variable strength orthogonal arrays with strength l containing strength greater than l by Fan-construction.

key words: *variable strength orthogonal arrays, Galois field, Fan-construction, covering array*

1. Introduction

An orthogonal array (OA) $OA(N, m_1^{k_1} m_2^{k_2} \cdots m_v^{k_v}, t)$ is an array of size $N \times n$, where $n = k_1 + k_2 + \cdots + k_v$ is the total number of factors, in which the first k_1 columns have symbols from $\{0, 1, \ldots, m_1 - 1\}$, the next k_2 columns have symbols from $\{0, 1, \ldots, m_2 - 1\}$, and so on, with the property that in any $N \times t$ subarray every possible t-tuple occurs an equal number of times as a row. An OA with $m_1 = m_2 = \cdots = m_v = m$ is called symmetric and denoted by $OA(N, m^n, t)$ for short; otherwise, the array is called asymmetric.

OAs are of great importance in statistics and combinatorics, and they are widely used in computer science, coding theory, cryptography, information sciences and quantum information theory [1]–[6]. Recently, the use of OAs has been extended to software testing [7], [8]. One of OAs' advantages is making it relatively easy to identify the particular combination that caused a failure. Soon covering arrays (CAs) the natural generalizations of OAs are introduced in software testing. Motivated by the effectiveness of CAs, a number of recent studies have focused on the construction of CAs [9]–[17].

In some complex software testing, interactions do not often exist uniformly between parameters. Some parameters have strong interactions with each other while others may have few or no interactions. For this reason, Cohen et al. [18] proposed a new object, variable strength covering array (VCA), which provides a more robust environment for software interaction testing. In this case, VCAs may be more effective and efficient in comparison with CAs. Cohen et al. [19] developed Simulated Annealing to support VCA construction. Chen et al. [20] adopted an improved version of Ant Colony Algorithm in a strategy called Ant Colony System to support VCA construction. Raaphorst et al. [21] introduced a special form of VCA, said variable strength orthogonal array (VOA) and they used linear feedback shift registers to construct VOAs.

However, above results were focused on strengths of two and three only. Recent studies demonstrate the need to go up to $t = 6$ in order to capture most fault. Other than the results reported by Ahmed et al. [22], little is known regarding the construction methods for VCAs including strength higher than three. Consequently, despite the needs in practical applications, there are still challenging unsolved problems in this area. The construction of OAs has been made new progress recently. New construction methods are constantly proposed in Pang et al. [29], Wang et al. [30], Pang et al. [31], Pang et al. [32], Zhang et al. [33], Pang et al. [34], Du et al. [35], Pang et al. [36] and Pang et al. [37], which could facilitate the construction of related structures to OAs. Moreover, communications and computer sciences often benefit from OAs and related structures. The study of constructions of VOAs is conducive to promoting the constructions of such VCAs.

Compared with VOAs presented in [21] whose strength did not consider higher than three, we extend the strengths to four, five and even arbitrary strength. And most of VOAs constructed by Galois field are optimal VCAs, namely N is equal to the product of the largest t numbers of levels. The majority of VCAs constructed in [22] comprise strengths four and five, but there are few results about VCAs containing strength six. In this paper, we construct three families of VOAs containing arbitrary strength.

As a special form of VCA, VOA plays an important role in computer software testing and cryptography. Therefore, it is not only of theoretical importance but also of great application value to construct VOAs. In this paper, we study the construction of VOAs with strength two containing strength greater than two by Galois field and construct some VOAs with strength l containing strength greater than l by Fan-construction.

2. Preliminaries

In this section, we introduce relevant notations, definitions,

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†The authors are with the School of Economics and Statistics, Guangzhou University, Guangzhou, 510000, China.
††The author is with the College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, China.

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a) E-mail: shanqipang@126.com

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and lemmas.

In this work, 0_r represents the $r \times 1$ vector of zeros, for a matrix A, A' denotes its transpose, and we denote a Galois field of order m by $GF(m)$.

Definition 1: A VOA, denoted by $VOA(N; t, m_1^{k_1} m_2^{k_2} \cdots m_v^{k_v}, D)$, is an $N \times n$ OA of strength t containing a submatrix D which is an $N \times n'$ OA of strength t', where $n = k_1 + k_2 + \cdots + k_v$, $t \leq t'$, $n' \leq n$ and $t' \leq n'$.

Definition 2: ([23]) Let A be an $OA(N, m_1^{k_1} \cdots m_v^{k_v}, t)$ for $n = k_1 + k_2 + \ldots + k_v$. Suppose that B is an arbitrary $F \times n$ subarray of A. We say that B is a fan of A, if B is an OA of strength $t-1$. And we say that two row-disjoint fans of A are uniform if any $t-1$ columns in both arrays cover all $(t-1)$-tuples of values from the $t-1$ columns an equal number of times. If the N rows of A can be partitioned into M uniformly row-disjoint fans, then we refer to A as an M-divisible OA.

Lemma 1: ([24]) Consider an $r \times \sum_{i=1}^{n} u_i$ matrix $C = [P_1 \vdots P_2 \vdots \cdots \vdots P_n]$, $P_i = [\mathbf{p}_{i1}, \mathbf{p}_{i2}, \ldots, \mathbf{p}_{iu_i}]$, $1 \leq i \leq n$, such that for every choice of t matrices $P_{i_1}, \ldots, P_{i_t}$ from $P_1, \ldots, P_n$, the $N \times \sum_{j=1}^{t} u_{i_j}$ matrix $[P_{i_1} \vdots P_{i_2} \vdots \cdots \vdots P_{i_t}]$ has full column rank over GF(m). Then an $OA(m^r, n, (m^{u_1})(m^{u_2}) \cdots (m^{u_n}), t)$ can be constructed.

Lemma 2: ([25]) If $m \geq 2$ is a prime power, then the array $OA(m^{t+1}, (m^2)m^{m+1}, t)$ can be constructed whenever $m \geq t \geq 1$.

Lemma 3: ([26]) An $OA(m^5, (m^2)m^{m^2+m+1}, 3)$ can be constructed for any prime power m.

Lemma 4: ([24]) If m is a prime power, then an $OA(m^6, (m^2)^2 m^{m+1}, 4)$ can be constructed.

Lemma 5: ([25]) If $m \geq 4$ is a power of two, then the array $OA(m^7, (m^2)^2 m^{m+1}, 5)$ can be constructed.

Lemma 6: ([23])(Fan-construction) Suppose that there is an M-divisible $OA(N, m_1^{u_1} m_2^{u_2} \cdots m_v^{u_v}, t)$. Then there exist

(i) an $OA(N, m_1^{u_1} m_2^{u_2} \cdots m_v^{u_v} M, t)$;

(ii) an $OA(N, m_1^{u_1} m_2^{u_2} \cdots m_v^{u_v} d_1^{h_1} d_2^{h_2} \cdots d_s^{h_s}, h)$ with $h = min\{t, l\}$, provided that an $OA(M, d_1^{h_1} d_2^{h_2} \cdots d_s^{h_s}, l)$ exists;

(iii) an $OA(N, m_1^{u_1} m_2^{u_2} \cdots m_v^{u_v} d_1^{h_1} d_2^{h_2} \cdots d_s^{h_s}, t)$, if $\prod_{i=1}^{s} d_i^{h_i} | M$.

Lemma 7: ([27]) If $m \geq 2$ is a prime power, then an $OA(m^t, m^{m+1}, t)$ of index unity exists whenever $m \geq t-1 \geq 0$.

3. Main Results

3.1 Construction of VOAs by Galois Field

Theorem 1: If $m \geq 2$ is a prime power and $t \geq 3$ is an integer, then we can construct the array $VOA(m^{t+1}; 2, (m^2)m^{m^t+\cdots+m^2}, D)$, where D is an $OA(m^{t+1}, (m^2)m^{m+1}, t)$.

Proof. Let $\gamma_s = m^t + m^{t-1} + \cdots + m^{t-(s-1)} + 2$ for $s = 1, 2, \ldots, t-1$. Let $P_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}'$, $P_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}'$, $P_i = \begin{bmatrix} \alpha_{i-2}^{t-2} & \alpha_{i-2}^{t-1} & 1 & \alpha_{i-2} & \cdots & \alpha_{i-2}^{t-2} \end{bmatrix}'$ for $3 \leq i \leq m+2$, where $\alpha_1, \ldots, \alpha_m$ are distinct elements of $GF(m)$, $P_l = \begin{bmatrix} x_{1(l-m-2)} & \cdots & x_{(t+1)(l-m-2)} \end{bmatrix}'$ for $m+3 \leq l \leq m^t+2$, where $x_{j(l-m-2)} \in GF(m)$ is the jth component of the vector P_l for $1 \leq j \leq t+1$, $x_{3(l-m-2)} = 1$, and $P_l \neq P_i$ for $3 \leq i \leq m+2$, $P_q = \begin{bmatrix} x_{1(q-m-2)} & x_{2(q-m-2)} & \cdots & x_{(t+1)(q-m-2)} \end{bmatrix}'$ for $\gamma_{p-1}+1 \leq q \leq \gamma_p$ and $2 \leq p \leq t-2$, where $x_{j(q-m-2)} \in GF(m)$ is the jth component of the vector P_q for $1 \leq j \leq t+1$, $x_{3(q-m-2)} = \cdots = x_{(p+1)(q-m-2)} = 0$ and $x_{(p+2)(q-m-2)} = 1$, $P_w = \begin{bmatrix} x_{1(w-m-2)} & \cdots & x_{(t+1)(w-m-2)} \end{bmatrix}'$ for $\gamma_{t-1}+1 \leq w \leq \gamma_t - 1$, where $x_{j(w-m-2)} \in GF(m)$ is the jth component of the vector P_w for $1 \leq j \leq t+1$, $x_{3(w-m-2)} = x_{4(w-m-2)} = \cdots = x_{t(w-m-2)} = 0$, $x_{(t+1)(w-m-2)} = 1$, and $P_w \neq P_2$.

From Lemma 2, we know the matrix consisting of any t of matrices $P_1, P_2, P_3, \ldots, P_{m+2}$ has full column rank. Next, we need to prove the matrix $[P_{i_1} \vdots P_{i_2}]$ consisting of any two of matrices $P_1, P_2, \ldots, P_{m^t+m^{t-1}+\cdots+m^2+1}$ has full column rank. In the following, we denote $x_{j(i-m-2)}$ by x_{ji} and α_{i-2}^j by α_i^j.

(i) Let $i_1 = 1$ and $i_2 \in \{m+3, m+4, \ldots, m^t+2\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ x_{1i_2} & x_{2i_2} & 1 & x_{4i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

The determinant of the 3×3 submatrix composed of the first three rows of the matrix $[P_{i_1} \vdots P_{i_2}]$ is 1. Hence, the matrix $[P_{i_1} \vdots P_{i_2}]$ has full column rank.

(ii) Let $i_1 = 1$ and $i_2 \in \{\gamma_{p-1}+1, \gamma_{p-1}+2, \ldots, \gamma_p\}$ for $2 \leq p \leq t-2$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} 1 & 0 & 0'_{p-1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0'_{p-1} & 0 & 0 & \cdots & 0 \\ x_{1i_2} & x_{2i_2} & 0'_{p-1} & 1 & x_{(p+3)i_2)} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

By similar arguments as in (i), the matrix $[P_{i_1} \vdots P_{i_2}]$ has rank 3.

(iii) Let $i_1 = 1$ and $i_2 \in \{\gamma_{t-2}+1, \gamma_{t-2}+2, \ldots, \gamma_{t-1}-1\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} 1 & 0 & 0'_{t-1} & 0 \\ 0 & 1 & 0'_{t-1} & 0 \\ x_{1i_2} & x_{2i_2} & 0'_{t-1} & 1 \end{bmatrix}'.$$

It is obvious that this matrix has rank 3.

(iv) Let $i_1 = 2$ and $i_2 \in \{m+3, m+4, \ldots, m^t+2\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 1 \\ x_{1i_2} & x_{2i_2} & 1 & x_{4i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

The 2×2 submatrix formed by the third and $(t+1)$th rows is clearly nonsingular.

(v) Let $i_1 = 2$ and $i_2 \in \{\gamma_{p-1}+1, \gamma_{p-1}+2, \ldots, \gamma_p\}$ for $2 \le p \le t-2$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} 0 & 0 & 0'_{p-1} & 0 & 0 & \cdots & 1 \\ x_{1i_2} & x_{2i_2} & 0'_{p-1} & 1 & x_{(p+3)i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

The determinant of the 2×2 submatrix formed by the $(p+2)$th and $(t+1)$th rows is -1.

(vi) Let $i_1 = 2$ and $i_2 \in \{\gamma_{t-2}+1, \gamma_{t-2}+2, \ldots, \gamma_{t-1}-1\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} 0 & 0 & 0'_{t-1} & 1 \\ x_{1i_2} & x_{2i_2} & 0'_{t-1} & 1 \end{bmatrix}'.$$

Without loss of generality, one can assume that $x_{1i_2} \neq 0$ since $P_{i_1} \neq P_{i_2}$. The determinant of the 2×2 submatrix formed by the first and $(t+1)$th rows is $-x_{1i_2}$.

(vii) Let $i_1 \in \{3, 4, \ldots, m+2\}$ and $i_2 \in \{m+3, m+4, \ldots, m^t+2\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} \alpha_{i_1}^{t-2} & \alpha_{i_1}^{t-1} & 1 & \alpha_{i_1} & \cdots & \alpha_{i_1}^{t-2} \\ x_{1i_2} & x_{2i_2} & 1 & x_{4i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

Naturally, we can suppose that $x_{1i_2} \neq \alpha_{i_1}^{t-2}$ since $P_{i_1} \neq P_{i_2}$ for $3 \le i_1 \le m+2$. The 2×2 submatrix given by the first and third rows is nonsingular.

(viii) Let $i_1 \in \{3, 4, \ldots, m+2\}$ and $i_2 \in \{\gamma_{p-1}+1, \gamma_{p-1}+2, \ldots, \gamma_p\}$ for $2 \le p \le t-2$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} \alpha_{i_1}^{t-2} & \alpha_{i_1}^{t-1} & 1 & \alpha_{i_1} & \cdots & \alpha_{i_1}^{p-2} & \alpha_{i_1}^{p-1} & \alpha_{i_1}^{p} & \cdots & \alpha_{i_1}^{t-2} \\ x_{1i_2} & x_{2i_2} & 0 & 0 & \cdots & 0 & 1 & x_{(p+3)i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

The 2×2 submatrix given by the third and $(p+2)$th rows is seen to be nonsingular.

(ix) Let $i_1 \in \{3, 4, \ldots, m+2\}$ and $i_2 \in \{\gamma_{t-2}+1, \gamma_{t-2}+2, \ldots, \gamma_{t-1}-1\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} \alpha_{i_1}^{t-2} & \alpha_{i_1}^{t-1} & 1 & \alpha_{i_1} & \cdots & \alpha_{i_1}^{t-3} & \alpha_{i_1}^{t-2} \\ x_{1i_2} & x_{2i_2} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}'.$$

The rank of this matrix is 2 since the 2×2 submatrix given by the third and $(t+1)$th rows is nonsingular.

(x) Let $\{i_1, i_2\} \subseteq \{m+3, m+4, \ldots, m^t+2\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 1 & x_{4i_1} & \cdots & x_{(t+1)i_1} \\ x_{1i_2} & x_{2i_2} & 1 & x_{4i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

Suppose that $x_{1i_1} \neq x_{1i_2}$ since $P_{i_1} \neq P_{i_2}$. The determinant of the 2×2 submatrix composed of the first and third rows of the matrix $[P_{i_1} \vdots P_{i_2}]$ is $x_{1i_1} - x_{1i_2}$.

(xi) Let $i_1 \in \{m+3, m+4, \ldots, m^t+2\}$ and $i_2 \in \{\gamma_{p-1}+1, \gamma_{p-1}+2, \ldots, \gamma_p\}$ for $2 \le p \le t-2$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 1 & x_{4i_1} & \cdots & x_{(p+1)i_1} & x_{(p+2)i_1} & x_{(p+3)i_1} & \cdots & x_{(t+1)i_1} \\ x_{1i_2} & x_{2i_2} & 0 & 0 & \cdots & 0 & 1 & x_{(p+3)i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

It is easy to observe that the above matrix has rank 2.

(xii) Let $i_1 \in \{m+3, m+4, \ldots, m^t+2\}$ and $i_2 \in \{\gamma_{t-2}+1, \gamma_{t-2}+2, \ldots, \gamma_{t-1}-1\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 1 & x_{4i_1} & \cdots & x_{ti_1} & x_{(t+1)i_1} \\ x_{1i_2} & x_{2i_2} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}'.$$

Obviously, the rank of this matrix is 2.

(xiii) Let $i_1 \in \{\gamma_{p_1-1}+1, \gamma_{p_1-1}+2, \ldots, \gamma_{p_1}\}$ and $i_2 \in \{\gamma_{p_2-1}+1, \gamma_{p_2-1}+2, \ldots, \gamma_{p_2}\}$ for $\{p_1, p_2\} \subseteq \{2, \ldots, t-2\}$ $(p_1 < p_2)$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 0'_{p_1-1} & 1 & x_{(p_1+3)i_1} & \cdots & x_{(p_2+1)i_1)} & x_{(p_2+2)i_1} & x_{(p_2+3)i_1} & \cdots & x_{(t+1)i_1} \\ x_{1i_2} & x_{2i_2} & 0'_{p_1-1} & 0 & 0 & \cdots & 0 & 1 & x_{(p_2+3)i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

Clearly, the determinant of the 2×2 submatrix given by the (p_1+2)th and (p_2+2)th rows is 1.

(xiv) Let $\{i_1, i_2\} \subseteq \{\gamma_{p-1}+1, \gamma_{p-1}+2, \ldots, \gamma_p\}$ for $2 \le p \le t-2$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 0'_{p-1} & 1 & x_{(p+3)i_1} & \cdots & x_{(t+1)i_1} \\ x_{1i_2} & x_{2i_2} & 0'_{p-1} & 1 & x_{(p+3)i_2} & \cdots & x_{(t+1)i_2} \end{bmatrix}'.$$

Assume that $x_{1i_1} \neq x_{1i_2}$, then the determinant of the 2×2 submatrix composed of the first and $(p+2)$th rows is $x_{1i_1} - x_{1i_2}$.

(xv) Let $i_1 \in \{\gamma_{p-1}+1, \gamma_{p-1}+2, \ldots, \gamma_p\}$ for $2 \le p \le t-2$ and $i_2 \in \{\gamma_{t-2}+1, \gamma_{t-2}+2, \ldots, \gamma_{t-1}-1\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 0'_{p-1} & 1 & x_{(p+3)i_1} & \cdots & x_{ti_1} & x_{(t+1)i_1} \\ x_{1i_2} & x_{2i_2} & 0'_{p-1} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}'.$$

The determinant of the 2×2 submatrix composed of the $(p+2)$th and $(t+1)$th rows is 1.

(xvi) Let $\{i_1, i_2\} \subseteq \{\gamma_{t-2}+1, \gamma_{t-2}+2, \ldots, \gamma_{t-1}-1\}$. Then

$$[P_{i_1} \vdots P_{i_2}] = \begin{bmatrix} x_{1i_1} & x_{2i_1} & 0'_{t-2} & 1 \\ x_{1i_2} & x_{2i_2} & 0'_{t-2} & 1 \end{bmatrix}'.$$

Due to $P_{i_1} \neq P_{i_2}$, we can assume that $x_{1i_1} \neq x_{1i_2}$. The determinant of the 2×2 submatrix composed of the first and $(t+1)$th rows is $x_{1i_1} - x_{1i_2}$.

(xvii) Let $\{i_1, i_2\} \subseteq \{1, 2, \ldots, m+2\}$. From Lemma 2, we know the matrix $[P_{i_1} \vdots P_{i_2}]$ has full column rank.

The array $OA(m^{t+1}, (m^2)m^{m^t+m^{t-1}+\cdots+m^2}, 2)$ can be constructed via Lemma 1. According to Lemma 2, we know the matrix D consisting of the first $(m+2)$ columns of the $OA(m^{t+1}, (m^2)m^{m^t+m^{t-1}+\cdots+m^2}, 2)$ is an $OA(m^{t+1}, (m^2)m^{m+1}, t)$. Hence, it is a $VOA(m^{t+1}; 2, (m^2)m^{m^t+\cdots+m^2}, D)$.

Example 1: Let $m = 3$ and $t = 3$. Let B be an $3^4 \times 4$ matrix whose rows are all possible 4-tuples over $GF(3)$ and C be a 4×38 matrix which can be written as $C = [P_1 \vdots P_2 \vdots \ldots \vdots P_{37}]$. From Theorem 1, we can take

$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}'$, $P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}'$, $P_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}'$,
$P_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}'$, $P_5 = \begin{bmatrix} 2 & 1 & 1 & 2 \end{bmatrix}'$, $P_6 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}'$,
$P_7 = \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}'$, $P_8 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}'$, $P_9 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}'$,
$P_{10} = \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}'$, $P_{11} = \begin{bmatrix} 0 & 2 & 1 & 0 \end{bmatrix}'$, $P_{12} = \begin{bmatrix} 0 & 2 & 1 & 1 \end{bmatrix}'$,

$P_{13} = \begin{bmatrix} 0 & 2 & 1 & 2 \end{bmatrix}'$, $P_{14} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}'$, $P_{15} = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}'$, $P_{16} = \begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix}'$, $P_{17} = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}'$, $P_{18} = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix}'$, $P_{19} = \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}'$, $P_{20} = \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix}'$, $P_{21} = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}'$, $P_{22} = \begin{bmatrix} 2 & 0 & 1 & 0 \end{bmatrix}'$, $P_{23} = \begin{bmatrix} 2 & 0 & 1 & 1 \end{bmatrix}'$, $P_{24} = \begin{bmatrix} 2 & 0 & 1 & 2 \end{bmatrix}'$, $P_{25} = \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}'$, $P_{26} = \begin{bmatrix} 2 & 1 & 1 & 1 \end{bmatrix}'$, $P_{27} = \begin{bmatrix} 2 & 2 & 1 & 0 \end{bmatrix}'$, $P_{28} = \begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix}'$, $P_{29} = \begin{bmatrix} 2 & 2 & 1 & 2 \end{bmatrix}'$, $P_{30} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}'$, $P_{31} = \begin{bmatrix} 0 & 2 & 0 & 1 \end{bmatrix}'$, $P_{32} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}'$, $P_{33} = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}'$, $P_{34} = \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}'$, $P_{35} = \begin{bmatrix} 2 & 0 & 0 & 1 \end{bmatrix}'$, $P_{36} = \begin{bmatrix} 2 & 1 & 0 & 1 \end{bmatrix}'$, $P_{37} = \begin{bmatrix} 2 & 2 & 0 & 1 \end{bmatrix}'$.

Then, on computing BC and replacing the 9 combinations $(0,0)$, $(0,1)$, $\ldots$, $(2,2)$ under the first two columns of BC by 9 distinct symbols $0,1,\cdots,8$, we can obtain the $OA(3^4,(9)^1 3^{36},2)$. From Lemma 2, we know the matrix D consisting of the first five columns of the $OA(3^4,(9)^1 3^{36},2)$ is an $OA(3^4,(9)^1 3^4,3)$. Thus the array $OA(3^4,(9)^1 3^{36},2)$ is a $VOA(3^4;2,9^1 3^{36},D)$.

Theorem 2: A $VOA(m^5;2,(m^2)m^{m^4+\cdots+m^2},D)$ can be constructed for any prime power m, where D is an $OA(m^5,(m^2)m^{m^2+m+1},3)$.

Proof. Let $\tau = (m^2+m+2)$. Let $P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}'$, $P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix}'$, $P_i = \begin{bmatrix} 1 & \alpha_{i-2}^2 & 0 & 1 & \alpha_{i-2} \end{bmatrix}'$ for $3 \le i \le m+2$, where $\alpha_1,\ldots,\alpha_m$ are distinct elements of $GF(m)$, $P_j = \begin{bmatrix} 0 & f(\beta_{j-(m+2)},\xi_{j-(m+2)}) & 1 & \beta_{j-(m+2)} & \xi_{j-(m+2)} \end{bmatrix}'$ for $m+3 \le j \le m^2+m+2$, where f is an irreducible binary quadratic form over $GF(m)$ and $\beta_{j-(m+2)}$ and $\xi_{j-(m+2)}$ denote any two elements of $GF(m)$, $P_k = \begin{bmatrix} x_{1(k-\tau)} & x_{2(k-\tau)} & 1 & x_{4(k-\tau)} & x_{5(k-\tau)} \end{bmatrix}'$ for $m^2+m+3 \le k \le m^4+m+2$, where $x_{p(k-\tau)} \in GF(m)$ is the pth component of the vector P_k for $1 \le p \le 5$, $x_{3(k-\tau)} = 1$, and $P_k \ne P_j$ for $m+3 \le j \le m^2+m+2$, $P_q = \begin{bmatrix} x_{1(q-\tau)} & x_{2(q-\tau)} & 0 & 1 & x_{5(q-\tau)} \end{bmatrix}'$ for $m^4+m+3 \le q \le m^4+m^3+2$, where $x_{p(q-\tau)} \in GF(m)$ is the pth component of the vector P_q for $1 \le p \le 5$, $x_{3(q-\tau)} = 0$, $x_{4(q-\tau)} = 1$, and $P_q \ne P_i$ for $3 \le i \le m+2$, $P_u = \begin{bmatrix} x_{1(u-\tau)} & x_{2(u-\tau)} & 0 & 0 & 1 \end{bmatrix}'$ for $m^4+m^3+3 \le u \le m^4+m^3+m^2+1$, where $x_{p(u-\tau)} \in GF(m)$ is the pth component of the vector P_u for $1 \le p \le 5$, $x_{3(u-\tau)} = x_{4(u-\tau)} = 0$, $x_{5(u-\tau)} = 1$, and $P_u \ne P_2$.

Suppose P_{i_1}, P_{i_2} are any two of $P_1,\ldots,P_{m^4+m^3+m^2+1}$. The proof that the matrix $[P_{i_1} \vdots P_{i_2}]$ has full column rank follows as it did in Theorem 1. Thus we can derive the array $OA(m^5,(m^2)m^{m^4+m^3+m^2},2)$. We know the matrix D given by the first (m^2+m+2) columns of the $OA(m^5,(m^2)m^{m^4+m^3+m^2},2)$ is an $OA(m^5,(m^2)m^{m^2+m+1},3)$ from Lemma 3. Therefore, the array we derived is a $VOA(m^5;2,(m^2)m^{m^4+m^3+m^2},D)$.

Example 2: Let $m=2$ and $t=3$. Let B be an $2^5 \times 5$ matrix whose rows are all possible 5-tuples over $GF(2)$ and C be a 5×30 matrix. From Theorem 2, we can take

$$C = \begin{bmatrix} 1000010111111111111111000000000000000111111110000000111100110 \\ 0100010111111100000000111111110000000111100001111000110011101 \\ 0010010111000011110000111100001111000110011001100110101010111 \\ 0001010100110011001100110011001100110101010101010101111111000 \\ 0000011111111111111111111111111111111000000000000000000000000 \\ 0000110010101010101010101010101010101111111111111111000000000 \end{bmatrix}.$$

Then, we can obtain the array $VOA(2^5;2,4^1 2^{28},D)$ using the same methodology as used in Example 1, where the matrix D given by the first eight columns of the $VOA(2^5;2,4^1 2^{28},D)$ is an $OA(2^5,(4)^1 2^7,3)$.

Theorem 3: A $VOA(m^6;2,(m^2)^2 m^{m^5+m^4+m^3+m^2-m-1},D)$ can be constructed for any prime power m, where D is an $OA(m^6,(m^2)^2 m^{m+1},4)$.

Proof. Let $\tau = m+3$ and $1 \le p \le 6$. Let $P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}'$, $P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}'$, $P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}'$, $P_i = \begin{bmatrix} \alpha_{i-3} & \alpha_{i-3}^2 & \alpha_{i-3}^2 & \alpha_{i-3}^3 & 1 & \alpha_{i-3} \end{bmatrix}'$ for $4 \le i \le m+3$, where $\alpha_1,\ldots,\alpha_m$ are distinct elements of $GF(m)$, $P_j = \begin{bmatrix} x_{1(j-\tau)} & x_{2(j-\tau)} & x_{3(j-\tau)} & x_{4(j-\tau)} & 1 & x_{6(j-\tau)} \end{bmatrix}'$ for $m+4 \le j \le m^5+3$, where $x_{p(j-\tau)} \in GF(m)$ is the pth component of the vector P_j, $x_{5(j-\tau)} = 1$, and $P_j \ne P_i$ for $4 \le i \le m+3$, $P_k = \begin{bmatrix} x_{1(k-\tau)} & x_{2(k-\tau)} & x_{3(k-\tau)} & x_{4(k-\tau)} & 0 & 1 \end{bmatrix}'$ for $m^5+4 \le k \le m^5+m^4+2$, where $x_{p(k-\tau)} \in GF(m)$ is the pth component of the vector P_k, $x_{5(k-\tau)} = 0$, $x_{6(k-\tau)} = 1$, and $P_k \ne P_3$, $P_u = \begin{bmatrix} x_{1(u-\tau)} & x_{2(u-\tau)} & x_{3(u-\tau)} & 1 & 0 & 0 \end{bmatrix}'$ for $m^5+m^4+3 \le u \le m^5+m^4+m^3-m+2$, where $x_{p(u-\tau)} \in GF(m)$ is the pth component of the vector P_u, $x_{4(u-\tau)} = 1$, $x_{5(u-\tau)} = x_{6(u-\tau)} = 0$, and $(x_{1(u-\tau)}, x_{2(u-\tau)}) \ne (0,0)$, $P_v = \begin{bmatrix} x_{1(v-\tau)} & x_{2(v-\tau)} & 1 & 0 & 0 & 0 \end{bmatrix}'$ for $m^5+m^4+m^3-m+3 \le v \le m^5+m^4+m^3+m^2-m+1$, where $x_{p(v-\tau)} \in GF(m)$ is the pth component of the vector P_v, $x_{3(v-\tau)} = 1$, $x_{4(v-\tau)} = x_{5(v-\tau)} = x_{6(v-\tau)} = 0$, and $(x_{1(v-\tau)}, x_{2(v-\tau)}) \ne (0,0)$.

The proof that the matrix $[P_{i_1} \vdots P_{i_2}]$ has full column rank is perfectly analogous to the argument used in the proof of Theorem 1, where P_{i_1}, P_{i_2} are any two of $P_1,\ldots,P_{m^5+m^4+m^3+m^2-m+1}$. Then the array $OA(m^6,(m^2)^2 m^{m^5+m^4+m^3+m^2-m-1},2)$ can be obtained. By applying Lemma 4, we have that the matrix D, the first $(m+3)$ columns of the $OA(m^6,(m^2)^2 m^{m^5+m^4+m^3+m^2-m-1},2)$, is an $OA(m^6,(m^2)^2 m^{m+1},4)$. Thus we get a $VOA(m^6;2,(m^2)^2 m^{m^5+m^4+m^3+m^2-m-1},D)$.

Example 3: Let $m=2$ and $t=4$. Let B be an $2^6 \times 6$ matrix whose rows are all possible 6-tuples over $GF(2)$ and C be a 6×61 matrix. From Theorem 3, we can take

$$C = \begin{bmatrix} 1000010111111111111111000000000000000111111110000000111100110 \\ 0100010111111100000000111111110000000111100001111000110011101 \\ 0010010111000011110000111100001111000110011001100110101010111 \\ 0001010100110011001100110011001100110101010101010101111111000 \\ 0000011111111111111111111111111111111000000000000000000000000 \\ 0000110010101010101010101010101010101111111111111111000000000 \end{bmatrix}.$$

With the similar method presented in Example 1, we can then get the array $VOA(2^6;2,(4)^2 2^{57},D)$, more specifically, the matrix D, the first five columns of the $VOA(2^6;2,(4)^2 2^{57},D)$, is an $OA(2^6,(4)^2 2^3,4)$.

Theorem 4: A $VOA(m^7;2,(m^2)^2 m^{m^6+m^5+m^4+m^3+m^2-m-1},D)$ can be constructed if $m \ge 4$ is a power of two, where D

is an $OA(m^7, (m^2)^2 m^{m+1}, 5)$.

Proof. Let $\tau = m+3$. Let $P_1 = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \end{bmatrix}'$, $P_2 = \begin{bmatrix} 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \end{bmatrix}'$, $P_3 = \begin{bmatrix} 0&0&0&0&0&0&1 \end{bmatrix}'$, $P_i = \begin{bmatrix} \alpha_{i-3}^3 & \alpha_{i-3} & \alpha_{i-3}^3 & \alpha_{i-3}^4 & 1 & \alpha_{i-3}^2 & \alpha_{i-3} \end{bmatrix}'$ for $4 \le i \le m+3$, where $\alpha_1, \ldots, \alpha_m$ are distinct elements of $GF(m)$, $P_j = \begin{bmatrix} x_{1(j-\tau)} & \ldots & x_{4(j-\tau)} & 1 & x_{6(j-\tau)} & x_{7(j-\tau)} \end{bmatrix}'$ for $m+4 \le j \le m^6+3$, where $x_{p(j-\tau)} \in GF(m)$ is the pth component of the vector P_j for $1 \le p \le 7$, $x_{5(j-\tau)} = 1$, and $P_j \ne P_i$ for $4 \le i \le m+3$, $P_k = \begin{bmatrix} x_{1(k-\tau)} & x_{2(k-\tau)} & x_{3(k-\tau)} & x_{4(k-\tau)} & 0 & 1 & x_{7(k-\tau)} \end{bmatrix}'$ for $m^6+4 \le k \le m^6+m^5+3$, where $x_{p(k-\tau)} \in GF(m)$ is the pth component of the vector P_k for $1 \le p \le 7$, $x_{5(k-\tau)} = 0$, $x_{6(k-\tau)} = 1$, $P_u = \begin{bmatrix} x_{1(u-\tau)} & x_{2(u-\tau)} & x_{3(u-\tau)} & x_{4(u-\tau)} & 0 & 0 & 1 \end{bmatrix}'$ for $m^6+m^5+4 \le u \le m^6+m^5+m^4+2$, where $x_{p(u-\tau)} \in GF(m)$ is the pth component of the vector P_u for $1 \le p \le 7$, $x_{5(u-\tau)} = x_{6(u-\tau)} = 0$, $x_{7(u-\tau)} = 1$, and $P_u \ne P_3$, $P_v = \begin{bmatrix} x_{1(v-\tau)} & x_{2(v-\tau)} & x_{3(v-\tau)} & 1 & 0 & 0 & 0 \end{bmatrix}'$ for $m^6+m^5+m^4+3 \le v \le m^6+m^5+m^4+m^3-m+2$, where $x_{p(v-\tau)} \in GF(m)$ is the pth component of the vector P_v for $1 \le p \le 7$, $x_{4(v-\tau)} = 1$, $x_{5(v-\tau)} = x_{6(v-\tau)} = x_{7(v-\tau)} = 0$, and $(x_{1(v-\tau)}, x_{2(v-\tau)}) \ne (0,0)$. $P_w = \begin{bmatrix} x_{1(w-\tau)} & x_{2(w-\tau)} & 1 & 0 & 0 & 0 & 0 \end{bmatrix}'$ for $m^6+m^5+m^4+m^3-m+3 \le w \le m^6+m^5+m^4+m^3+m^2-m+1$, where $x_{p(w-\tau)} \in GF(m)$ is the pth component of the vector P_w for $1 \le p \le 7$, $x_{3(w-\tau)} = 1$, $x_{4(w-\tau)} = x_{5(w-\tau)} = x_{6(w-\tau)} = x_{7(w-\tau)} = 0$, and $(x_{1(w-\tau)}, x_{2(w-\tau)}) \ne (0,0)$.

Let P_{i_1}, P_{i_2} be any two of $P_1, \ldots, P_{m^6+m^5+m^4+m^3+m^2-m+1}$.

The matrix $[P_{i_1} \vdots P_{i_2}]$ has full column rank by similar arguments as in Theorem 1. Then we can construct the array $OA(m^7, (m^2)^2 m^{m^6+m^5+m^4+m^3+m^2-m-1}, 2)$. By Lemma 5, we know the matrix D formed by the first $(m+3)$ columns of the $OA(m^7, (m^2)^2 m^{m^6+m^5+m^4+m^3+m^2-m-1}, 2)$ is an $OA(m^7, (m^2)^2 m^{m+1}, 5)$. Consequently, a $VOA(m^7; 2, (m^2)^2 m^{m^6+m^5+m^4+m^3+m^2-m-1}, D)$ can be constructed.

From Theorem 4, we can construct a $VOA(4^7; 2, (16)^2 4^{5453}, D)$ when $m = 4$, where D is an $OA(4^7, (16)^2 4^5, 5)$.

3.2 Construction of VOAs by Fan-Construction

Theorem 5: Let $m \ge 2$ be a prime power, and let u, l and t be positive integers such that $u|m$ and $m \ge t-1$. For any positive integer $l \le t$, if there is an $OA(u, w_1^{k_1} w_2^{k_2} \cdots w_s^{k_s}, l)$, then a $VOA(m^t; l, m^m w_1^{k_1} w_2^{k_2} \cdots w_s^{k_s}, D)$ exists, where D is an $OA(m^t, m^m, t)$.

Proof. For stated values of m and t, an m-divisible $OA(m^t, m^m, t)$ exists from Lemma 7. By assumption, we know that an $OA(u, w_1^{k_1} w_2^{k_2} \cdots w_s^{k_s}, l)$ exists and $u|m$. Hence, the $OA(m, w_1^{k_1} w_2^{k_2} \cdots w_s^{k_s}, l)$ exists. By Lemma 6 (ii), we can obtain a $VOA(m^t; l, m^m w_1^{k_1} \cdots w_s^{k_s}, D)$, where D is an $OA(m^t, m^m, t)$.

Example 4: Let $m = 16$, $t = 4$, $u = 16$, and $l = 3$. From Lemma 7, an $OA(16^4, 16^{17}, 4)$ exists. Hence, a 16-divisible $OA(16^4, 16^{16}, 4)$ exists. From [28], we know that the array $OA(16, 4^1 2^3, 3)$ exists. We can obtain a $VOA(16^4; 3, 16^{16} 4^1 2^3, D)$ from Theorem 5, where D is an $OA(16^4, 16^{16}, 4)$.

Theorem 6: Let $m \ge 2$ and $u \ge 2$ be prime powers, and let l and t be positive integers such that $u^l|m$, $m \ge t-1$ and $u \ge l-1$. Then for any positive integer $l \le t$ a $VOA(m^t; l, m^m u^{u+1}, D)$ exists, where D is an $OA(m^t, m^m, t)$.

Proof. For stated values of m, u, t and l, both an m-divisible $OA(m^t, m^m, t)$ and an $OA(u^l, u^{u+1}, l)$ exist from Lemma 7. Since $u^l|m$ by assumption, we know that an $OA(u^l, u^{u+1}, l)$ implies the existence of an $OA(m, u^{u+1}, l)$. We now start with an m-divisible $OA(m^t, m^m, t)$ and apply Lemma 6 (ii) with an $OA(m, u^{u+1}, l)$. This gives a $VOA(m^t; l, m^m u^{u+1}, D)$, where D is an $OA(m^t, m^m, t)$.

Example 5: Let $m = 8$, $t = 5$, $u = 2$, and $l = 3$. From Lemma 7, both an $OA(8^5, 8^9, 5)$ and an $OA(2^3, 2^4, 3)$ exist. We know that an $OA(8^5, 8^9, 5)$ implies the existence of an 8-divisible $OA(8^5, 8^8, 5)$. Start with an 8-divisible $OA(8^5, 8^8, 5)$ and apply Lemma 6 (ii) with an $OA(2^3, 2^4, 3)$. This gives a $VOA(8^5; 3, 8^8 2^4, D)$, where D is an $OA(8^5, 8^8, 5)$.

4. Conclusion

VOA, a special form of VCA, has the potential for use in software testing, allowing the engineer to omit the parameter combinations known to not interact in order to reduce the number of tests required. To capture most fault in computer software testing, we study the construction of VOAs by Galois field and Fan-construction and obtain some VOAs with strength l ($l \ge 2$) containing strength greater than l. In the future, we will construct VOAs with more different numbers of levels.

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Qingjuan Zhang was born in Henan, China, in 1992. She is now a Ph.D. candidate of Guangzhou University. Her present research interests include Design of Experiment (DOE) and combinatorial design. Email: qingjuanzhang2021@163.com.

Shanqi Pang received the B.S. in mathematics from Henan Normal University, Xinxiang, China in 1986, and the PH.D. Degree in applied mathematics from Xidian University, Xi'an, China, in 2003. He is now a professor of Henan Normal University. His present research interests include Design of Experiment (DOE) and combination of design. He is a member of IEICE. Email: shanqipang@126.com.

Yuan Li He is now a professor of Guangzhou University. His present research interests include time series and nonparametric statistics. Email: mathly@gzhu.edu.cn.