LETTER

High-Order Bi-orthogonal Fourier Transform and Its Applications in Non-stability Signal Analysis

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SUMMARY This paper presents a novel signal analysis algorithm, named High-order Bi-orthogonal Fourier Transform (HBFT), which can be seen as an expansion of Fourier transform. The HBFT formula and discrete HBFT formula are derived, some of their main characteristics are briefly discusses. This paper also uses HBFT to analyze the multi-LFM signals, obtain the modulate rate parameters, analyze the high dynamic signals, and obtain the accelerated and varying accelerated motion parameters. The result proves that HBFT is suitable for analysis of the non-stability signals with high-order components.

key words: *high-order bi-orthogonal Fourier transform, Bi-orthogonal base, non-stability signal, multi-LFM signal, high dynamic signals*

1. Introduction

Fourier analysis is suitable for analyzing certainty or stationary signals, and it plays an important role in the fields of signal processing, image processing, and microwave and antenna technology [1]. In practical applications, the signals are always the non-stability signals whose frequency is variable over time, such as linear frequency modulated (LFM) signals or signals of high dynamic receivers. This kind of signal is needed to analyze the second-order modulated rate parameter, even the second and third-order accelerated and varying accelerated motion parameters, Fourier analysis is no longer the best tool. So varieties of derivative signal analysis algorithms based on Fourier Transform have been developed, such as short time Fourier transform (STFT) [2], Gabor transform (GT) [3], [4], fractional Fourier Transform (FRFT) [5] and the fractional Gabor transform [6]. They all improve Fourier transform in some aspects, for example time-frequency characteristic, but they are not suitable for high-order parameter analysis of non-stability signals.

High-order Bi-orthogonal Fourier Transform (HBFT) presented in this paper can be seen as high-order expansion of Fourier transform, since the first-order HBFT is the same as Fourier Transform. The transform kernel of second-order HBFT has the same form with modulated rate parameter of LFM signal, so it's good for analysis FM signals. And the HBFT is also suitable for analysis the accelerated and varying accelerated motion parameter of high dynamic signals.

Manuscript received April 18, 2014.
Manuscript revised August 11, 2014.

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DOI: 10.1587/transinf.2014EDL8077

2. High-order Bi-orthogonal Fourier Transform

There exist two function series $\{\phi_k(t)|k \in Z\}$ and $\{\tilde{\phi}_l(t)|l \in Z\}$ of $L^2[0,T]$,

$$< \phi_k, \tilde{\phi}_l >= \int_0^T \phi_k, \tilde{\phi}_l^* \mathrm{d}t = \delta(k-l). \tag{1}$$

$\{\phi_k(t)\}$ and $\{\tilde{\phi}_l(t)\}$ compose normative bi-orthogonal basis functions of $L^2[0,T]$. Let

$$\begin{aligned} \phi_k(t) &= \exp(\mathrm{i}2\pi k \frac{t^n}{T^n}) \\ \tilde{\phi}_l(t) &= \frac{nt^{n-1}}{T^n} \exp(\mathrm{i}2\pi l \frac{t^n}{T^n}) \end{aligned}, \tag{2}$$

where $n \in Z^+, n \neq +\infty$. Now it is possible to expand these signal functions into a bi-orthogonal series.

Let $f(t)$ be a continuous function of $L^2[0,T]$, and then it can be expressed using linear combination of above bi-orthogonal basis function, namely

$$f(t) = \sum_{k=-\infty}^{+\infty} < f(t), \tilde{\phi}_k(t) > \phi_k(t). \tag{3}$$

So Eq. (3) can be written as follow

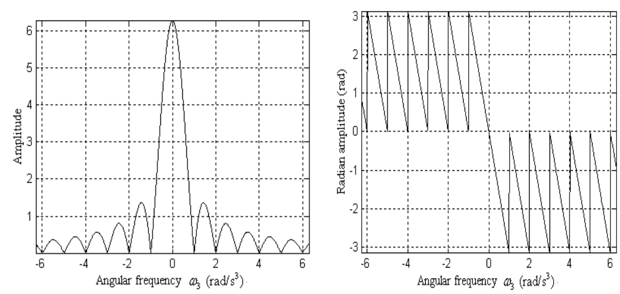
$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{+\infty} F(k)\phi_k(t) = \sum_{k=-\infty}^{+\infty} F(k) \exp(\mathrm{i}2\pi k \frac{t^n}{T^n}) \\ F(k) &=< f(t), \tilde{\phi}_k(t) >= \frac{n}{T^n} \int_0^T f(t)t^{n-1} \exp(-\mathrm{i}2\pi k \frac{t^n}{T^n})\mathrm{d}t \end{aligned} \tag{4}$$

When $T \to +\infty$, it results

$$\begin{aligned} F(\omega_n) &= n \int_0^{+\infty} f(t)t^{n-1} \exp(-\mathrm{i}\omega_n t^n)\mathrm{d}t \equiv \mathrm{HBFT}_n[f(t)] \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega_n) \exp(\mathrm{i}\omega_n t^n)\mathrm{d}\omega_n \equiv \mathrm{IHBFT}_n[F(\omega_n)] \end{aligned} \tag{5}$$

Equation (5) is called nth-order bi-orthogonal Fourier transform and its inverse transform. Variable ω_n is nth-order bi-orthogonal frequency and $F(\omega_n)$ is called nth-order bi-orthogonal frequency spectrum.

In practice, the signal after sampled is discrete, so discrete HBFT is presented refers to discrete Fourier transform. The sampling frequency is f_s, sampling point number is N, and then translating the variables into discrete values, namely $t = l\Delta t$ and $\omega_n = k\Delta\omega_n$, signal time length $T = N\Delta t$



(a) Amplitude spectrum (b) Phase spectrum

Fig. 1 The third-order HBFT spectrum of rectangular pulse function.

and spectrum length $\Omega_n = N\Delta\omega_n$ are given by

$$\Delta t = \frac{1}{f_s},\ \Delta\omega_n = \frac{2\pi}{T^n},\ \Omega_n = \frac{2\pi}{\Delta t T^{n-1}} = \frac{2\pi}{T^{n-1}} f_s. \tag{6}$$

When a discrete signal is $f(l)$ and discrete spectrum is $F(k)$, the discrete HBFT is

$$\begin{aligned} F(k) &= \frac{n}{f_s^{n-1}} \sum_{l=0}^{N} f(l) l^{n-1} \exp(-\mathrm{i}\frac{2\pi}{N^n} k l^n) \\ f(l) &= \frac{1}{N^n} \sum_{k=0}^{N} F(k) \exp(\mathrm{i}\frac{2\pi}{N^n} k l^n) \end{aligned} . \tag{7}$$

From Eq. (6), it can be seen that the frequency resolution $\Delta\omega_n$ is only related with the signal length T, while the max nth-order frequency Ω_n is not only dominated by the time interval Δt (or sampling frequency f_s), but also the signal length T.

A normalized rectangular pulse function is defined by

$$E(t) = \begin{cases} 1 & 0 < t < T \\ 0 & t \geq T \end{cases} . \tag{8}$$

Its nth-order HBFT is

$$\begin{aligned} \mathrm{HBFT}_n[E(t)] &= \frac{\mathrm{i}}{\omega_n}[\exp(-\mathrm{i}\omega_n T^n) - 1] \\ &= T^n \mathrm{sinc}(\frac{\omega_n T^n}{2}) \exp(-\mathrm{i}\frac{\omega_n T^n}{2}). \end{aligned} \tag{9}$$

Taking absolute value, the nth-order amplitude spectrum is

$$|\mathrm{HBFT}_n[E(t)]| = T^n \mathrm{sinc}(\frac{\omega_n T^n}{2}). \tag{10}$$

The nth-order amplitude spectrum of rectangular pulse function is shown as sinc function.

Setting $n = 3$, and the time length of rectangular pulse is $T = \sqrt[3]{2\pi}$ s for example. The third-order HBFT spectrums are shown in Fig. 1.

By simulation, even higher order HBFT spectrums of rectangular pulse function are similar to Fig. 1.

Let $T \to +\infty$, then the nth-order HBFT amplitude spectrum of E(t) is

$$\mathrm{HBFT}_n[1] = \lim_{T\to\infty} \mathrm{HBFT}[E(t)] = 2\pi\delta(\omega_n). \tag{11}$$

For nth-order FM signal whose expression is

$$f(t) = A\exp(\mathrm{i}Kt^n). \tag{12}$$

Where K is FM rate parameter and A is envelope, its nth-order HBFT amplitude spectrum is

$$F(\omega_n) = \mathrm{HBFT}_n[A\exp(\mathrm{i}Kt^n)] = 2\pi A\delta(\omega_n - K) \tag{13}$$

3. HBFT of Non-stability Signal

Like other algorithms, HBFT can be applied for filtering, denoising, analysis and synthesis of signals. Moreover HBFT has the unique feature that other algorithms do not have, such as FM rate estimation, and the accelerated and varying accelerated motion parameters analysis, so it is particularly suitable for analysis the non-stability signals [7], [8].

3.1 FM Rate Estimation of Multi-LFM Signal

An important application of HBFT is to use the second-order discrete HBFT for detection and analysis different FM rate multi-LFM signal. If a multi-LFM signal is defined as

$$f_m(t) = \sum_{i=1}^{m} A_i \exp(\mathrm{i}(f_0 t + K_i t^2), \tag{14}$$

Where K_i is FM rate and A_i is corresponding normalized amplitude. After down conversion, the fundamental frequency component f_0 is removed. Setting m=9,

$$\begin{aligned} K_9 &= \{100, 120, 160, 450, 455, 560, 570, 840, 880\} \\ A_9 &= \{1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.6, 0.7, 0.8\} \end{aligned},$$

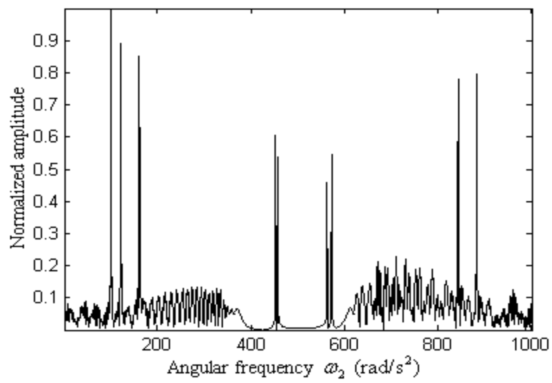
and sampling time length $T_0 = \sqrt{2\pi}$ s, the spectrum resolution $\Delta\omega_2 = 1$ rad/s^2, the sampling point number of signals $N_0 = 1000$, then $\Omega_2 = 1000$ rad/s^2 and the sampling frequency $f_s = 282Hz$. The second-order discrete HBFT (2nd HBFT) of signal is shown in Fig. 2 (a).

When the simulation conditions are replaced with $T = 2T_0$ and $N = N_0$, then $\Delta\omega_2 = 1/4$ rad/s2 and $\Omega_2 = 250$ rad/s^2, the 2nd HBFT is shown in Fig. 2 (b); when $T = T_0$ and $N = 2N_0$, that $\Delta\omega_2 = 1$ rad/s2 and $\Omega_2 = 2000$ rad/s^2, the 2nd HBFT is shown in Fig. 2 (c); when $T = 2T_0$ and $N = 2N_0$, that $\Delta\omega_2 = 1/4$ rad/s2 and $\Omega_2 = 500$ rad/s^2, 2nd HBFT is shown in Fig. 2 (d).

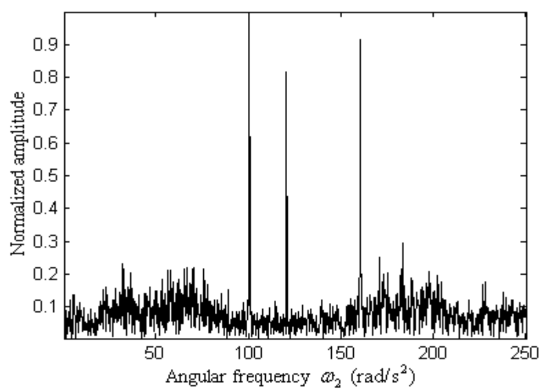
From the Fig. 2, we can see there is no cross-term between the multi-LFM signals, and using HBFT provides better performance estimating the FM rate. Figure 2 (b) and Fig. 2 (d) show that when some FM rate components exceed the maximum spectrum range, it does not affect the analysis of the other components.

In fact, even if sampling frequency meets sampling law, the discrete HBFT spectrum will still have noise because bi-orthogonal basis function of HBFT does not fit cyclic convolution condition (except for first-order).

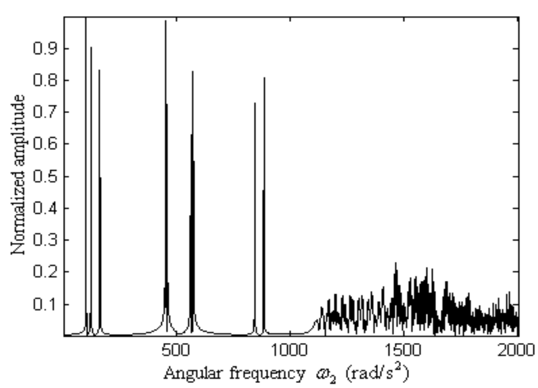
Nonlinear FM signals are common in nature, such as bat sonar system and in artificial applications that use



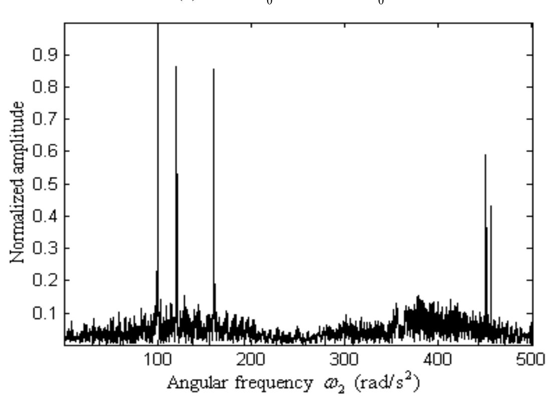
(a) $T = T_0$, $N = N_0$



(b) $T = 2T_0$, $N = N_0$



(c) $T = T_0$, $N = 2N_0$



(d) $T = 2T_0$, $N = 2N_0$

Fig. 2 The second-order discrete HBFT spectrums of 9 LFM signals.

second-order or the third-order FM signals for echo location [8], [9]. Very good results can be achieved by using the second-order or higher order HBFT to process these signals; the basic idea is similar to LFM signal analysis.

3.2 HBFT of High Dynamic Signals

Another application of HBFT is high-order accelerated and varying accelerated motion parameters analysis. The accelerated motion always happens in high dynamic carriers, such as aircraft and missile. The signal-to-noise ratio of the receivers on carriers is always worse for the high order modulated components which come from varying accelerated motion. For example missile-borne SAR, from the imaging principle of SAR, the carrier should be uniform motion, but the missile is always in accelerated motion, or even varying accelerated motion. The motion leads to range migration of missile borne SAR, and the imaging precision of some algorithms decrease, even no image [9], [10]. So the high-order accelerated and varying accelerated motion parameters analysis is demanded, which can be based on to de-noise or filter.

Suppose there is no noise outside, after demodulated, the signal of high dynamic receiver is

$$s(t) = Ae^{j\theta(t)}. \tag{15}$$

Where A is amplitude, and phase $\theta(t)$ is defined as integration of Doppler frequency $f_d(t)$

$$\theta(t) = \int_0^t 2\pi f_d(\tau)d\tau. \tag{16}$$

Doppler frequency (suppose all the k-order varying rates are existing) can be expressed as the Taylor series expansion, when the time is t_0, as below

$$f_d(t) = \sum_{k=0}^{\infty} f_k \frac{(t-t_0)^k}{k!}, \text{ where } f_k = \frac{\partial^k f_d(t)}{\partial t^k}\Big|_{t=t_0}. \tag{17}$$

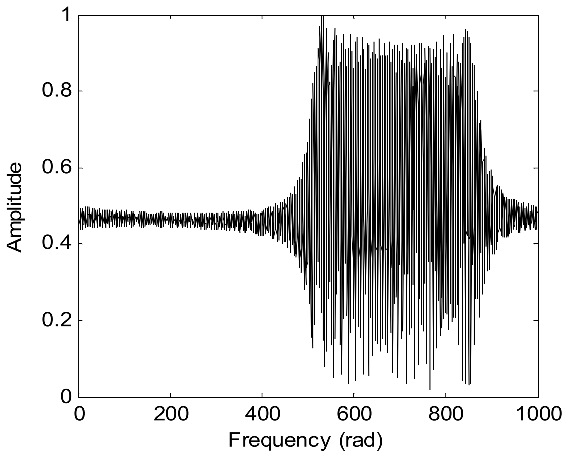
When $t_0 = 0$,

$$\begin{aligned} \theta(t) &= \theta_0 + 2\pi\left[f_0 t + \frac{f_1}{2}t^2 + \frac{f_2}{6}t^3\right] \\ &= \theta_0 + \omega_0 t + \frac{\omega_1}{2}t^2 + \frac{\omega_2}{6}t^3 \end{aligned}. \tag{18}$$

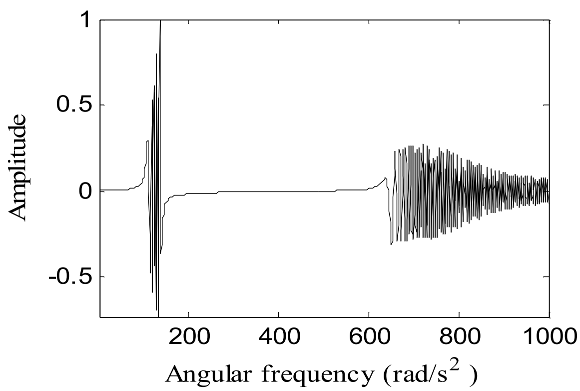
Where f_0 is Doppler frequency (Hz), f_1 is second order varying rate (Hz/s), f_2 is third order varying rate (Hz/s^2). And $\omega_0, \omega_1, \omega_2$ are angular frequency, respectively related to dynamic speed, acceleration, and varying acceleration of receiver.

When the receiver is in high dynamic moving, normalized the amplitude to 1 and omitted the constant term, the instantaneous angular frequency $[\omega_0, \omega_1, \omega_2] = [500, 200, 60]$. The duration of the signal $T_0 = \sqrt{2\pi}$, spectrum resolution $\Delta\omega_2 = 1$ rad/s^2, sample point $N_0 = 1000$. The first order, second order and third order HBFT are shown in Fig. 3 (a), (b), (c).

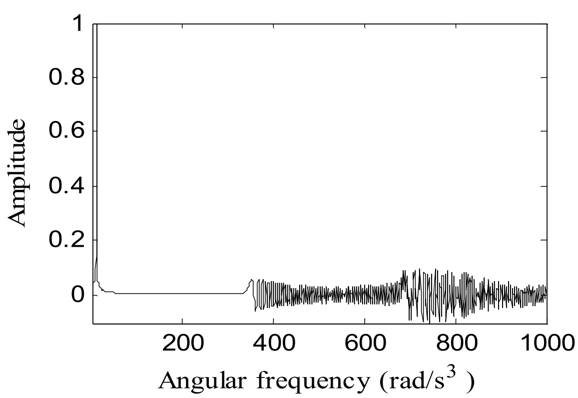
We can see from Fig. 3, for the change of acceleration



(a) First order HBFT



(b) Second order HBFT



(c) Third order HBFT

Fig. 3 HBFT of high dynamic signal.

and varying acceleration are great, this not only lead to the expansion of spectrum, but also emerge the high-order noise modulation, as shown in Fig. 3 (a). The second order acceleration spectrum also emerge the high-order noise modulation, for the third order varying acceleration term, as shown in Fig. 3 (b). The third order varying acceleration parameters is shown in Fig. 3 (c).

After using HBFT, the high-order parameters of the signal are obtained, and can be filtered or de-noised according to requirement.

4. Conclusions

This paper introduces HBFT, discusses some major characteristics of this new algorithm and applies HBFT to analyze the non-stability signals. The results indicate that HBFT is good at analyzing the high-order parameters of some FM signals and high dynamic signals. In the above discussion, the order n has been restrained to positive integer, when in fact the values of n could be extended to rational domain, leading to rational order bi-orthogonal Fourier transform.

Acknowledgments

The authors would like to thank the anonymous reviewers and editors for their helpful suggestions. This work is supported by National Natural Science Foundation of China under Grants 61371038.

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