# Exponent Function for Stationary Memoryless Channels with Input Cost at Rates above the Capacity 

Yasutada Oohama


#### Abstract

We consider the stationary memoryless channels with input cost. We prove that for transmission rates above the capacity the correct probability of decoding tends to zero exponentially as the block length $n$ of codes tends to infinity. In the case where both of channel input and output sets are finite, we determine the optimal exponent function on the above exponential decay of the correct probability. To derive this result we use a new technique called the recursive method, which is based on the information spectrum approach. The recursive method utilizes a certain recursive structure on the information spectrum quantities.


Keywords—Stationary memoryless channels, Strong converse theorem, Information spectrum approach

## I. Introduction

A certain class of noisy channels has a property that the error probability of decoding goes to one as the block length $n$ of transmitted codes tends to infinity at rates above the channel capacity. This property is called the strong converse property. In the case of DMCs without cost Arimoto [2] proved that the error probability of decoding goes to one exponentially and derived a lower bound of the exponent function. Subsequently, Dueck and Körner [3] determined the optimal exponent function for the error probability of decoding to go to one. They derived the result by using a combinatorial method base on the type of sequences [1]. The equality of the lower bound of Arimoto [2] to that of the optimal bound of Dueck and Körner [3] was proved by the author [4]. A simple derivation of the exponent function in the problem set up of quantum channel coding was given by Nagaoka [5], Hayashi and Nagaoka [6]. In the derivation they used the information spectrum method introduced by Han [7] and a min-max expression of the channel capacity.

In this paper, we determine the optimal exponent function on the correct probability of decoding at rates above capacity for DMCs with input cost. This result can be obtained by a method quite parallel with the method Dueck and Körner [3] used to obtain the optimal exponent function in the case without input cost. Instead of using their method, we use a new method based on the information spectrum method. A main contribution of this paper is that we establish a new powerful method to derive a tight exponent function at rates above the capacity for DMCs.

[^0]As we mentioned previously, there have been three different methods by Arimoto [2], Dueck and Körner [3] and Nagaoka [5], Hayashi and Nagaoka [6] to derive the result. Our method can be regarded as the fourth new method, having the following two merits:

1. Our method and the method of Nagaoka [5], Hayashi and Nagaoka [6] are based on the information spectrum method. Those two methods have a common advantage that they also work for the derivation of the exponent function for general memoryless channels(GMCs), where the channel input and outputs are real lines. On the other hand, the method of type used by Dueck and Körner [3] only works for DMCs where channel input and output sets are finite.
2. The recursive method is a general powerful tool to prove strong converse theorems for several coding problems in information theory. In fact, this method played important roles in deriving exponential strong converse exponent for communication systems treated in [8]-[12].
By the first merit, we derive a lower bound of the optimal exponent function for GMCs. This lower bound is thought to be useful for deriving explicit lower bounds of the optimal exponent functions for several examples of GMCs.

## II. Capacity Results for the Discrete Memoryless Channels with Input Cost

We consider a stationary discrete memoryless channel(DMC) with the input set $\mathcal{X}$ and the output set $\mathcal{Y}$. We assume that $\mathcal{X}$ and $\mathcal{Y}$ are finite sets. A case where $\mathcal{X}$ and $\mathcal{Y}$ are real lines will be treated in Section VI

The SDMC is specified by the following stochastic matrix:

$$
\begin{equation*}
W \triangleq\{W(y \mid x)\}_{(x, y) \in \mathcal{X} \times \mathcal{Y}} \tag{1}
\end{equation*}
$$

Let $X^{n}$ be a random variable taking values in $\mathcal{X}^{n}$. We write an element of $\mathcal{X}^{n}$ as $x^{n}=x_{1} x_{2} \cdots x_{n}$. Suppose that $X^{n}$ has a probability distribution on $\mathcal{X}^{n}$ denoted by $p_{X^{n}}=$ $\left\{p_{X^{n}}\left(x^{n}\right)\right\}_{x^{n} \in \mathcal{X}^{n}}$. Similar notations are adopted for other random variables. Let $Y^{n} \in \mathcal{Y}^{n}$ be a random variable obtained as the channel output by connecting $X^{n}$ to the input of channel. We write a conditional distribution of $Y^{n}$ on given $X^{n}$ as

$$
W^{n}=\left\{W^{n}\left(y^{n} \mid x^{n}\right)\right\}_{\left(x^{n}, y^{n}\right) \in \mathcal{X}^{n} \times \mathcal{Y}^{n}}
$$

Since the channel is memoryless, we have

$$
\begin{equation*}
W^{n}\left(y^{n} \mid x^{n}\right)=\prod_{t=1}^{n} W\left(y_{t} \mid x_{t}\right) \tag{2}
\end{equation*}
$$

Let $K_{n}$ be uniformly distributed random variables taking values in message sets $\mathcal{K}_{n}$.

The random variable $K_{n}$ is a message sent to the receiver. A sender transforms $K_{n}$ into a transmitted sequence $X^{n}$ using an encoder function and sends it to the receiver. In this paper we assume that the encoder function $\varphi^{(n)}$ is a deterministic encoder. In this case, $\varphi^{(n)}$ is a one-to-one mapping from $\mathcal{K}_{n}$ into $\mathcal{X}^{n}$. The joint probability mass function on $\mathcal{X}^{n} \times \mathcal{Y}^{n}$ is given by

$$
\operatorname{Pr}\left\{\left(X^{n}, Y^{n}\right)=\left(x^{n}, y^{n}\right)\right\}=\frac{1}{\left|\mathcal{K}_{n}\right|} \prod_{t=1}^{n} W\left(y_{t} \mid x_{t}(k)\right)
$$

where $x_{t}(k)=\left[\varphi^{(n)}(k)\right]_{t}, t=1,2, \cdots, n$ are the $t$-th components of $x^{n}=x^{n}(k)=\varphi^{(n)}(k)$ and $\left|\mathcal{K}_{n}\right|$ is a cardinality of the set $\mathcal{K}_{n}$. The decoding function at the receiver is denoted by $\psi^{(n)}$. This function is formally defined by $\psi^{(n)}: \mathcal{Y}^{n} \rightarrow \mathcal{K}_{n}$. Let $c: \mathcal{X} \rightarrow[0, \infty)$ be a cost function. The average cost on output of $\varphi^{(n)}$ must not exceed $\Gamma$. This condition is given by $\varphi^{(n)}\left(K_{n}\right) \in \mathcal{S}_{\Gamma}^{(n)}$, where

$$
\mathcal{S}_{\Gamma}^{(n)} \triangleq\left\{x^{n} \in \mathcal{X}^{n}: \frac{1}{n} \sum_{t=1}^{n} c\left(x_{t}\right) \leq \Gamma\right\}
$$

The average error probabilities of decoding at the receiver is defined by

$$
\begin{aligned}
\mathrm{P}_{\mathrm{e}}^{(n)} & =\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \triangleq \operatorname{Pr}\left\{\psi^{(n)}\left(Y^{n}\right) \neq K_{n}\right\} \\
& =1-\operatorname{Pr}\left\{\psi^{(n)}\left(Y^{n}\right)=K_{n}\right\}
\end{aligned}
$$

For $k \in \mathcal{K}_{n}$, set $\mathcal{D}(k) \triangleq\left\{y^{n}: \psi^{(n)}\left(y^{n}\right)=k\right\}$. The families of sets $\{\mathcal{D}(k)\}_{k \in \mathcal{K}_{n}}$ is called the decoding regions. Using the decoding region, $\mathrm{P}_{\mathrm{e}}^{(n)}$ can be written as

$$
\begin{aligned}
\mathrm{P}_{\mathrm{e}}^{(n)} & \left.=1-\frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} \operatorname{Pr}\left\{Y^{n} \in \mathcal{D}(k) \mid X^{n}=\varphi^{(n)}(k)\right)\right\} \\
& =1-\frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} \sum_{y^{n} \in \mathcal{D}(k)} W^{n}\left(y^{n} \mid \varphi^{(n)}(k)\right) \\
& =1-\frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \mid \varphi^{(n)}(k)\right)
\end{aligned}
$$

Set

$$
\mathrm{P}_{\mathrm{c}}^{(n)}=\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \triangleq 1-\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right)
$$

The quantity $\mathrm{P}_{\mathrm{c}}^{(n)}$ is called the average correct probability of decoding. This quantity has the following form

$$
\mathrm{P}_{\mathrm{c}}^{(n)}=\frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \mid \varphi^{(n)}(k)\right)
$$

For given $\varepsilon \in(0,1), R$ is $\varepsilon$-achievable under $\Gamma$ if for any $\delta>0$, there exist a positive integer $n_{0}=n_{0}(\varepsilon, \delta)$ and a sequence of pairs $\left\{\left(\varphi^{(n)}, \psi^{(n)}\right): \varphi^{(n)}\left(K_{n}\right) \in \mathcal{S}_{\Gamma}^{(n)}\right\}_{n=1}^{\infty}$ such that for any $n \geq n_{0}(\varepsilon, \delta)$,

$$
\begin{equation*}
\mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \leq \varepsilon, \quad \frac{1}{n} \log \left|\mathcal{K}_{n}\right| \geq R-\delta \tag{3}
\end{equation*}
$$

The supremum of all $\varepsilon$-achievable $R$ under $\Gamma$ is denoted by $C_{\text {DMC }}(\varepsilon, \Gamma \mid W)$. We set

$$
C_{\mathrm{DMC}}(\Gamma \mid W) \triangleq \inf _{\varepsilon \in(0,1)} C_{\mathrm{DMC}}(\varepsilon, \Gamma \mid W)
$$

which is called the channel capacity. The maximum error probability of decoding is defined by as follows:

$$
\begin{aligned}
\mathrm{P}_{\mathrm{e}, \mathrm{~m}}^{(n)} & =\mathrm{P}_{\mathrm{e}, \mathrm{~m}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
& \triangleq \max _{k \in \mathcal{K}_{n}} \operatorname{Pr}\left\{\psi^{(n)}\left(Y^{n}\right) \neq k \mid K_{n}=k\right\}
\end{aligned}
$$

Based on this quantity, we define the maximum capacity as follows. For a given $\varepsilon \in(0,1), R$ is $\varepsilon$-achievable under $\Gamma$, if for any $\delta>0$, there exist a positive integer $n_{0}=n_{0}(\varepsilon, \delta)$ and a sequence of pairs $\left\{\left(\varphi^{(n)}, \psi^{(n)}\right): \varphi^{(n)}\left(K_{n}\right) \in \mathcal{S}_{\Gamma}^{(n)}\right\}_{n=1}^{\infty}$ such that for any $n \geq n_{0}(\varepsilon, \delta)$,

$$
\begin{equation*}
\mathrm{P}_{\mathrm{e}, \mathrm{~m}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \leq \varepsilon, \quad \frac{1}{n} \log \left|\mathcal{K}_{n}\right| \geq R-\delta \tag{4}
\end{equation*}
$$

The supremum of all $\varepsilon$-achievable rates under $\Gamma$ is denoted by $C_{\mathrm{m}, \mathrm{DMC}}(\varepsilon, \Gamma \mid W)$. We set

$$
C_{\mathrm{m}, \mathrm{DMC}}(\Gamma \mid W)=\inf _{\varepsilon \in(0,1)} C_{\mathrm{m}, \mathrm{DMC}}(\varepsilon, \Gamma \mid W)
$$

which is called the maximum capacity of the DMC. Set

$$
\begin{equation*}
C(\Gamma \mid W)=\max _{\substack{p_{X} \in \mathcal{P}(\mathcal{X}): \\ \mathrm{E}_{p_{X}} c(X) \leq \Gamma}} I\left(p_{X}, W\right) \tag{5}
\end{equation*}
$$

where $\mathcal{P}(\mathcal{X})$ is a set of probability distribution on $\mathcal{X}$ and $I\left(p_{X}, W\right)$ stands for a mutual information between $X$ and $Y$ when input distribution of $X$ is $p_{X}$. The following is a well known result.

Theorem 1: For any DMC $W$, we have

$$
C_{\mathrm{m}, \mathrm{DMC}}(\Gamma \mid W)=C_{\mathrm{DMC}}(\Gamma \mid W)=C(\Gamma \mid W)
$$

Han [7] established the strong converse theorem for DMCs with input cost. His result is as follows.

Theorem 2 (Han [7]): If $R>C(\Gamma \mid W)$, then for any $\left\{\left(\varphi^{(n)}, \psi^{(n)}\right): \varphi^{(n)}\left(K_{n}\right) \in \mathcal{S}_{\Gamma}^{(n)}\right\}_{n=1}^{\infty}$ satisfying

$$
\frac{1}{n} \liminf _{n \rightarrow \infty} M_{n} \geq R
$$

we have

$$
\lim _{n \rightarrow \infty} \mathrm{P}_{\mathrm{e}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right)=1
$$

The following corollary immediately follows from this theorem.

Corollary 1: For each fixed $\varepsilon \in(0,1)$ and any DMC $W$, we have

$$
C_{\mathrm{m}, \mathrm{DMC}}(\varepsilon, \Gamma \mid W)=C_{\mathrm{DMC}}(\varepsilon, \Gamma \mid W)=C(\Gamma \mid W)
$$

To examine an asymptotic behavior of $\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)}\right)$ for large $n$ at $R>C(\Gamma \mid W)$, we define the following quantities:

$$
\begin{aligned}
& G^{(n)}(R, \Gamma \mid W) \\
& \triangleq \min _{\substack{\left(\varphi^{(n)}, \psi^{(n)}\right): \\
\varphi^{(n)}\left(K_{n}\right) \in \mathcal{S}_{\Gamma}^{(n)},(1 / n) \log M_{n} \geq R}}\left(-\frac{1}{n}\right) \log \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
& G^{*}(R, \Gamma \mid W) \triangleq \lim _{n \rightarrow \infty} G^{(n)}(R, \Gamma \mid W)
\end{aligned}
$$

On the above exponent functions, we have the following property.

Property 1:
a) By definition we have that for each fixed $n \geq 1$, $G^{(n)}(R, \Gamma \mid W)$ is a monotone increasing function of $R \geq 0$ and satisfies $G^{(n)}(R, \Gamma \mid W) \leq R$.
b) The sequence $\left\{G^{(n)}(R, \Gamma \mid W)\right\}_{n \geq 1}$ of exponent functions satisfies the following subadditivity property:

$$
\begin{align*}
& G^{(n+m)}(R, \Gamma \mid W) \\
\leq & \frac{n G^{(n)}(R, \Gamma \mid W)+m G^{(m)}(R, \Gamma \mid W)}{n+m} \tag{6}
\end{align*}
$$

from which we have that $G^{*}(R, \Gamma \mid W)$ exists and is equal to $\inf _{n \geq 1} G^{(n)}(R, \Gamma \mid W)$.
c) For fixed $R>0$, the function $G^{*}(R, \Gamma \mid W)$ is a monotone decreasing function of $\Gamma$. For fixed $\Gamma>\Gamma_{0}=$ $\min _{x \in \mathcal{X}} c(x)$, the function $G^{*}(R, \Gamma \mid W)$ a monotone increasing function of $R$ and satisfies

$$
\begin{equation*}
G^{*}(R, \Gamma \mid W) \leq R \tag{7}
\end{equation*}
$$

d) The function $G^{*}(R, \Gamma \mid W)$ is a convex function of $(R, \Gamma)$.
Proof of Property 1 is given in Appendix $A$

## III. Main Result

In this section we state our main result. Define

$$
\begin{aligned}
G_{\mathrm{DK}}(R, \Gamma \mid W) \triangleq \min _{\substack{\left.q_{X Y} \in \mathcal{P} \mathcal{X} \times \mathcal{Y}\right): \\
\mathrm{E}_{q_{X}}[c(X)] \leq \Gamma}}\{ & {\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+} } \\
& \left.+D\left(q_{Y \mid X} \| W \mid q_{X}\right)\right\}
\end{aligned}
$$

where $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is the set of joint probability distributions on $\mathcal{X} \times \mathcal{Y},[t]^{+}=\max \{0, t\}$, and

$$
\begin{aligned}
I\left(q_{X}, q_{Y \mid X}\right) & =\mathrm{E}_{q}\left[\log \frac{q_{Y \mid X}(Y \mid X)}{q_{Y}(Y)}\right] \\
D\left(q_{Y \mid X} \| W \mid q_{X}\right) & =\mathrm{E}_{q}\left[\log \frac{q_{Y \mid X}(Y \mid X)}{W(Y \mid X)}\right]
\end{aligned}
$$

Using the standard method developed by Csiszár and Körner [1], we can prove the following theorem.

Theorem 3: For any $R>0$,

$$
G^{*}(R, \Gamma \mid W) \leq G_{\mathrm{DK}}(R, \Gamma \mid W)
$$

Proof of this theorem is given in Appendix B Let $\Gamma_{\text {max }} \triangleq$ $\max _{x \in \mathcal{X}} c(x)$. The case $\Gamma \geq \Gamma_{\max }$ corresponds to the case without cost. In this case Dueck and Körner [3] show that

$$
G^{*}(R, \Gamma \mid W)=G_{\mathrm{DK}}(R, \Gamma \mid W)
$$

They derived the bound $G^{*}(R, \Gamma \mid W) \leq G_{\mathrm{DK}}(R, \Gamma \mid W)$ by using a combinatorial method based on the type of sequences. Our method to prove Theorem 3 is different from their method since we do not use a particular structure of types.

We next derive a lower bound of $G^{*}(R, \Gamma \mid W)$. To this end we define several quantities. Define

$$
\begin{aligned}
& \Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right) \\
& \triangleq \log \left[\sum_{(x, y) \in \mathcal{X} \times \mathcal{Y}} q_{X}(x) W(y \mid x) \frac{W^{\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}}{Q^{\lambda}(y)}\right] \\
& \Omega^{(\mu, \lambda)}(W) \triangleq \max _{q_{X} \in \mathcal{P}(\mathcal{X})} \min _{Q \in \mathcal{P}(\mathcal{Y})} \Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right) \\
& G^{(\mu, \lambda)}(R, \Gamma \mid W) \triangleq \frac{\lambda(R-\mu \Gamma)-\Omega^{(\mu, \lambda)}(W)}{1+\lambda} \\
& G(R, \Gamma \mid W) \triangleq \sup _{\mu, \lambda \geq 0} G^{(\mu, \lambda)}(R, \Gamma \mid W)
\end{aligned}
$$

Our main result is the following.
Theorem 4: For any DMC $W$, we have

$$
\begin{equation*}
G^{*}(R, \Gamma \mid W) \geq G(R, \Gamma \mid W) \tag{8}
\end{equation*}
$$

Proof of this theorem will be given in Section IV Arimoto [2] derived a lower bound of $G^{*}(R, \Gamma \mid W)$, which we denote by $G_{\mathrm{AR}}(R, \Gamma \mid W)$. To describe this exponent function we define some functions. For $\lambda \in[0,1)$, define

$$
\begin{aligned}
& J^{(\mu, \lambda)}\left(q_{X} \mid W\right) \\
& \triangleq \log \sum_{y \in \mathcal{Y}}\left[\sum_{x \in \mathcal{X}} q_{X}(x)\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}}\right]^{1-\lambda} \\
& G_{\mathrm{AR}}^{(\mu, \lambda)}\left(R, \Gamma, q_{X} \mid W\right) \triangleq \lambda(R-\mu \Gamma)-J^{(\mu, \lambda)}\left(q_{X} \mid W\right) \\
& G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W) \triangleq \min _{q_{X} \in \mathcal{P}(\mathcal{X})} G_{\mathrm{AR}}^{(\mu, \lambda)}\left(R, \Gamma, q_{X} \mid W\right)
\end{aligned}
$$

Furthermore, set

$$
\begin{aligned}
& G_{\mathrm{AR}}(R, \Gamma \mid W) \triangleq \sup _{\substack{\mu \geq 0, \lambda \in[0,1)}} G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W) \\
= & \sup _{\substack{\mu \geq 0, \lambda \in[0,1)}} \min _{q_{X} \in \mathcal{P}(\mathcal{X})} G_{\mathrm{AR}}^{(\lambda)}\left(R, \Gamma, q_{X} \mid W\right) \\
= & \sup _{\substack{\mu \geq 0, \lambda \in[0,1)}}\left[\lambda(R-\mu \Gamma)-\max _{q_{X} \in \mathcal{P}(\mathcal{X})} J^{(\mu, \lambda)}\left(q_{X} \mid W\right)\right] .
\end{aligned}
$$

Then we have the following proposition.
Proposition 1: For any DMC $W$ and for any $\mu, \lambda \geq 0$, we have the following:

$$
\begin{equation*}
G^{(\mu, \lambda)}(R, \Gamma \mid W)=G_{\mathrm{AR}}^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}(R, \Gamma \mid W) \tag{9}
\end{equation*}
$$

In particular, we have

$$
\begin{equation*}
G(R, \Gamma \mid W)=G_{\mathrm{AR}}(R, \Gamma \mid W) \tag{10}
\end{equation*}
$$

Proof of this proposition is given in Section V We next state a relation between $G_{\mathrm{AR}}(R, \Gamma \mid W)$ and $G_{\mathrm{DK}}(R, \Gamma \mid W)$. To this end we present a lemma stating that $G_{\mathrm{DK}}(R, \Gamma \mid W)$ has two parametric expressions. For $\mu>0$, we define

$$
\begin{align*}
& G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) \\
& \begin{array}{l}
\triangleq \min _{q}\left\{\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}+D\left(q_{Y \mid X}| | W \mid q_{X}\right)\right. \\
\\
\left.\quad-\mu\left(\Gamma-\mathrm{E}_{q_{X}}[c(X)]\right)\right\} .
\end{array}
\end{align*}
$$

For $\mu, \lambda \geq 0$, we define

$$
\begin{align*}
& G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W) \\
& \begin{array}{l}
\triangleq \min _{q}\left\{\lambda\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]-\mu \Gamma+\mu \mathrm{E}_{q_{X}}[c(X)]\right. \\
\left.\quad+D\left(q_{Y \mid X}| | W \mid q_{X}\right)\right\}
\end{array}
\end{align*}
$$

Then we have the following lemma.
Lemma 1: For any $R>0$, we have

$$
\begin{equation*}
G_{\mathrm{DK}}(R, \Gamma \mid W)=\max _{\mu \geq 0} G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) \tag{13}
\end{equation*}
$$

For any $\mu \geq 0$, any $R>0$, we have

$$
\begin{equation*}
G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)=\max _{0 \leq \lambda \leq 1} G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W) \tag{14}
\end{equation*}
$$

The two equalities (13) and (14) imply that

$$
\begin{equation*}
G_{\mathrm{DK}}(R, \Gamma \mid W)=\max _{\substack{\mu \geq 0 \\ \lambda \in[0,1]}} G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W) . \tag{15}
\end{equation*}
$$

Proof of this lemma will be given in Appendix C. The following proposition states that the two quantities $G_{\mathrm{AR}}(R, \Gamma \mid W)$ and $G_{\mathrm{DK}}(R, \Gamma \mid W)$ match.

Proposition 2: For any $\mu, \lambda \geq 0$, we have the following:

$$
\begin{equation*}
G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W)=G_{\mathrm{DK}}^{(\mu \lambda, \lambda)}(R, \Gamma \mid W) \tag{16}
\end{equation*}
$$

In particular, we have

$$
\begin{equation*}
G_{\mathrm{AR}}(R, \Gamma \mid W)=G_{\mathrm{DK}}(R, \Gamma \mid W) \tag{17}
\end{equation*}
$$

Proof of this proposition is given in Section $\bar{\square}$ From Theorems 3, 4 and Propositions 1, 2, we immediately obtain the following theorem.

Theorem 5: For any DMC $W$, we have

$$
\begin{align*}
& G^{*}(R, \Gamma \mid W) \\
= & G(R, \Gamma \mid W)=G_{\mathrm{AR}}(R, \Gamma \mid W)=G_{\mathrm{DK}}(R, \Gamma \mid W) . \tag{18}
\end{align*}
$$

## IV. Proof of the Results

We first prove the following lemma.
Lemma 2: For any $\eta>0$ and for any $\left(\varphi^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left|\mathcal{K}_{n}\right| \geq R$, we have

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \leq p_{X^{n} Y^{n}}\{ \\
& \left.R \leq \frac{1}{n} \log \frac{W^{n}\left(Y^{n} \mid X^{n}\right)}{Q_{Y^{n}}\left(Y^{n}\right)}+\eta, \Gamma \geq \frac{1}{n} c\left(X^{n}\right)\right\}+\mathrm{e}^{-n \eta} . \tag{19}
\end{align*}
$$

In (19) we can choose any probability distribution $Q_{Y^{n}}$ on $\mathcal{Y}^{n}$.

Proof: For $x^{n}$ in $^{n}$, set

$$
\mathcal{A}\left(x^{n}\right) \triangleq\left\{y^{n}: W^{n}\left(y^{n} \mid x^{n}\right) \geq\left|\mathcal{K}_{n}\right| \mathrm{e}^{-n \eta} Q_{Y^{n}}\left(y^{n}\right)\right\}
$$

Let $\overline{\mathcal{A}\left(x^{n}\right)}$ stand for $\mathcal{Y}^{n}-\mathcal{A}\left(x^{n}\right)$. Then we have the following:

$$
\begin{aligned}
\mathrm{P}_{\mathrm{c}}^{(n)}= & \frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \cap \mathcal{A}\left(\varphi^{(n)}(k)\right) \mid \varphi^{(n)}(k)\right) \\
& +\frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \cap \overline{\mathcal{A}\left(\varphi^{(n)}(k)\right)} \mid \varphi^{(n)}(k)\right) \\
\leq & \Delta_{0}+\Delta_{1}
\end{aligned}
$$

where

$$
\begin{aligned}
& \Delta_{0} \triangleq \frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{A}\left(\varphi^{(n)}(k)\right) \mid \varphi^{(n)}(k)\right) \\
& \Delta_{1} \triangleq \frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \cap \overline{\mathcal{A}\left(\varphi^{(n)}(k)\right)} \mid \varphi^{(n)}(k)\right) .
\end{aligned}
$$

On the quantity $\Delta_{0}$, we have

$$
\begin{gather*}
\Delta_{0} \stackrel{(\mathrm{a})}{=} p_{X^{n} Y^{n}}\left\{\frac{1}{n} \log \left|\mathcal{K}_{n}\right| \leq \frac{1}{n} \log \frac{W^{n}\left(Y^{n} \mid X^{n}\right)}{Q_{Y^{n}}\left(Y^{n}\right)}+\eta\right\} \\
\stackrel{(\mathrm{b})}{=} p_{X^{n} Y^{n}}\left\{\frac{1}{n} \log \left|\mathcal{K}_{n}\right| \leq \frac{1}{n} \log \frac{W^{n}\left(Y^{n} \mid X^{n}\right)}{Q_{Y^{n}}\left(Y^{n}\right)}+\eta\right. \\
\left.\Gamma \geq \frac{1}{n} c\left(X^{n}\right)\right\} \\
\stackrel{(c)}{\leq} p_{X^{n} Y^{n}}\left\{R \leq \frac{1}{n} \log \frac{W^{n}\left(Y^{n} \mid X^{n}\right)}{Q_{Y^{n}}\left(Y^{n}\right)}+\eta\right. \\
\left.\Gamma \geq \frac{1}{n} c\left(X^{n}\right)\right\} \tag{20}
\end{gather*}
$$

Step (a) follows from the definition of $\Delta$. Step (b) follows from $X^{n}=\varphi^{(n)}\left(K_{n}\right) \in \mathcal{S}_{\Gamma}^{(n)}$. Step (c) follows from ( $1 / n$ ) $\log \left|\mathcal{K}_{n}\right| \geq R$. Hence it suffices to show $\Delta_{1} \leq \mathrm{e}^{-n \eta}$ to prove Lemma 2 We have the following chain of inequalities:

$$
\begin{aligned}
& \Delta_{1} \stackrel{(\mathrm{a})}{\leq} \frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}}\left|\mathcal{K}_{n}\right| \mathrm{e}^{-n \eta} Q_{Y^{n}}\left(\mathcal{D}(k) \cap \overline{\mathcal{A}\left(\varphi^{(n)}(k)\right)}\right) \\
& \leq \mathrm{e}^{-n \eta} \sum_{k \in \mathcal{K}_{n}} Q_{Y^{n}}(\mathcal{D}(k))=\mathrm{e}^{-n \eta} Q_{Y^{n}}\left(\bigcup_{k \in \mathcal{K}_{n}} \mathcal{D}(k)\right) \\
& \leq \mathrm{e}^{-n \eta}
\end{aligned}
$$

Step (a) follows from that for every $y^{n} \in \mathcal{D}(k) \cap \overline{\mathcal{A}\left(\varphi^{(n)}(k)\right)}$, we have $W^{n}\left(y^{n} \mid \varphi^{(n)}(k)\right)<\mathrm{e}^{-n \eta}\left|\mathcal{K}_{n}\right| Q_{Y^{n}}\left(y^{n}\right)$.

From Lemma 2 we have the following lemma
Lemma 3: For any $\eta>0$ and for any $\left(\varphi^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left|\mathcal{K}_{n}\right| \geq R$, we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \leq p_{X^{n} Y^{n}}\{ \\
& \left.R \leq \frac{1}{n} \sum_{t=1}^{n} \log \frac{W\left(Y_{t} \mid X_{t}\right)}{Q_{t}\left(Y_{t}\right)}+\eta, \Gamma \geq \frac{1}{n} \sum_{t=1}^{n} c\left(X_{t}\right)\right\}+\mathrm{e}^{-n \eta}
\end{aligned}
$$

Proof: In 19) in Lemma2 we choose $Q_{Y^{n}}$ having the form

$$
Q_{Y^{n}}\left(Y^{n}\right)=\prod_{t=1}^{n} Q_{t}\left(Y_{t}\right)
$$

Then from the bound (19) in Lemma 2, we obtain

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \leq p_{X^{n} Y^{n}}\{ \\
& \left.R \leq \frac{1}{n} \sum_{t=1}^{n} \log \frac{W\left(Y_{t} \mid X_{t}\right)}{Q_{t}\left(Y_{t}\right)}+\eta, \Gamma \geq \frac{1}{n} \sum_{t=1}^{n} c\left(X_{t}\right)\right\}+\mathrm{e}^{-n \eta}
\end{aligned}
$$

completing the proof.
We use the following lemma, which is well known as the Cramèr's bound in the large deviation principle.

Lemma 4: For any real valued random variable $Z$ and any $\theta>0$, we have

$$
\operatorname{Pr}\{Z \geq a\} \leq \exp [-(\theta a-\log \mathrm{E}[\exp (\theta Z)])]
$$

Here we define a quantity which serves as an exponential upper bound of $\mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right)$. Let $\mathcal{P}^{(n)}(W)$ be a set of all probability distributions $p_{X^{n} Y^{n}}$ on $\mathcal{X}^{n} \times \mathcal{Y}^{n}$ having the form:

$$
\begin{aligned}
& p_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right) \\
= & \prod_{t=1}^{n} p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right)
\end{aligned}
$$

For simplicity of notation we use the notation $p^{(n)}$ for $p_{X^{n} Y^{n}}$ $\in \mathcal{P}^{(n)}(W)$. For $p^{(n)} \in \mathcal{P}^{(n)}(W)$ and $Q^{n}=\left\{Q_{t}\right\}_{t=1}^{n} \in$ $\mathcal{P}^{n}(\mathcal{Y})$, we define

$$
\Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right) \triangleq \log \mathrm{E}_{p^{(n)}}\left[\prod_{t=1}^{n} \frac{W^{\lambda}\left(Y_{t} \mid X_{t}\right) \mathrm{e}^{-\mu \lambda c\left(X_{t}\right)}}{Q_{t}^{\lambda}\left(Y_{t}\right)}\right]
$$

By Lemmas 3 and 4 we have the following proposition.
Proposition 3: For any $\lambda>0$, any $Q^{n} \in \mathcal{P}^{n}(\mathcal{Y})$, and any $\left(\varphi^{(n)}, \psi^{(n)}\right)$ satisfying $(1 / n) \log \left|\mathcal{K}_{n}\right| \geq R$, we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
\leq & 2 \exp \left\{-n \frac{\lambda(R-\mu \Gamma)-\frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)}{1+\lambda}\right\}
\end{aligned}
$$

for some $p^{(n)} \in \mathcal{P}^{(n)}(W)$ and for any $Q^{n} \in \mathcal{P}^{n}(\mathcal{Y})$.

Proof: Under the condition $(1 / n) \log \left|\mathcal{K}_{n}\right| \geq R$, we have the following chain of inequalities:

$$
\begin{align*}
& \quad \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi \mid W\right) \stackrel{(\mathrm{a})}{\leq} p_{X^{n} Y^{n}}\{ \\
& \left.R \leq \frac{1}{n} \sum_{t=1}^{n} \log \frac{W\left(Y_{t} \mid X_{t}\right)}{Q_{t}\left(Y_{t}\right)}+\eta, \Gamma \geq \frac{1}{n} \sum_{t=1}^{n} c\left(X_{t}\right)\right\} \\
& +\mathrm{e}^{-n \eta} \\
& \leq p_{X^{n} Y^{n}}\left\{(R-\mu \Gamma)-\eta \leq \frac{1}{n} \sum_{t=1}^{n} \log \left[\frac{W\left(Y_{t} \mid X_{t}\right)}{Q_{t}\left(Y_{t}\right)}\right]\right. \\
& \left.\quad-\frac{\mu}{n} \sum_{t=1}^{n} c\left(X_{t}\right)\right\}+\mathrm{e}^{-n \eta} \\
& \begin{array}{l}
\text { (b) } \\
\leq \\
\quad
\end{array} \quad+\mathrm{e}^{-n \eta}\left[n\left\{-\lambda(R-\mu \Gamma)+\lambda \eta+\frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)\right\}\right]
\end{align*}
$$

Step (a) follows from Lemma 3 Step (b) follows from Lemma 4. We choose $\eta$ so that

$$
\begin{equation*}
-\eta=-\lambda(R-\mu \Gamma)+\lambda \eta+\frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right) \tag{22}
\end{equation*}
$$

Solving (22) with respect to $\eta$, we have

$$
\eta=\frac{\lambda(R-\mu \Gamma)-\frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)}{1+\lambda}
$$

For this choice of $\eta$ and (21), we have

$$
\begin{aligned}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \leq 2 \mathrm{e}^{-n \eta} \\
= & 2 \exp \left\{-n \frac{\lambda(R-\mu \Gamma)-\frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)}{1+\lambda}\right\},
\end{aligned}
$$

completing the proof.
Set

$$
\begin{aligned}
& \bar{\Omega}^{(\mu, \lambda)}(W) \\
\triangleq & \sup _{n \geq 1 p^{(n)} \in \mathcal{P}^{(n)}(W)} \max _{Q^{n} \in \mathcal{P}^{n}(\mathcal{Y})} \min _{n} \frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right) .
\end{aligned}
$$

By the above definition of $\bar{\Omega}^{(\mu, \lambda)}(W)$ and Proposition 3, we have

$$
\begin{equation*}
G^{(n)}(R, \Gamma \mid W) \geq \frac{\lambda(R-\mu \Gamma)-\bar{\Omega}^{(\mu, \lambda)}(W)}{1+\lambda}-\frac{1}{n} \log 2 \tag{23}
\end{equation*}
$$

Then from (23), we obtain the following corollary.
Corollary 2: For any $\mu, \lambda>0$, we have

$$
G^{*}(R, \Gamma \mid W) \geq \frac{\lambda(R-\mu \Gamma)-\bar{\Omega}^{(\mu, \lambda)}(W)}{1+\lambda}
$$

We shall call $\bar{\Omega}^{(\mu, \lambda)}(W)$ the communication potential. The above corollary implies that the analysis of $\bar{\Omega}^{(\mu, \lambda)}(W)$ leads to an establishment of a strong converse theorem for the DMC.

In the following argument we drive an explicit upper bound of $\bar{\Omega}^{(\mu, \lambda)}(W)$. For each $t=1,2, \cdots, n$, define the function of $\left(x_{t}, y_{t}\right) \in \mathcal{X} \times \mathcal{Y}$ by

$$
f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) \triangleq \frac{W^{\lambda}\left(y_{t} \mid x_{t}\right) \mathrm{e}^{-\mu \lambda c\left(x_{t}\right)}}{Q_{t}^{\lambda}\left(y_{t}\right)}
$$

For each $t=1,2, \cdots, n$, we define the probability distribution

$$
p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)} \triangleq\left\{p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)}\left(x^{t}, y^{t}\right)\right\}_{\left(x^{t}, y^{t}\right) \in \mathcal{X}^{t} \times \mathcal{Y}^{t}}
$$

by

$$
\begin{aligned}
& p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)}\left(x^{t}, y^{t}\right) \\
\triangleq & C_{t}^{-1} p_{X^{t} Y^{t}}\left(x^{t}, y^{t}\right) \prod_{i=1}^{t} f_{Q_{i}}^{(\mu, \lambda)}\left(x_{i}, y_{i}\right) \\
= & C_{t}^{-1} p_{X^{t}}\left(x^{t}\right) \prod_{i=1}^{t}\left\{W\left(y_{i} \mid x_{i}\right) f_{Q_{i}}^{(\mu, \lambda)}\left(x_{i}, y_{i}\right)\right\}
\end{aligned}
$$

where

$$
C_{t} \triangleq \mathrm{E}_{p_{X^{t} Y^{t}}}\left[\prod_{i=1}^{t} f_{Q_{i}}^{(\mu, \lambda)}\left(X_{i}, Y_{i}\right)\right]
$$

are constants for normalization. For each $t=1,2, \cdots, n$, set

$$
\begin{equation*}
\Phi_{t, Q^{t}}^{(\mu, \lambda)} \triangleq C_{t} C_{t-1}^{-1} \tag{24}
\end{equation*}
$$

where we define $C_{0}=1$. Then we have the following lemma.

## Lemma 5:

$$
\begin{equation*}
\Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)=\sum_{t=1}^{n} \log \Phi_{t, Q^{t}}^{(\mu, \lambda)} \tag{25}
\end{equation*}
$$

Proof: From (24) we have

$$
\begin{equation*}
\log \Phi_{t, Q^{t}}^{(\mu, \lambda)}=\log C_{t}-\log C_{t-1} \tag{26}
\end{equation*}
$$

Furthermore, by definition we have

$$
\begin{equation*}
\Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)=\log C_{n}, C_{0}=1 \tag{27}
\end{equation*}
$$

From (26) and (27), 25) is obvious.
The following lemma is useful for the computation of $\Phi_{t, Q^{t}}^{(\mu, \lambda)}$ for $t=1,2, \cdots, n$.

Lemma 6: For each $t=1,2, \cdots, n$, and for any $\left(x^{t}, y^{t}\right) \in$ $\mathcal{X}^{t} \times \mathcal{Y}^{t}$ we have

$$
\begin{align*}
& p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)}\left(x^{t}, y^{t}\right) \\
& =\left(\Phi_{t, Q^{t}}^{(\mu, \lambda)}\right)^{-1} p_{X^{t-1} Y^{t-1} ; Q^{t-1}}^{(\mu, \lambda)}\left(x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) . \tag{28}
\end{align*}
$$

Furthermore, we have

$$
\begin{align*}
& \Phi_{t, Q^{t}}^{(\mu, \lambda)}=\sum_{x^{t}, y^{t}} p_{X^{t-1} Y^{t-1} ; Q^{t-1}}^{(\mu, \lambda)}\left(x^{t-1}, y^{t-1}\right) \\
& \quad \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) \tag{29}
\end{align*}
$$

Proof: By the definition of $p_{X^{t} Y^{t} ; Q^{t}}^{(\mu,,)}\left(x^{t}, y^{t}\right), t=1,2$, $\cdots, n$, we have

$$
\begin{align*}
& p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)}\left(x^{t}, y^{t}\right) \\
= & C_{t}^{-1} p_{X^{t}}\left(x^{t}\right) \prod_{i=1}^{t}\left\{W\left(y_{i} \mid x_{i}\right) f_{Q_{i}}^{(\mu, \lambda)}\left(x_{i}, y_{i}\right)\right\} . \tag{30}
\end{align*}
$$

Then we have the following chain of equalities:

$$
\begin{align*}
& p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)}\left(x^{t}, y^{t}\right) \\
\stackrel{(\mathrm{a})}{=} & C_{t}^{-1} p_{X^{t}\left(x^{t}\right)} \prod_{i=1}^{t}\left\{W\left(y_{i} \mid x_{i}\right) f_{Q_{i}}^{(\mu, \theta)}\left(x_{i}, y_{i}\right)\right\} \\
= & C_{t}^{-1} p_{X^{t-1}}\left(x^{t-1}\right) \prod_{i=1}^{t-1}\left\{W\left(y_{i} \mid x_{i}\right) f_{Q_{i}}^{(\mu, \lambda)}\left(x_{i}, y_{i}\right)\right\} \\
& \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) \\
\stackrel{(\mathrm{b})}{=} & C_{t}^{-1} C_{t-1} p_{X^{t-1} Y^{t-1} ; Q^{t-1}}^{(\mu, \lambda)}\left(x^{t-1}, y^{t-1}\right) \\
& \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) \\
= & \left(\Phi_{t, Q^{t}}^{(\mu, \lambda)}\right)^{-1} p_{X^{t-1} Y^{t-1} ; Q^{t-1}}^{(\mu, \lambda)}\left(x^{t-1}, y^{t-1}\right) \\
& \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) . \tag{31}
\end{align*}
$$

Steps (a) and (b) follow from (30). From (31), we have

$$
\begin{align*}
& \Phi_{t, Q^{t}}^{(\mu, \lambda)} p_{X^{t} Y^{t} ; Q^{t}}^{(\mu, \lambda)}\left(x^{t}, y^{t}\right)  \tag{32}\\
= & p_{X^{t-1} Y^{t-1} ; Q^{t-1}}^{(\mu, \lambda)}\left(x^{t-1}, y^{t-1}\right) \\
& \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right) \tag{33}
\end{align*}
$$

Taking summations of (32) and (33) with respect to $x^{t}, y^{t}$, we obtain

$$
\begin{aligned}
\Phi_{t, Q^{t}}^{(\mu, \lambda)}= & \sum_{x^{t}, y^{t}} p_{X^{t-1} Y^{t-1} ; Q^{t-1}}^{(\mu, \lambda)}\left(x^{t-1}, y^{t-1}\right) \\
& \times p_{X_{t} \mid X^{t-1}}\left(x_{t} \mid x^{t-1}\right) W\left(y_{t} \mid x_{t}\right) f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right)
\end{aligned}
$$

completing the proof.
We set

$$
=\sum_{x^{t-1}, y^{t-1}} p_{X_{t} ; Q^{t-1}}^{(\mu, \lambda)}\left(x_{t}\right) .
$$

Then by (29) in Lemma6 and the definition of $f_{Q_{t}}^{(\mu, \lambda)}\left(x_{t}, y_{t}\right)$, we have

$$
\begin{align*}
\Phi_{t, Q^{t}}^{(\mu, \lambda)}= & \sum_{x_{t}, y_{t}} p_{X_{t} ; Q^{t-1}}^{(\mu, \lambda)}\left(x_{t}\right) W\left(y_{t} \mid x_{t}\right) \\
& \times \frac{W^{\lambda}\left(y_{t} \mid x_{t}\right) \mathrm{e}^{-\mu \lambda c\left(x_{t}\right)}}{Q_{t}^{\lambda}\left(y_{t}\right)} \tag{34}
\end{align*}
$$

The following proposition is a mathematical core to prove our main result.

Proposition 4: For any $\lambda>0$, we have

$$
\bar{\Omega}^{(\mu, \lambda)}(W) \leq \Omega^{(\mu, \lambda)}(W)
$$

Proof: We first observe that by (25) in Lemma 5 and (34), we have

$$
\begin{gather*}
\Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right)=\sum_{t=1}^{n} \log \Phi_{t, Q^{t}}^{(\mu, \lambda)}  \tag{35}\\
\Phi_{t, Q^{t}}^{(\mu, \lambda)}=\sum_{x_{t}, y_{t}} p_{X_{t} ; Q^{t-1}}^{(\mu, \lambda)}\left(x_{t}\right) W\left(y_{t} \mid x_{t}\right) \\
\times \frac{W^{\lambda}\left(y_{t} \mid x_{t}\right) \mathrm{e}^{-\mu \lambda c\left(x_{t}\right)}}{Q_{t}^{\lambda}\left(y_{t}\right)} \tag{36}
\end{gather*}
$$

In (36), we set $q_{X_{t}}\left(x_{t}\right)=p_{X_{t} ; Q^{t-1}}^{(\mu, \lambda)}\left(x_{t}\right)$. Note that $q_{X_{t}}$ is a function of $Q^{t-1}$. We define a joint distribution $q_{t}=q_{X_{t} Y_{t}}$ on $\mathcal{X} \times \mathcal{Y}$ by

$$
q_{t}\left(x_{t}, y_{t}\right)=q_{X_{t} Y_{t}}\left(x_{t}, y_{t}\right)=q_{X_{t}}\left(x_{t}\right) W\left(y_{t} \mid x_{t}\right)
$$

Then we have

$$
\Phi_{t, Q^{t}}^{(\mu, \lambda)}=\mathrm{E}_{q_{t}}\left[\frac{W^{\lambda}\left(Y_{t} \mid X_{t}\right) \mathrm{e}^{-\mu \lambda c\left(X_{t}\right)}}{Q_{t}^{\lambda}\left(Y_{t}\right)}\right] .
$$

We define $Q^{n}=\left\{Q_{t}\right\}_{t=1}^{n}$ recursively. For each $t=$ $1,2, \cdots, n$, we choose $Q_{t}$ so that it minimizes $\Phi_{t, Q^{t}}^{(\mu, \lambda)}$. Let $Q_{\mathrm{opt}, t}$ be one of the minimizes on the above optimization problem. We set $Q_{\mathrm{opt}}^{t} \triangleq\left\{Q_{\mathrm{opt}, i}\right\}_{i=1}^{t}$. Note that $Q_{\mathrm{opt}, t}$ can be determined recursively depending on the $t-1$ previous minizers $Q_{\text {opt }}^{t-1}$. Then we have the following:

$$
\begin{align*}
& \log \Phi_{t, Q_{\mathrm{opt}}^{t}}^{(\mu, \lambda)}=\log \mathrm{E}_{q_{t}}\left[\frac{W^{\lambda}\left(Y_{t} \mid X_{t}\right) \mathrm{e}^{-\mu \lambda c\left(X_{t}\right)}}{Q_{\mathrm{opt}, t}^{\lambda}\left(Y_{t}\right)}\right] \\
& =\min _{Q \in \mathcal{P}(\mathcal{Y})} \Omega^{(\mu, \lambda)}\left(q_{X_{t}}, Q \mid W\right) \\
& \leq \max _{q_{X_{t}}} \min _{Q \in \mathcal{P}(\mathcal{Y})} \Omega^{(\mu, \lambda)}\left(q_{X_{t}}, Q \mid W\right)=\Omega^{(\mu, \lambda)}(W) . \tag{37}
\end{align*}
$$

Hence we have the following:

$$
\begin{align*}
& \min _{Q^{n} \in \mathcal{P}^{n}(\mathcal{Y})} \frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right) \leq \frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q_{\mathrm{opt}}^{n}\right) \\
& \stackrel{(\mathrm{a})}{=} \frac{1}{n} \sum_{t=1}^{n} \log \Phi_{t, Q_{\mathrm{opt}}^{( }}^{(\mu, \lambda)} \stackrel{(\mathrm{b})}{\leq} \Omega^{(\mu, \lambda)}(W) \tag{38}
\end{align*}
$$

Step (a) follows from (35). Step (b) follows from (37). Since (38) holds for any $n \geq 1$ and for any $p^{(n)} \in \mathcal{P}^{(n)}(W)$, we have

$$
\begin{aligned}
& \bar{\Omega}^{(\mu, \lambda)}(W) \\
= & \sup _{n \geq 1} \max _{p^{(n)} \in \mathcal{P}^{(n)}(W)} \min _{Q^{n} \in \mathcal{P}^{n}(\mathcal{Y})} \frac{1}{n} \Omega^{(\mu, \lambda)}\left(p^{(n)}, Q^{n}\right) \\
\leq & \Omega^{(\mu, \lambda)}(W),
\end{aligned}
$$

completing the proof.
Proof of Theorem 4. From Corollary 2 and Proposition 4 we have $G^{*}(R, \Gamma \mid W) \geq G^{(\mu, \lambda)}(R, \Gamma \mid W)$ for any $\mu, \lambda \geq 0$. Hence we have the bound $G^{*}(R, \Gamma \mid W) \geq G(R, \Gamma \mid W)$.

## V. Equivalence of Three Exponent Functions

In this section we prove Propositions 1 and 2 stated in Section III We first prove Proposition (1) The following is a key lemma to prove this proposition.

Lemma 7: For any $q_{X} \in \mathcal{P}(\mathcal{X})$

$$
\min _{Q \in \mathcal{P}(\mathcal{Y})} \Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right)=(1+\lambda) J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}\left(q_{X} \mid W\right)
$$

The distribution $Q \in \mathcal{P}(\mathcal{Y})$ attaining $(1+\lambda) J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}\left(q_{X} \mid W\right)$ is given by

$$
Q(y)=\kappa\left[\sum_{x \in \mathcal{X}} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}}
$$

where $\kappa$ is a constant for normalization, having the form

$$
\begin{align*}
\kappa^{-1} & =\sum_{y \in \mathcal{Y}}\left[\sum_{x \in \mathcal{X}} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}} \\
& =\exp \left[J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}\left(q_{X} \mid W\right)\right] . \tag{39}
\end{align*}
$$

Proof: We observe that

$$
\begin{gather*}
\Omega^{(\mu, \lambda)}(W)=\max _{q_{X} \in \mathcal{P}(\mathcal{X})} \log \left\{\min _{Q \in \mathcal{P}(\mathcal{Y})} \sum_{x, y} q_{X}(x) W(y \mid x)\right. \\
\left.\times\left[\frac{W(y \mid x) \mathrm{e}^{-\mu c(x)}}{Q(y)}\right]^{\lambda}\right\} . \tag{40}
\end{gather*}
$$

On the objective function of the minimization problem inside the logarithm function in (40), we have the following chain of inequalities:

$$
\begin{align*}
& \sum_{x, y} q_{X}(x) W(y \mid x)\left[\frac{W(y \mid x) \mathrm{e}^{-\mu c(x)}}{Q(y)}\right]^{\lambda} \\
= & \sum_{y}\left[\sum_{x} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{(a)} Q^{-\lambda}(y) \\
\geq & \left\{\sum_{y}\left[\sum_{x} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}}\right\}^{1+\lambda} \\
& \times\left\{\sum_{y} Q(y)\right\}^{-\lambda} \\
= & \left\{\sum_{y}\left[\sum_{x} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}}\right\}^{1+\lambda} \\
= & \exp \left\{(1+\lambda) J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}\left(q_{X} \mid W\right)\right\} . \tag{41}
\end{align*}
$$

In (a), we have used the reverse Hölder inequality

$$
\sum_{i} a_{i} b_{i} \geq\left(\sum_{i} a_{i}^{\frac{1}{\alpha}}\right)^{\alpha}\left(\sum_{i} b_{i}^{\frac{1}{\beta}}\right)^{\beta}
$$

which holds for nonegative $a_{i}, b_{i}$ and for $\alpha+\beta=1$ such that either $\alpha>1$ or $\beta>1$. In our case we have applied the inequality to

$$
\left.\begin{array}{rl}
i & \rightarrow y \\
a_{i} & \rightarrow \sum_{x} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}, \\
b_{i} & \rightarrow Q^{-\lambda}(y), \\
(\alpha, \beta) & \rightarrow(1+\lambda,-\lambda) .
\end{array}\right\}
$$

In the reverse Hölder inequality the equality holds if and only if $a_{i}^{\frac{1}{\alpha}}=\kappa b_{i}^{\frac{1}{\beta}}$ for some constant $\kappa$. In 41), the equality holds for

$$
Q(y)=\kappa\left[\sum_{x} q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}}
$$

where $\kappa$ is a normalized constant. From (41), we have

$$
\Omega^{(\mu, \lambda)}(W)=(1+\lambda) \max _{q_{X} \in \mathcal{P}(\mathcal{X})} J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}\left(q_{X} \mid W\right)
$$

completing the proof.
Proof of Proposition (1) The equality (9) in Proposition 1 immediately follows from Lemma 7 Using (9), we prove $G(R, \Gamma \mid W)=G_{\mathrm{DK}}(R, \Gamma \mid W)$. We have the following chain of inequalities:

$$
\begin{aligned}
& \quad G(R, \Gamma \mid W)=\max _{\mu \geq 0, \lambda \geq 0} G^{(\mu, \lambda)}(R, \Gamma \mid W) \\
& \stackrel{(a)}{=} \max _{\mu \geq 0} \max _{\rho=\frac{\lambda}{1+\lambda} \in[0,1)} G_{\mathrm{AR}}^{(\mu, \rho)}(R, \Gamma \mid W)=G_{\mathrm{AR}}(R, \Gamma \mid W) .
\end{aligned}
$$

Step (a) followns from (9) in Proposition 1.
We next prove Proposition 2. We can show that $G_{\mathrm{AR}}(R$, $\Gamma \mid W)$ and $G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W)$ satisfies the following property.

Property 2:
a) The function $G_{\mathrm{AR}}(R, \Gamma \mid W)$ is monotone increasing function of $R$ and is positive if and only if $R>$ $C(\Gamma \mid W)$.
b) For $y \in \mathcal{Y}$, set

$$
\Lambda(y) \triangleq \sum_{x \in \mathcal{X}} q_{X}(x)\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}}
$$

Then, for $\lambda \in(0,1]$, necessary and sufficient conditions on the probability distribution $q_{X} \in \mathcal{P}(\mathcal{X})$ that minimizes $J^{(\mu, \lambda)}\left(q_{X} \mid W\right)$ is

$$
\sum_{y \in \mathcal{Y}}\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}} \Lambda(y)^{-\lambda} \leq \sum_{y \in \mathcal{Y}} \Lambda(y)^{1-\lambda}
$$

for any $x \in \mathcal{X}$ with equality if $q_{X}(x) \neq 0$.
We now proceed to the proof of Proposition 2
Proof of (16) in Proposition 2. We prove $G_{\mathrm{DK}}^{(\mu \lambda, \lambda)}(R, \Gamma \mid W)$ $=G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W)$. For a given joint distribution

$$
\left(q_{X}, q_{Y \mid X}\right)=\left\{q_{X}(x) q_{Y \mid X}(y \mid x)\right\}_{(x, y) \in \mathcal{X} \times \mathcal{Y}}
$$

we introduce the stochastic matrix $q_{X \mid Y}=\left\{q_{X \mid Y}(x \mid y)\right\}$ $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and the probability distribution $q_{Y}=\left\{q_{Y}(y)\right\}_{y \in \mathcal{Y}}$ by

$$
q_{X}(x) q_{Y \mid X}(y \mid x)=q_{Y}(y) q_{X \mid Y}(x \mid y), \quad(x, y) \in \mathcal{X} \times \mathcal{Y}
$$

The above $q_{X \mid Y}$ is called a backward channel. Using $\left(q_{Y}\right.$, $\left.q_{X \mid Y}\right)$, we obtain the following chain of equalities:

$$
\begin{align*}
& -\lambda\left\{I\left(q_{X}, q_{Y \mid X}\right)-\mu \mathrm{E}_{q_{X}}[c(X)]\right\} \\
& +D\left(q_{Y \mid X}| | W \mid q_{X}\right) \\
= & \lambda D\left(q_{X \mid Y} \| q_{X} \mid q_{Y}\right)+D\left(q_{Y}, q_{X \mid Y} \| q_{X}, W\right) \\
& +\mu \lambda \mathrm{E}_{\left(q_{Y}, q_{X \mid Y}\right)}[c(X)] \\
= & \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} q_{Y}(y) q_{X \mid Y}(x \mid y) \log \left\{\frac{q_{X \mid Y}^{-\lambda}(x \mid y)}{q_{X}^{-\lambda}(x)}\right\} \\
& +\sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} q_{Y}(y) q_{X \mid Y}(x \mid y) \\
& \times \log \left\{\frac{q_{X \mid Y}(x \mid y) q_{Y}(y)}{q_{X}(x) W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}}\right\} \\
= & \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} q_{Y}(y) q_{X \mid Y}(x \mid y) \\
& \times \log \left\{\frac{q_{X \mid Y}^{1-\lambda}(x \mid y)}{\left.q_{X}^{1-\lambda}(x) W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}}\right\} \\
& +\sum_{y \in \mathcal{Y}} q_{Y}(y) \log q_{Y}(y) \\
= & (1-\lambda) \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} q_{Y}(y) q_{X \mid Y}(x \mid y) \\
& \times \log \left\{\frac{q_{X \mid Y}(x \mid y)}{q_{X}(x)\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}}\right\} \\
& +\sum_{y \in \mathcal{Y}} q_{Y}(y) \log q_{Y}(y) \\
= & (1-\lambda) D\left(q_{X \mid Y} \| \hat{q}_{X \mid Y} \mid q_{Y}\right)+D\left(q_{Y} \| \hat{q}_{Y}\right) \\
& -J(\mu, \lambda)\left(q_{X} \mid W\right) \tag{42}
\end{align*}
$$

where $\hat{q}_{X \mid Y}=\left\{\hat{q}_{X \mid Y}(x \mid y)\right\}(x, y) \in \mathcal{X} \times \mathcal{Y}$ is a stochastic matrix whose components are

$$
\begin{gather*}
\hat{q}_{X \mid Y}(x \mid y)=\frac{1}{\Lambda(y)} q_{X}(x)\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\} \\
(x, y) \in \mathcal{X} \times \mathcal{Y} \tag{43}
\end{gather*}
$$

and $\hat{q}_{Y}=\left\{\hat{q}_{Y}(y)\right\}_{y \in \mathcal{Y}}$ is a probability distribution whose components are

$$
\begin{equation*}
\hat{q}_{Y}(y)=\frac{\Lambda(y)^{1-\lambda}}{\sum_{y \in \mathcal{Y}} \Lambda(y)^{1-\lambda}}, \quad y \in \mathcal{Y} \tag{44}
\end{equation*}
$$

Hence, by (42) and the non-negativity of divergence, we obtain

$$
G_{\mathrm{DK}}^{(\mu \lambda, \lambda)}\left(R, \Gamma, q_{X} \mid W\right) \geq G_{\mathrm{AR}}^{(\mu, \lambda)}\left(R, \Gamma, q_{X} \mid W\right)
$$

for any $q_{X} \in \mathcal{P}(\mathcal{X})$. Next, we prove

$$
G_{\mathrm{DK}}^{(\mu \lambda, \lambda)}(R, \Gamma \mid W)=G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W)
$$

To this end it suffices to show that for any $\lambda \geq 0$,

$$
G_{\mathrm{DK}}^{(\mu \lambda, \lambda)}(R, \Gamma \mid W) \leq G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W)
$$

Let $q_{X}$ be a probability distribution that attains the minimum of $G_{\mathrm{AR}}^{(\mu, \lambda)}\left(R, \Gamma, q_{X} \mid W\right)$. Then, by Property 2 we have

$$
\begin{equation*}
\sum_{y \in \mathcal{Y}}\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}} \Lambda(y)^{-\lambda} \leq \sum_{y \in \mathcal{Y}} \Lambda(y)^{1-\lambda} \tag{45}
\end{equation*}
$$

for any $x \in \mathcal{X}$ with equality if $q_{X}(x) \neq 0$. For $x \in \mathcal{X}$ with $q_{X}(x)>0$ and $y \in \mathcal{Y}$, define the matrix $V=\{V(y \mid x)\}$ $(x, y) \in \mathcal{X} \times \mathcal{Y}$ by

$$
\begin{equation*}
V(y \mid x)=\frac{\hat{q}_{Y}(y) \hat{q}_{X \mid Y}(x \mid y)}{q_{X}(x)}, \quad(x, y) \in \mathcal{X} \times \mathcal{Y} \tag{46}
\end{equation*}
$$

By (43) and (44), each $V(y \mid x)$ has the following form:

$$
\begin{align*}
& V(y \mid x)=\frac{\Lambda(y)^{1-\lambda}}{\sum_{y \in \mathcal{Y}} \Lambda(y)^{1-\lambda}} \\
& \quad \times \frac{1}{\Lambda(y)} q_{X}(x)\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}} \cdot \frac{1}{q_{X}(x)} \\
& =\frac{\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}} \Lambda(y)^{-\lambda}}{\sum_{y \in \mathcal{Y}} \Lambda(y)^{1-\lambda}} \tag{47}
\end{align*}
$$

Taking summation of both sides of (47) with respect to $y \in \mathcal{Y}$ and taking (45) into account, we obtain

$$
\sum_{y \in \mathcal{Y}} V(y \mid x)=\frac{\sum_{y \in \mathcal{Y}}\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{1+\lambda} \Lambda(y)^{-\lambda}}{\sum_{y \in \mathcal{Y}} \Lambda(y)^{1-\lambda}}=1
$$

The above equality implies that $V$ is a stochastic matrix. Furthermore, note that from (46),

$$
q_{X}(x) V(y \mid x)=\hat{q}_{Y}(y) \hat{q}_{X \mid Y}(x \mid y), \quad(x, y) \in \mathcal{X} \times \mathcal{Y}
$$

Then, choosing $q_{Y}=\hat{q}_{Y}, q_{X \mid Y}=\hat{q}_{X \mid Y}$ in 42), we have, for $\lambda \geq 0$,

$$
\begin{aligned}
& G_{\mathrm{DK}}^{(\mu \lambda, \lambda)}(R, \Gamma \mid W) \\
\leq & \lambda\left\{(R-\mu \Gamma)-I\left(q_{X}, V\right)+\mu \mathrm{E}_{q_{X}}[c(X)]\right\} \\
& +D\left(V| | W \mid q_{X}\right) \\
= & \lambda(R-\mu \Gamma)-J^{(\mu, \lambda)}\left(q_{X} \mid W\right)=G_{\mathrm{AR}}^{(\mu, \lambda)}(R, \Gamma \mid W),
\end{aligned}
$$

completing the proof.
We prove (17) in Proposition 2 by (16).
Proof of (17) in Proposition [2; We prove $G_{\mathrm{AR}}(R, \Gamma \mid$ $W)=G_{\mathrm{DK}}(R, \Gamma \mid W)$. Let $q_{X}^{*}$ be an input distribution attaining $C(\Gamma \mid W)$. Then, by the definition of $G_{\mathrm{DK}}^{(\mu, 0)}(R, \Gamma \mid W)$, we have

$$
\begin{equation*}
G_{\mathrm{DK}}^{(\mu, 0)}(R, \Gamma \mid W) \leq-\mu\left(\Gamma-\mathrm{E}_{q_{X}^{*}}[c(X)]\right) \leq 0 \tag{48}
\end{equation*}
$$

for any $\mu \geq 0$. Hence we have

$$
\max _{\mu \geq 0} G_{\mathrm{DK}}^{(\mu, 0)}(R, \Gamma \mid W)=0
$$

Then we have the following chain of inequalities:

$$
\begin{aligned}
& G_{\mathrm{DK}}(R, \Gamma \mid W)=\max _{\mu \geq 0, \lambda>0} G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W) \\
&= \max _{\substack{\mu \geq 0, \lambda>0, \alpha=\frac{\mu}{\lambda} \geq 0}} G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W)=\max _{\substack{\alpha \geq 0, \lambda>0, \mu=\alpha \lambda}} G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W) \\
&= \max _{\alpha \geq 0, \lambda>0} G_{\mathrm{DK}}^{(\alpha \lambda, \lambda)}(R, \Gamma \mid W) \stackrel{(\mathrm{a})}{=} \max _{\alpha \geq 0, \lambda>0} G_{\mathrm{AR}}^{(\alpha, \lambda)}(R, \Gamma \mid W) \\
& \stackrel{(\mathrm{b})}{=} \max _{\alpha \geq 0, \lambda \geq 0} G_{\mathrm{AR}}^{(\alpha, \lambda)}(R, \Gamma \mid W)=G_{\mathrm{AR}}(R, \Gamma \mid W) .
\end{aligned}
$$

Step (a) followns from (16) in Proposition 2 Step (b) follows from $G_{\mathrm{AR}}^{(\alpha, 0)}(R, \Gamma \mid W)=0$ for any $\alpha \geq 0$.

## VI. Extention to Genaral Memoryless Channels

In this section we consider a stationary general memoryless channel(GMC), where $\mathcal{X}$ and $\mathcal{Y}$ are real lines. The GMC is specified with a noisy channel $W$. We assume that for each $X=x \in \mathcal{X}, W$ has a conditonal density function $W(\mathrm{~d} y \mid x)$. Except for Theorem 3. Property 2 part b), and Proposition 2) the results we have presented so far also hold for this general case. Let $q_{X}$ be a probability measure on $\mathcal{X}$ having the density $q_{X}(\mathrm{~d} x)$. Let $Q$ be a probability measure on $\mathcal{Y}$ having the density $Q(\mathrm{~d} y)$. In the case of GMC, the definitions of $\Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right)$ and $\Omega^{(\mu, \lambda)}(W)$ are

$$
\begin{aligned}
& \Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right) \\
& \triangleq \log \left[\iint \mathrm{d} x \mathrm{~d} y q_{X}(x) \frac{W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}}{Q^{\lambda}(y)}\right] \\
& \Omega^{(\mu, \lambda)}(W) \triangleq \max _{q_{X}} \min _{Q} \Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right)
\end{aligned}
$$

For GMC $W$, we define the exponent functions $G^{(\mu, \lambda)}$ ( $R, \Gamma \mid W)$ and $G(R, \Gamma \mid W)$ in a manner similar to the definitions of those exponent functions in the case of DMC. The following theorem is a generalization of Theorem 4 to the case of GMC.

Theorem 6: For any GMC $W$, we have

$$
\begin{equation*}
G^{*}(R, \Gamma \mid W) \geq G(R, \Gamma \mid W) \tag{49}
\end{equation*}
$$

We next describe a lemma which is a generalization of Lemma 7 to the case of GMC. For $\lambda \in[0,1)$, define

$$
\begin{aligned}
& J^{(\mu, \lambda)}\left(q_{X} \mid W\right) \\
& \triangleq \log \int \mathrm{d} y\left[\int \mathrm{~d} x q_{X}(x)\left\{W(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right\}^{\frac{1}{1-\lambda}}\right]^{1-\lambda}
\end{aligned}
$$

Then we have the following lemma.
Lemma 8: For any probability densitity function $q_{X}=$ $\left.q_{( } \mathrm{d} x\right)$ on $\mathcal{X}$, we have

$$
\min _{Q} \Omega^{(\mu, \lambda)}\left(q_{X}, Q \mid W\right)=(1+\lambda) J^{\left(\mu, \frac{\lambda}{1+\lambda)}\right.}\left(q_{X} \mid W\right)
$$

The probability density function $Q$ attaining $(1+\lambda) J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}($ $\left.q_{X} \mid W\right)$ is given by

$$
Q(y)=\kappa\left[\int \mathrm{d} x q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}}
$$

where $\kappa$ is a constant for normalization, having the form

$$
\begin{align*}
\kappa^{-1} & =\int \mathrm{d} y\left[\int \mathrm{~d} x q_{X}(x) W^{1+\lambda}(y \mid x) \mathrm{e}^{-\mu \lambda c(x)}\right]^{\frac{1}{1+\lambda}} \\
& =\exp \left[J^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}\left(q_{X} \mid W\right)\right] \tag{50}
\end{align*}
$$

For GMC $W$, we define the exponent functions $G_{\mathrm{AR}}^{(\mu, \lambda)}($ $R, \Gamma \mid W)$ and $G_{\mathrm{AR}}(R, \Gamma \mid W)$ in a manner similar to the definitions of those exponent functions in the case of DMC. From Lemma 8 we have the following proposition, which is a generalization of Proposition 1 to the case of GMC.

Proposition 5: For any GMC $W$ and for any $\mu, \lambda \geq 0$, we have the following:

$$
\begin{equation*}
G^{(\mu, \lambda)}(R, \Gamma \mid W)=G_{\mathrm{AR}}^{\left(\mu, \frac{\lambda}{1+\lambda}\right)}(R, \Gamma \mid W) \tag{51}
\end{equation*}
$$

In particular, we have

$$
\begin{equation*}
G(R, \Gamma \mid W)=G_{\mathrm{AR}}(R, \Gamma \mid W) \tag{52}
\end{equation*}
$$

From Theorem 6 and Proposition 5, we immediately obtain the following result.

Theorem 7: For any GMC $W$, we have

$$
\begin{equation*}
G^{*}(R, \Gamma \mid W) \geq G(R, \Gamma \mid W)=G_{\mathrm{AR}}(R, \Gamma \mid W) \tag{53}
\end{equation*}
$$

Theorem 3 is related to the upper bound of $G^{*}(R, \Gamma \mid W)$. Proof of this theorem depends heavily on a finiteness of $\mathcal{X}$. We have no result on the upper bound of $G^{*}(R, \Gamma \mid W)$ and the tightness of the bound $G(R, \Gamma \mid W)$. In the case of GMC, $G(R, \Gamma \mid W)$ and $G_{\mathrm{AR}}(R, \Gamma \mid W)$ are not computable since those are variational problems. On the other hand, $G(R, \Gamma \mid W)$ has a min-max expression. In [13], the author succeeded in obtaining an explicit form of $G(R, \Gamma \mid W)$ for additive white Gaussian noise channels(AWGNs) by utilizing the min-max property of $G(R, \Gamma \mid W)$.

## Appendix

## A. General Properties on $G^{*}(R, \Gamma \mid W)$

In this appendix we prove Property 1 describing general properties on $G^{*}(R, \Gamma \mid W)$.

Proof of Property ㄱ. By definition it is obvious that for fixed $\Gamma>0, G^{(n)}(R, \Gamma \mid W)$ is a monotone increasing function of $R>0$ and that for fixed $R>0, G^{(n)}(R, \Gamma \mid W)$ is a monotone increasing function of $\Gamma>0$. We prove the part b). By time sharing we have that

$$
\begin{align*}
& G^{(n+m)}\left(\frac{n R+m R^{\prime}}{n+m}, \left.\frac{n \Gamma+m \Gamma^{\prime}}{n+m} \right\rvert\, W\right) \\
\leq & \frac{n G^{(n)}(R, \Gamma \mid W)+m G^{(m)}\left(R^{\prime}, \Gamma^{\prime} \mid W\right)}{n+m} \tag{54}
\end{align*}
$$

The part b) follows by letting $R=R^{\prime}$ and $\Gamma=\Gamma^{\prime}$ in (54). We next prove the part c). By definition it is obvious that for fixed $\Gamma>0, G^{*}(R, \Gamma \mid W)$ is a monotone decreasing function of $R>0$ and that for fixed $R>0, G^{*}(R, \Gamma \mid W)$ is a monotone increasing function of $\Gamma>0$. It is obvious that the worst pair
of $\left(\varphi^{(n)}, \psi^{(n)}\right)$ is that for $M_{n}=\left\lfloor\mathrm{e}^{n R}\right\rfloor$, the decoder $\psi^{(n)}$ always outputs a constant message $m_{0} \in \mathcal{M}_{n}$. In this case we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty}\left(-\frac{1}{n}\right) \log \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
= & \lim _{n \rightarrow \infty}\left(-\frac{1}{n}\right) \log M_{n}=R
\end{aligned}
$$

Hence we have (7) in the part c). We finally prove the part d). Let $\lfloor a\rfloor$ be an integer part of $a$. Fix any $\alpha \in[0,1]$. Let $\bar{\alpha}=1-\alpha$. We choose $(n, m)$ so that

$$
n=k_{\alpha} \triangleq\lfloor k \alpha\rfloor, m=k_{\bar{\alpha}} \triangleq\lfloor k \bar{\alpha}\rfloor .
$$

For this choice of $n$ and $m$, we have

$$
\left.\begin{array}{l}
\left(1-\frac{1}{k}\right) \alpha \leq \frac{n}{n+m} \leq \frac{k}{k-1} \alpha  \tag{55}\\
\left(1-\frac{1}{k}\right) \bar{\alpha} \leq \frac{m}{n+m} \leq \frac{k}{k-1} \bar{\alpha}
\end{array}\right\}
$$

Fix small positive $\tau$ arbitrary. Then, for any

$$
k>\max \left\{\left(\alpha R+\bar{\alpha} R^{\prime}\right) / \tau,\left(\alpha \Gamma+\bar{\alpha} \Gamma^{\prime}\right) / \tau\right\},
$$

we have the following chain of inequalities:

$$
\begin{align*}
& \quad G^{\left(k_{\alpha}+k_{\bar{\alpha}}\right)}\left(\alpha R+\bar{\alpha} R^{\prime}-\tau, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime}-\tau \mid W\right) \\
& \stackrel{\text { (a) }}{\leq} G^{\left(k_{\alpha}+k_{\bar{\alpha}}\right)}\left(\left(1-\frac{1}{k}\right)\left(\alpha R+\bar{\alpha} R^{\prime}\right)\right. \\
& \left.\left.\quad\left(1-\frac{1}{k}\right)\left(\alpha \Gamma+\bar{\alpha} \Gamma^{\prime}\right) \right\rvert\, W\right) \\
& \stackrel{\text { (b) }}{\leq} G^{(n+m)}\left(\frac{n R+m R^{\prime}}{n+m}, \left.\frac{n \Gamma+m \Gamma^{\prime}}{n+m} \right\rvert\, W\right) \\
& \stackrel{\text { (c) }}{\leq} \frac{n G^{(n)}(R, \Gamma \mid W)+m G^{(m)}\left(R^{\prime}, \Gamma^{\prime} \mid W\right)}{n+m} \\
& \stackrel{\text { (d) }}{\leq}\left(\frac{k}{k-1}\right)\left[\alpha G^{\left(k_{\alpha}\right)}(R, \Gamma \mid W)+\bar{\alpha} G^{\left(k_{\bar{\alpha}}\right)}\left(R^{\prime}, \Gamma^{\prime} \mid W\right)\right] . \tag{56}
\end{align*}
$$

Step (a) follows from the part a) and

$$
k>\max \left\{\left(\alpha R+\bar{\alpha} R^{\prime}\right) / \tau,\left(\alpha \Gamma+\bar{\alpha} \Gamma^{\prime}\right) / \tau\right\}
$$

Step (b) follows from the part a). Step (c) follows from (54).
Step (d) follows from (55). Letting $k \rightarrow \infty$ in (56), we have

$$
\begin{align*}
& G^{*}\left(\alpha R+\bar{\alpha} R^{\prime}-\tau, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime}-\tau \mid W\right) \\
\leq & \alpha G^{*}(R, \Gamma \mid W)+\bar{\alpha} G^{*}\left(R^{\prime}, \Gamma^{\prime} \mid W\right), \tag{57}
\end{align*}
$$

where $\tau$ can be taken arbitrary small. We choose $R^{\prime}, \Gamma^{\prime}$, and $\alpha$, as

$$
\left.\begin{array}{l}
R^{\prime}=R+2 \sqrt{\tau}, \quad \Gamma^{\prime}=\Gamma+2 \sqrt{\tau}  \tag{58}\\
\alpha=1-\sqrt{\tau}
\end{array}\right\}
$$

For the above choice of $R^{\prime}, \Gamma^{\prime}$, and $\alpha$, we have

$$
\begin{equation*}
\alpha R+\bar{\alpha} R^{\prime}=R+2 \tau, \quad \alpha \Gamma+\bar{\alpha} \Gamma^{\prime}=\Gamma+2 \tau \tag{59}
\end{equation*}
$$

Then we have the following chain of inequalities:

$$
\begin{align*}
& \quad G^{*}(R+\tau, \Gamma+\tau \mid W) \\
& \stackrel{(\mathrm{a})}{=} G^{*}\left(\alpha R+\bar{\alpha} R^{\prime}-\tau, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime}-\tau \mid W\right) \\
& \stackrel{\text { (b) }}{\leq} \alpha G^{*}(R, \Gamma \mid W)+\bar{\alpha} G^{*}\left(R^{\prime}, \Gamma^{\prime} \mid W\right) \\
& \stackrel{\text { (c) }}{\leq} \alpha G^{*}(R, \Gamma \mid W)+\bar{\alpha} R^{\prime} \\
& \stackrel{\text { (d) }}{=}(1-\sqrt{\tau}) G^{*}(R, \Gamma \mid W)+\sqrt{\tau} R+2 \tau \\
& \leq G^{*}(R, \Gamma \mid W)+\sqrt{\tau} R+2 \tau .
\end{align*}
$$

Step (a) follows from (59). Step (b) follows from (57). Step (c) follows from (7). Step (d) follows from (58). For any positive $\tau$, we have the following chain of inequalities:

$$
\begin{align*}
& G^{*}\left(\alpha R+\bar{\alpha} R^{\prime}, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime} \mid W\right) \\
& =G^{*}\left(\alpha R+\bar{\alpha} R^{\prime}-\tau+\tau, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime}-\tau+\tau \mid W\right) \\
& \text { (a) } \\
& \stackrel{(a)}{\leq} G^{*}\left(\alpha R+\bar{\alpha} R^{\prime}-\tau, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime}-\tau \mid W\right) \\
& +\sqrt{\tau}\left(\alpha R+\bar{\alpha} R^{\prime}-\tau\right)+2 \tau \\
& \text { (b) } \\
& \stackrel{\text { (b) }}{\leq} G^{*}(R, \Gamma \mid W)+\bar{\alpha} G^{*}\left(R^{\prime}, \Gamma^{\prime} \mid W\right) \\
& +\sqrt{\tau}\left(\alpha R+\bar{\alpha} R^{\prime}\right)+\tau(2-\sqrt{\tau}) . \tag{61}
\end{align*}
$$

Step (a) follows from (60). Step (b) follows from (57). Since $\tau>0$ can be taken arbitrary small in 61, we have

$$
\begin{gathered}
G^{*}\left(\alpha R+\bar{\alpha} R^{\prime}, \alpha \Gamma+\bar{\alpha} \Gamma^{\prime} \mid W\right) \\
\leq \alpha G^{*}(R, \Gamma \mid W)+\bar{\alpha} G^{*}\left(R^{\prime}, \Gamma^{\prime} \mid W\right),
\end{gathered}
$$

which implies the convexity of $G^{*}(R, \Gamma \mid W)$ on $(R, \Gamma)$.

## B. Proof of Theorem 3

In this appendix we prove Theorem 3 We first describe some definitions necessary for the proof. For $x^{n} \in \mathcal{X}^{n}$, set

$$
p_{x^{n}}(x) \triangleq \frac{\left|\left\{t: x_{t}=x\right\}\right|}{n}, x \in \mathcal{X}
$$

The probability distribution $p_{x^{n}} \triangleq\left\{p_{x^{n}}(x)\right\}_{x \in \mathcal{X}}$ on $\mathcal{X}$ is called the type of sequences in $\mathcal{X}^{n}$. Let $\mathcal{P}_{n}(\mathcal{X})$ be a set of all types of sequences in $\mathcal{X}^{n}$. Let $\mathcal{P}(\mathcal{Y} \mid \mathcal{X})$ be a set of all conditional distributions $q_{Y \mid X}$ on $\mathcal{Y}$ for given $X \in \mathcal{X}$. We fix $\delta \in[0,1 / 2)$. We consider any pair $\left(q_{X}, q_{Y \mid X}\right) \in$ $\mathcal{P}_{n}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y} \mid \mathcal{X})$ satisfying $\mathrm{E}_{q_{X}} c(X) \leq \Gamma$. For such pair of $\left(q_{X}, q_{Y \mid X}\right)$, we can construct an $n$-length block code $\left(\phi^{(n)}, \psi^{(n)}\right)$ with message set $\mathcal{K}_{n}$ satisfying:
a) $\quad \mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid q_{Y \mid X}\right) \geq 1-\delta$.
b) all codewords $\phi^{(n)}(k), k \in \mathcal{K}_{n}$ have the identical type $q_{X}$.
c) $\frac{1}{n} \log \left|\mathcal{K}_{n}\right| \geq \min \left\{R, I\left(q_{X}, q_{Y \mid X}\right)-\delta\right\}$.

By the condition b), we have $c\left(\phi^{(n)}(k)\right)=\mathrm{E}_{q_{X}} c(X) \leq \Gamma$. Hence the $n$-length block code $\left(\phi^{(n)}, \psi^{(n)}\right)$ satisfies the cost constraint. Furthermore, by this condition we can obtain the following result.

Lemma 9: For every $k \in \mathcal{K}_{n}$, we have

$$
\begin{align*}
& \sum_{y^{n} \in \mathcal{Y}^{n}} q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right) \log \frac{q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)}{W^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)} \\
= & n D\left(q_{Y \mid X}| | W \mid q_{X}\right) \tag{62}
\end{align*}
$$

Proof: For each $k \in \mathcal{K}_{n}$, we set

$$
\phi^{(n)}(k)=x^{n}(k)=x_{1}(k) x_{2}(k) \cdots x_{n}(k)
$$

For each $k \in \mathcal{K}_{n}$, we have the following chain of equalities:

$$
\begin{aligned}
& \sum_{y^{n} \in \mathcal{Y}^{n}} q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right) \log \frac{q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)}{W^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)} \\
\stackrel{(\mathrm{a})}{=} & \sum_{t=1}^{n} \sum_{y_{t} \in \mathcal{Y}} q_{Y \mid X}\left(y_{t} \mid x_{t}(k)\right) \log \frac{q_{Y \mid X}\left(y_{t} \mid x_{t}(k)\right)}{W\left(y_{t} \mid x_{t}(k)\right)} \\
= & \sum_{a \in \mathcal{X}} \sum_{y \in \mathcal{Y}}\left|\left\{t: x_{t}(k)=a\right\}\right| q_{Y \mid X}(y \mid a) \log \frac{q_{Y \mid X}(y \mid a)}{W(y \mid a)} \\
= & n \sum_{a \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{x^{n}(k)}(a) q_{Y \mid X}(y \mid a) \log \frac{q_{Y \mid X}(y \mid a)}{W(y \mid a)} \\
\stackrel{(\mathrm{b})}{=} & n \sum_{a \in \mathcal{X}} \sum_{y \in \mathcal{Y}} q_{X}(a) q_{Y \mid X}(y \mid a) \log \frac{q_{Y \mid X}(y \mid a)}{W(y \mid a)} \\
= & n D\left(q_{Y \mid X} \| W \mid q_{X}\right) .
\end{aligned}
$$

Step (a) follows from the memoryless property of the noisy channel. Step (b) follows from that $p_{x^{n}(k)}=q_{X} \in \mathcal{P}_{n}(\mathcal{X})$.

For $k \in \mathcal{K}_{n}$, we set
$\alpha_{n}(k) \triangleq W^{n}\left(\mathcal{D}(k) \mid \phi^{(n)}(k)\right)=\sum_{y^{n} \in \mathcal{D}(k)} W^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)$,
$\beta_{n}(k) \triangleq q_{Y \mid X}^{n}\left(\mathcal{D}(k) \mid \phi^{(n)}(k)\right)=\sum_{y^{n} \in \mathcal{D}(k)} q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)$,
$\overline{\alpha_{n}(k)} \triangleq 1-\alpha_{n}(k)=q_{Y \mid X}^{n}\left(\overline{\mathcal{D}(k)} \mid \phi^{(n)}(k)\right)$,
$\overline{\beta_{n}(k)} \triangleq 1-\beta_{n}(k)=q_{Y \mid X}^{n}\left(\overline{\mathcal{D}(k)} \mid \phi^{(n)}(k)\right)$.
Furthermore, set

$$
\begin{aligned}
& \alpha_{n} \triangleq \sum_{k \in \mathcal{K}_{n}} \frac{1}{\left|\mathcal{K}_{n}\right|} \alpha_{n}(k)=\mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid W\right) \\
& \beta_{n} \triangleq \sum_{k \in \mathcal{K}_{n}} \frac{1}{\left|\mathcal{K}_{n}\right|} \beta_{n}(k)=\mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid q_{Y \mid X}\right)
\end{aligned}
$$

The quantity $\mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid W\right)$ has a lower bound given by the following Lemma.

Lemma 10: For any $\delta \in[0,1 / 2)$, we have

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid W\right)=\frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \mid \phi^{(n)}(k)\right) \\
& \geq \exp \left\{-n\left[(1-\delta)^{-1} D\left(q_{Y \mid X}| | W \mid q_{X}\right)+\eta_{n}(\delta)\right]\right\} \tag{63}
\end{align*}
$$

Here we set $\eta_{n}(\delta) \triangleq \frac{1}{n}(1-\delta)^{-1} h(1-\delta)$ and $h(\cdot)$ stands for a binary entropy function.

Proof: We have the following chain of inequalities:

$$
\begin{align*}
& n D\left(q_{Y \mid X}| | W \mid q_{X}\right) \\
& \stackrel{(\text { a) }}{=} \frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} \sum_{y^{n} \in \mathcal{Y}^{n}} q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right) \log \frac{q_{Y \mid X}^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)}{W^{n}\left(y^{n} \mid \phi^{(n)}(k)\right)} \\
& \stackrel{(\mathrm{b})}{\geq} \frac{1}{\left|\mathcal{K}_{n}\right|} \sum_{k \in \mathcal{K}_{n}}\left[\beta_{n}(k) \log \frac{\beta_{n}(k)}{\alpha_{n}(k)}+\overline{\beta_{n}(k)} \log \frac{\overline{\beta_{n}(k)}}{\overline{\alpha_{n}(k)}}\right] \\
& =\sum_{k \in \mathcal{K}_{n}}\left[\frac{\beta_{n}(k)}{\left|\mathcal{K}_{n}\right|} \log \frac{\frac{\beta_{n}(k)}{\left|\mathcal{K}_{n}\right|}}{\frac{\alpha_{n}(k)}{\left|\mathcal{K}_{n}\right|}}+\frac{\overline{\beta_{n}(k)}}{\left|\mathcal{K}_{n}\right|} \log \frac{\overline{\frac{\beta_{n}(k)}{\left|\mathcal{K}_{n}\right|}}}{\frac{\frac{\alpha_{n}(k)}{\left|\mathcal{K}_{n}\right|}}{}}\right] \\
& \stackrel{\text { (c) }}{\geq} \beta_{n} \log \frac{\beta_{n}}{\alpha_{n}}+\overline{\beta_{n}} \log \frac{\overline{\beta_{n}}}{\overline{\alpha_{n}}} \geq-h\left(\beta_{n}\right)-\beta_{n} \log \alpha_{n} \\
& \text { (d) }-h(1-\delta)-(1-\delta) \log \alpha_{n} . \tag{64}
\end{align*}
$$

Step (a) follows from Lemma 9 Steps (b) and (c) follow from the log-sum inequality. Step (d) follows from that

$$
\beta_{n}=\mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid q_{Y \mid X}\right) \geq 1-\delta
$$

and $\delta \in(0,1 / 2]$. From 64), we obtain

$$
\begin{aligned}
& \alpha_{n}=\mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid W\right) \\
& \geq \exp \left(-\frac{n D\left(q_{Y \mid X}| | W \mid q_{X}\right)+h(1-\delta)}{1-\delta}\right) \\
& =\exp \left\{-n\left[(1-\delta)^{-1} D\left(q_{Y \mid X}| | W \mid q_{X}\right)+\eta_{n}(\delta)\right]\right\}
\end{aligned}
$$

completing the proof.
Proof of Theorem 3. We first consider the case where $R \leq$ $I\left(q_{X}, q_{Y \mid X}\right)-\delta$. In this case we choose $\varphi^{(n)}=\phi^{(n)}$. Then we have

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right)=\mathrm{P}_{\mathrm{c}}^{(n)}\left(\phi^{(n)}, \psi^{(n)} \mid W\right) \\
& \stackrel{(\mathrm{a})}{=} \exp \left\{-n\left[R+\delta-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}\right. \\
& \left.\quad-n\left[(1-\delta)^{-1} D\left(q_{Y \mid X}| | W \mid q_{X}\right)+\eta_{n}(\delta)\right]\right\} \\
& \stackrel{(\mathrm{b})}{\geq} \exp \left\{-n\left[R-I\left(q_{X}, q_{Y \mid X}\right]^{+}\right.\right. \\
& \left.\quad-n\left[(1-\delta)^{-1} D\left(q_{Y \mid X}| | W \mid q_{X}\right)+\delta+\eta_{n}(\delta)\right]\right\} .
\end{align*}
$$

Step (a) follows from the condition $R+\delta-I\left(q_{X}, q_{Y \mid X}\right) \leq 0$. Step (b) follows from that

$$
\left[R+\delta-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+} \leq\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}+\delta
$$

We next consider the case where $R>I\left(q_{X}, q_{Y \mid X}\right)-\delta$. Consider the new message set $\widehat{\mathcal{K}}_{n}$ satisfying $\left|\widehat{\mathcal{K}}_{n}\right|=\mathrm{e}^{\lfloor n R\rfloor}$. For new message set $\widehat{\mathcal{K}}_{n}$, we define $\varphi^{(n)}(k)$ such that $\varphi^{(n)}(k)=$ $\phi^{(n)}(k)$ if $k \in \mathcal{K}_{n}$. For $k \in \widehat{\mathcal{K}}_{n}-\mathcal{K}_{n}$, we define $\varphi^{(n)}(k)$ arbitrary sequence of $\mathcal{X}^{n}$ having the type $q_{X}$. We use the same decoder $\psi^{(n)}$ as that of the message set $\mathcal{K}_{n}$. Then we
have the following:

$$
\begin{align*}
& \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
& =\frac{1}{\left|\widehat{\mathcal{K}}_{n}\right|}\left[\sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \mid \varphi^{(n)}(k)\right)\right. \\
& \left.+\sum_{k \in \widehat{\mathcal{K}}_{n}-\mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \mid \varphi^{(n)}(k)\right)\right] \\
& \geq \frac{1}{\left|\widehat{\mathcal{K}}_{n}\right|} \sum_{k \in \mathcal{K}_{n}} W^{n}\left(\mathcal{D}(k) \mid \varphi^{(n)}(k)\right) \\
& \stackrel{(\mathrm{a})}{\geq} \frac{\left|\mathcal{K}_{n}\right|}{\mathrm{e}^{n R}} \exp \left\{-n\left[(1-\delta)^{-1} D\left(q_{Y \mid X}| | W \mid q_{X}\right)+\eta_{n}(\delta)\right]\right\} \\
& \text { (b) } \\
& \stackrel{(\mathrm{b})}{\geq} \exp \left[-n\left\{R-\left(I\left(q_{X}, q_{Y \mid X}\right)-\delta\right)\right.\right. \\
& \left.\left.+(1-\delta)^{-1} D\left(q_{Y \mid X} \| W \mid q_{X}\right)+\eta_{n}(\delta)\right\}\right] \\
& \text { (c) } \\
& \stackrel{(c)}{\geq} \exp \left[-n\left\{\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}\right.\right. \\
& \left.\left.+(1-\delta)^{-1} D\left(q_{Y \mid X} \| W \mid q_{X}\right)+\delta+\eta_{n}(\delta)\right\}\right] . \tag{66}
\end{align*}
$$

Step (a) follows from (63) in Lemma 10. Step (b) follows from $\left|\mathcal{K}_{n}\right| \geq \mathrm{e}^{n\left[\left(I\left(q_{X}, q_{Y \mid X}\right)-\delta\right]\right.}$. Step (c) follows from $[a] \leq[a]^{+}$. Combining (65) and (66), we have

$$
\begin{align*}
& \quad \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
& \geq \exp \left[-n\left\{\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}\right.\right. \\
& \left.\left.\quad+(1-\delta)^{-1} D\left(q_{Y \mid X}| | W \mid q_{X}\right)+\delta+\eta_{n}(\delta)\right\}\right] \tag{67}
\end{align*}
$$

for any $q_{X} \in \mathcal{P}_{n}(\mathcal{X})$ with $\mathrm{E}_{q_{X}} c(X) \leq \Gamma$ and $q_{Y \mid X} \in \mathcal{P}($ $\mathcal{Y} \mid \mathcal{X})$. Hence from (67), we have

$$
\begin{align*}
- & \frac{1}{n} \log \mathrm{P}_{\mathrm{c}}^{(n)}\left(\varphi^{(n)}, \psi^{(n)} \mid W\right) \\
\leq & \min _{\substack{q_{X} \in \mathcal{P}_{n}(\mathcal{X}), \mathrm{E}_{q_{X}} c(X) \leq \Gamma, q_{Y \mid X} \in \mathcal{P}(\mathcal{Y} \mid \mathcal{X})}}\left\{\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}\right. \\
& \left.+(1-\delta)^{-1} D\left(q_{Y \mid X} \| W \mid q_{X}\right)+\delta+\eta_{n}(\delta)\right\} \\
\leq & (1-\delta)^{-1} \min _{\substack{q_{X} \in \mathcal{P}_{n}(\mathcal{X}), \mathrm{E}_{q_{X}} c(X) \leq \Gamma, q_{Y \mid X} \in \mathcal{P}(\mathcal{Y} \mid \mathcal{X})}}\left\{\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}\right. \\
& \left.+D\left(q_{Y \mid X}| | W \mid q_{X}\right)\right\}+\delta+\eta_{n}(\delta) \\
\leq & (1-\delta)^{-1} G_{\mathrm{DK}}(R, \Gamma \mid W)+\delta+\eta_{n}(\delta)+\varepsilon_{n} .
\end{align*}
$$

The quantity $\left\{\varepsilon_{n}\right\}_{n \geq 1}$ appearing in the last inequality is an error bound coming from an approximation of the marginal distribution $q_{X}^{*}$ of $q^{*}$ achieving $G_{\mathrm{DK}}(R, \Gamma \mid W)$ by some suitable type $q_{X} \in \mathcal{P}_{n}(\mathcal{X})$. Since $q_{X} \in \mathcal{P}_{n}(\mathcal{X})$ can be made arbitrary close to $q_{X}^{*}$ by letting $n$ sufficiently large, we can choose $\varepsilon_{n}$ so that $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$. We further note that $\eta_{n}(\delta) \rightarrow 0$ as $n \rightarrow \infty$. Hence by letting $n \rightarrow \infty$ in 68), we obtain

$$
G^{*}(R, \Gamma \mid W) \leq(1-\delta)^{-1} G_{\mathrm{DK}}(R, \Gamma \mid W)+\delta
$$

Since $\delta$ can be made arbitrary small, we conclude that $G^{*}(R$ $, \Gamma \mid W) \leq G_{\mathrm{DK}}(R, \Gamma \mid W)$.

## C. Proof of Lemma 7

In this appendix we prove Lemma 1 We can show that $G_{\mathrm{DK}}(R, \Gamma \mid W)$ satisfies the following property.

Property 3:
a) For every fixed $\Gamma>0$, the function $G_{\mathrm{DK}}(R, \Gamma \mid W)$ is monotone increasing for $R \geq 0$ and takes positive value if and only if $R>C(\Gamma \mid W)$. For every fixed $R \geq 0$, the function $G_{\mathrm{DK}}(R, \Gamma \mid W)$ is monotone decreasing for $\Gamma>0$.
b) $\quad G_{\mathrm{DK}}(R, \Gamma \mid W)$ is a convex function of $(R, \Gamma)$.
c) For $R, R^{\prime} \geq 0$

$$
\left|G_{\mathrm{DK}}(R, \Gamma \mid W)-G_{\mathrm{DK}}\left(R^{\prime}, \Gamma \mid W\right)\right| \leq\left|R-R^{\prime}\right|
$$

Property 3 part a) is obvious. Proof of the part b) is found in Appendix D. Proof of part c) is quite similar to that of the case without input cost given by Dueck and Körner [3]. We omit the detail.

We can show that $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)$ satisfies the following property.

Property 4:
a) For every fixed $\Gamma>0$, the function $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)$ is monotone increasing for $R \geq 0$. For every fixed $R \geq 0$, the function $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)$ is monotone decreasing for $\Gamma>0$.
b) For every fixed $\mu \geq 0$, the function $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)$ is a convex function of $(R, \Gamma)$.
c) For $R, R^{\prime} \geq 0$

$$
\left|G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)-G_{\mathrm{DK}}^{(\mu)}\left(R^{\prime}, \Gamma \mid W\right)\right| \leq\left|R-R^{\prime}\right|
$$

Property 4 part a) is obvious. Proof of the part b) is found in Appendix E Proof of part c) is quite similar to that of the case without input cost given by Dueck and Körner [3]. We omit the detail.

Proof of (13) in Lemma (7. From its formula, it is obvious that for any $\mu \geq 0$

$$
G_{\mathrm{DK}}(R, \Gamma \mid W) \geq G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)
$$

Hence it suffices to prove that for any $\Gamma>0$, there exists $\mu \geq 0$ such that

$$
\begin{equation*}
G_{\mathrm{DK}}(R, \Gamma \mid W) \leq G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) \tag{69}
\end{equation*}
$$

By Property 3 part b), $G_{\mathrm{DK}}(R, \Gamma \mid W)$ is a monotone decreasing and convex function of $\Gamma$. Then, there exists $\mu \geq 0$ such that for any $\Gamma^{\prime} \geq 0$, we have

$$
\begin{equation*}
G_{\mathrm{DK}}\left(R, \Gamma^{\prime} \mid W\right) \geq G_{\mathrm{DK}}(R, \Gamma \mid W)-\mu\left(\Gamma^{\prime}-\Gamma\right) \tag{70}
\end{equation*}
$$

Fix the above $\mu$. Let $q^{*} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ be a joint distribution that attains $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)$. Set $\Gamma^{\prime}=\mathrm{E}_{q^{*}}[c(X)]$. By the definition of $G_{\mathrm{DK}}\left(R, \Gamma^{\prime} \mid W\right)$, we have

$$
\begin{align*}
& G_{\mathrm{DK}}\left(R, \Gamma^{\prime} \mid W\right) \\
\leq & {\left[R-I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)\right]^{+}+D\left(q_{Y \mid X}^{*}| | W \mid q_{X}^{*}\right) } \tag{71}
\end{align*}
$$

Then, we the following chain of inequalities:

$$
\begin{align*}
& \quad G_{\mathrm{DK}}(R, \Gamma \mid W) \stackrel{(\mathrm{a})}{\leq} G_{\mathrm{DK}}\left(R, \Gamma^{\prime} \mid W\right)+\mu\left(\Gamma^{\prime}-\Gamma\right) \\
& \stackrel{\text { (b) }}{\leq}\left[R-I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)\right]^{+}+D\left(q_{Y \mid X}^{*}| | W \mid q_{X}^{*}\right)+\mu\left(\Gamma^{\prime}-\Gamma\right) \\
& \stackrel{(\mathrm{c})}{=}\left[R-I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)\right]^{+}+D\left(q_{Y \mid X}^{*}| | W \mid q_{X}^{*}\right) \\
& \quad-\mu\left(\Gamma-\mathrm{E}_{q^{*}}[c(X)]\right)=G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) . \tag{72}
\end{align*}
$$

Step (a) follows from 70). Step (b) follows from (71). Step (c) follows from the choice of $\Gamma^{\prime}=\mathrm{E}_{q^{*}}[c(X)]$. It follows from (72) that for $\Gamma>0$, (69) holds for some $\mu \geq 0$. This completes the proof.

Proof of (14) in Lemma 7 . Since $[a]^{+} \geq \lambda a$ for any $a$ and any $\lambda \in[0,1]$, it is obvious that

$$
G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) \geq \max _{0 \leq \lambda \leq 1} G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W)
$$

Hence it suffices to prove that for $R \geq 0$, there exists $\lambda \in[0,1]$ such that $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) \leq G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W)$. By Property 4 part b) $G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)$ is a monotone increasing and convex function of $R$. Then, by Property 3 part c), there exists $0 \leq$ $\lambda \leq 1$ such that for any $R^{\prime} \geq 0$, we have

$$
\begin{equation*}
G_{\mathrm{DK}}^{(\mu)}\left(R^{\prime}, \Gamma \mid W\right) \geq G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)+\lambda\left(R^{\prime}-R\right) \tag{73}
\end{equation*}
$$

Let $q^{*} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ be a joint distribution that attains $G_{\mathrm{DK}}^{(\mu, \lambda)}($ $R, \Gamma \mid W)$. Set $R^{\prime}=I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)$. Then we have the following chain of inequalities:

$$
\begin{aligned}
& G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W) \stackrel{(\mathrm{a})}{\leq} G_{\mathrm{DK}}^{(\mu)}\left(R^{\prime}, \Gamma \mid W\right)-\lambda\left(R^{\prime}-R\right) \\
&= \min _{q}\left\{\left[R^{\prime}-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}+D\left(q_{Y \mid X}| | W \mid q_{X}\right)\right. \\
&\left.-\mu\left(\Gamma-\mathrm{E}_{q_{X}}[c(X)]\right)\right\}-\lambda\left(R^{\prime}-R\right) \\
& \leq {\left[R^{\prime}-I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)\right]^{+}+D\left(q_{Y \mid X}^{*}| | W \mid q_{X}^{*}\right) } \\
&-\mu\left(\Gamma-\mathrm{E}_{q_{X}^{*}}[c(X)]\right)-\lambda\left(R^{\prime}-R\right) \\
& \stackrel{(\mathrm{b})}{=} D\left(q_{Y \mid X}^{*} \| W \mid q_{X}^{*}\right)+\lambda\left[R-I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)\right] \\
&-\mu\left(\Gamma-\mathrm{E}_{q_{X}^{*}}[c(X)]\right)=G_{\mathrm{DK}}^{(\mu, \lambda)}(R, \Gamma \mid W) .
\end{aligned}
$$

Step (a) follows from (73). Step (b) follows from the choice of $R^{\prime}=I\left(q_{X}^{*}, q_{Y \mid X}^{*}\right)$.

## D. Proof of Property 3 part b)

Proof of Property 3 part b): We first observe that

$$
\begin{align*}
& G_{\mathrm{DK}}(R, \Gamma \mid W) \\
= & \min _{q: \mathrm{E}_{q}[c(X)] \leq \Gamma}\left\{\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}+D\left(q_{Y \mid X}| | W \mid q_{X}\right)\right\} \\
= & \min _{q: \mathrm{E}_{q}[c(X)] \leq \Gamma} \Theta(R, q \mid W), \tag{74}
\end{align*}
$$

where we set

$$
\begin{aligned}
& \Theta(R, q \mid W) \\
\triangleq & {\left[R-I\left(q_{X}, q_{Y \mid X}\right)\right]^{+}+D\left(q_{Y \mid X}| | W \mid q_{X}\right) } \\
= & \max \left\{R-I\left(q_{X}, q_{Y \mid X}\right)+D\left(q_{Y \mid X} \| W \mid q_{X}\right)\right. \\
& \left.D\left(q_{Y \mid X}| | W \mid q_{X}\right)\right\}
\end{aligned}
$$

For each $i=0,1$, let $q_{X Y}^{(i)}$ be a probability distribution that attains $G_{\mathrm{DK}}\left(R_{i}, \Gamma_{i} \mid W\right)$. By definition we have

$$
\begin{equation*}
G_{\mathrm{DK}}\left(R_{i}, \Gamma_{i} \mid W\right)=\Theta\left(R_{i}, q^{(i)} \mid W\right) \text { for } i=0,1 \tag{75}
\end{equation*}
$$

For $\alpha_{0} \in[0,1]$, we set $q_{X Y}^{(\alpha)}=\alpha_{0} q_{X Y}^{(0)}+\alpha_{1} q_{X Y}^{(1)}$, where $\alpha_{1}=1-\alpha_{0}$. The quantities $q_{X}^{(\alpha)}$ and $q_{Y \mid X}^{(\alpha)}$ are probability and conditional probability distributions induced by $q_{X Y}^{(\alpha)}$. Set $\Gamma_{\alpha} \triangleq \alpha_{0} \Gamma_{0}+\alpha_{1} \Gamma_{1}$. By the linearity of $\mathrm{E}_{q}[c(X)]$ with respect to $q$, we have that

$$
\begin{equation*}
\mathrm{E}_{q^{(\alpha)}}[c(X)]=\sum_{i=0,1} \alpha_{i} \mathrm{E}_{q^{(i)}}[c(X)] \leq \Gamma_{\alpha} \tag{76}
\end{equation*}
$$

By the convex property of $-I\left(q_{X}, q_{Y \mid X}\right)+D\left(q_{Y \mid X}| | W \mid q_{X}\right)$ and $D\left(q_{Y \mid X}| | W \mid q_{X}\right)$ with respect to $q$, we have that

$$
\left.\begin{array}{l}
-I\left(q_{X}^{(\alpha)}, q_{Y \mid X}^{(\alpha)}\right)+D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right)  \tag{77}\\
\quad \leq \sum_{i=0,1} \alpha_{i}\left[-I\left(q_{X}^{(i)}, q_{Y \mid X}^{(i)}\right)+D\left(q_{Y \mid X}^{(i)}| | W \mid q_{X}^{(i)}\right)\right] \\
D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right) \leq \sum_{i=0,1} \alpha_{i} D\left(q_{Y \mid X}^{(i)} \| W \mid q_{X}^{(i)}\right)
\end{array}\right\}
$$

Set $R_{\alpha} \triangleq \alpha_{0} R_{0}+\alpha_{1} R_{1}$. We have the following two chains of inequalities:

$$
\begin{align*}
& \quad R_{\alpha}-I\left(q_{X}^{(\alpha)}, q_{Y \mid X}^{(\alpha)}\right)+D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right) \\
& \stackrel{(\mathrm{a})}{\leq} \sum_{i=0,1} \alpha_{i}\left[R_{i}-I\left(q_{X}^{(i)}, q_{Y \mid X}^{(i)}\right)+D\left(q_{Y \mid X}^{(i)} \| W \mid q_{X}^{(i)}\right)\right] \\
& \stackrel{\text { (b) }}{\leq} \sum_{i=0,1} \alpha_{i} \Theta\left(R_{i}, q^{(i)} \mid W\right) \\
& \\
& \quad D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right) \stackrel{(\mathrm{c})}{\leq} \sum_{i=0,1} \alpha_{i} D\left(q_{Y \mid X}^{(i)}| | W \mid q_{X}^{(i)}\right)  \tag{79}\\
& \text { (d) } \\
& \stackrel{\sum_{i=0,1}}{ } \alpha_{i} \Theta\left(R_{i}, q^{(i)} \mid W\right) .
\end{align*}
$$

Steps (a) and (c) follow from (77). Steps (b) and (d) follow from the definition of $\Theta\left(R_{i}, q^{(i)} \mid W\right), i=0,1$. From (78) and (79), we have that

$$
\begin{equation*}
\Theta\left(R_{\alpha}, q^{(\alpha)} \mid W\right) \leq \sum_{i=0,1} \alpha_{i} \Theta\left(R_{i}, q^{(i)} \mid W\right) \tag{80}
\end{equation*}
$$

Thus we have the following chain of inequalities

$$
\begin{aligned}
& G_{\mathrm{DK}}\left(R_{\alpha}, \Gamma_{\alpha} \mid W\right)=\min _{q: \mathrm{E}_{q}[c(X)] \leq \Gamma_{\alpha}} \Theta\left(R_{\alpha}, q \mid W\right) \\
& \stackrel{(\mathrm{a})}{\leq} \Theta\left(R_{\alpha}, q^{(\alpha)} \mid W\right) \stackrel{(\mathrm{b})}{\leq} \sum_{i=0,1} \alpha_{i} \Theta\left(R_{i}, q^{(i)} \mid W\right) \\
& \stackrel{(\mathrm{c})}{=} \sum_{i=0,1} \alpha_{i} G_{\mathrm{DK}}\left(R_{i}, \Gamma_{i} \mid W\right)
\end{aligned}
$$

Step (a) follows from (76). Step (b) follows from (80). Step (c) follows from (75).

## E. Proof of Property 4 part b)

Proof of Property 4 part b): We set

$$
\Theta^{(\mu)}(R, \Gamma, q \mid W) \triangleq \Theta(R, q \mid W)-\mu\left(\Gamma-\mathrm{E}_{q}[c(X)]\right)
$$

Then we have

$$
G_{\mathrm{DK}}^{(\mu)}(R, \Gamma \mid W)=\min _{q} \Theta^{(\mu)}(R, \Gamma, q \mid W)
$$

For each $i=0,1$, let $q_{X Y}^{(i)}$ be a probability distribution that attains $G_{\mathrm{DK}}\left(R_{i}, \Gamma_{i} \mid W\right)$. By definition we have

$$
\begin{equation*}
G_{\mathrm{DK}}\left(R_{i}, \Gamma_{i} \mid W\right)=\Theta^{(\mu)}\left(R_{i}, \Gamma_{i}, q^{(i)} \mid W\right) \text { for } i=0,1 \tag{81}
\end{equation*}
$$

For $\alpha_{0} \in[0,1]$, we set $q_{X Y}^{(\alpha)}=\alpha_{0} q_{X Y}^{(0)}+\alpha_{1} q_{X Y}^{(1)}$, where $\alpha_{1}=1-\alpha_{0}$. The quantities $q_{X}^{(\alpha)}$ and $q_{Y \mid X}^{(\alpha)}$ are probability and conditional probability distributions induced by $q_{X Y}^{(\alpha)}$. By the convex property of

$$
\begin{aligned}
& -I\left(q_{X}, q_{Y \mid X}\right)+D\left(q_{Y \mid X} \| W \mid q_{X}\right)+\mu \mathrm{E}_{q}[c(X)] \\
& \text { and } D\left(q_{Y \mid X} \| W \mid q_{X}\right)+\mu \mathrm{E}_{q}[c(X)]
\end{aligned}
$$

with respect to $q$, we have that

$$
\left.\begin{array}{l}
-I\left(q_{X}^{(\alpha)}, q_{Y \mid X}^{(\alpha)}\right)+D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right)+\mu \mathrm{E}_{q^{(\alpha)}}[c(X)] \\
\leq \sum_{i=0,1} \alpha_{i}\left[-I\left(q_{X}^{(i)}, q_{Y \mid X}^{(i)}\right)+D\left(q_{Y \mid X}^{(i)} \| W \mid q_{X}^{(i)}\right)\right. \\
\left.\quad+\mu \mathrm{E}_{q^{(i)}}[c(X)]\right] \\
D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right)+\mu \mathrm{E}_{q^{(\alpha)}}[c(X)] \\
\leq \sum_{i=0,1} \alpha_{i}\left[D\left(q_{Y \mid X}^{(i)}| | W \mid q_{X}^{(i)}\right)+\mu \mathrm{E}_{q^{(i)}}[c(X)]\right] . \tag{82}
\end{array}\right\}
$$

Then we have the following two chains of inequalities:

$$
\begin{align*}
& R_{\alpha}-I\left(q_{X}^{(\alpha)}, q_{Y \mid X}^{(\alpha)}\right)+D\left(q_{Y \mid X}^{(\alpha)}| | W \mid q_{X}^{(\alpha)}\right) \\
&-\mu\left(\Gamma_{\alpha}-\mathrm{E}_{q^{(\alpha)}}[c(X)]\right) \\
& \stackrel{\text { (a) }}{\leq} \sum_{i=0,1} \alpha_{i}\left[R_{i}-I\left(q_{X}^{(i)}, q_{Y \mid X}^{(i)}\right)+D\left(q_{Y \mid X}^{(i)}| | W \mid q_{X}^{(i)}\right)\right. \\
&\left.-\mu\left(\Gamma_{i}-\mathrm{E}_{q^{(i)}}[c(X)]\right)\right] \\
& \stackrel{\text { (b) }}{\leq} \sum_{i=0,1} \alpha_{i} \Theta^{(\mu)}\left(R_{i}, \Gamma_{i}, q \mid W\right),  \tag{83}\\
& D\left(q_{Y \mid X}^{(\alpha)} \| W \mid q_{X}^{(\alpha)}\right)-\mu\left(\Gamma_{\alpha}-\mathrm{E}_{q^{(\alpha)}}[c(X)]\right) \\
& \stackrel{\text { (c) }}{\leq} \sum_{i=0,1} \alpha_{i}\left[D\left(q_{Y \mid X}^{(i)}| | W \mid q_{X}^{(i)}\right)-\mu\left(\Gamma_{i}-\mathrm{E}_{q^{(i)}}[c(X)]\right)\right] \\
& \text { (d) } \sum_{i=0,1} \alpha_{i} \Theta^{(\mu)}\left(R_{i}, \Gamma_{i}, q^{(i)} \mid W\right) . \tag{84}
\end{align*}
$$

Steps (a) and (c) follow from 82). Steps (b) and (d) follow from the definition of $\Theta\left(R_{i}, \Gamma_{i}, q^{(i)} \mid W\right), i=0,1$. From (83)
and (84), we have that

$$
\begin{equation*}
\Theta^{(\mu)}\left(R_{\alpha}, \Gamma_{\alpha}, q^{(\alpha)} \mid W\right) \leq \sum_{i=0,1} \alpha_{i} \Theta^{(\mu)}\left(R_{i}, \Gamma_{i}, q^{(i)} \mid W\right) \tag{85}
\end{equation*}
$$

Thus we have the following chain of inequalities

$$
\begin{aligned}
& G_{\mathrm{DK}}^{(\mu)}\left(R_{\alpha}, \Gamma_{\alpha} \mid W\right)=\min _{q} \Theta^{(\mu)}\left(R_{\alpha}, \Gamma_{\alpha}, q \mid W\right) \\
\leq & \Theta^{(\mu)}\left(R_{\alpha}, \Gamma_{\alpha}, q^{(\alpha)} \mid W\right) \stackrel{(\mathrm{a})}{\leq} \sum_{i=0,1} \alpha_{i} \Theta^{(\mu)}\left(R_{i}, \Gamma_{i}, q^{(i)} \mid W\right) \\
\stackrel{(\mathrm{b})}{=} & \sum_{i=0,1} \alpha_{i} G_{\mathrm{DK}}^{(\mu)}\left(R_{i}, \Gamma_{i} \mid W\right) .
\end{aligned}
$$

Step (a) follows from 85). Step (b) follows from 81.

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[^0]:    Y. Oohama is with Dept. of Communication Engineering and Informatics, University of Electro-Communications, 1-5-1 Chofugaoka Chofu-shi, Tokyo 182-8585, Japan.

