PAPER Special Section on Foundations of Computer Science—New Spirits in Theory of Computation and Algorithm—

The Biclique Cover Problem and the Modified Galois Lattice*

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SUMMARY The minimum biclique cover problem is known to be NP-hard for general bipartite graphs. It can be solved in polynomial time for C4-free bipartite graphs, bipartite distance hereditary graphs and bipartite domino-free graphs. In this paper, we define the modified Galois lattice $G_m(B)$ for a bipartite graph B and introduce the redundant parameter $R(B)$. We show that $R(B) = 0$ if and only if B is domino-free. Furthermore, for an input graph such that $R(B) = 1$, we show that the minimum biclique cover problem can be solved in polynomial time.

key words: *biqlique cover, Galois lattice, domino-free*

1. Introduction

The problem of covering the edges of a graph has been studied in various ways. In this paper, we consider the covering problem in which all edges of an input bipartite graph are covered by the edges of bicliques (complete bipartite graphs). Covering a graph by bicliques arises in many areas [2]. From theoretical interests, Stockmeyer [3] investigated the computational complexities of the biclique cover problem and showed that it is equivalent to the set basis problem [3]. In computer graphics, bicliques are used to model the rectangle cover problem that asks if a rectilinear polygon can be expressed as the union of a minimum number of rectangles [4]. There are some applications in artificial intelligence, data mining [5] and biology [6].

The minimum biclique cover problem is NP-hard for general bibartite graphs [3], [7], [8] and it is also NP-hard for chordal bipartite graphs [9]. However, it can be solved in polynomial time for $C4$-free bipartite graphs [9], bipartite distance-hereditary graphs [9] and bipartite domino-free graphs [10]. A bipartite graph is $C4$-free if it has no cycle of length four as an induced subgraph. There are some characterizations for bipartite distance-hereditary graphs and we adopt the following definition: a bipartite graph is bipartite distance-hereditary if it is (6,2)-chordal, that is, every cycle of length at least 6 has at least 2 chords. A bipartite graph is domino-free if it has no domino as an induced subgraph, where a domino is a cycle of length six with exactly one chord as in Fig. 1. By definition, neither bipartite C_4-free graphs nor bipartite distance-hereditary graphs have any domino as an induced subgraph. Thus the class of bipartite domino-free graphs is a strict generalization of bipartite $C4$-free graphs and distance-hereditary bipartite graphs.

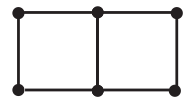

Fig. 1 A domino.

Amilhastre et al. [10] showed that the size of a minimum biclique cover and the size of a minimum biclique partition are equal if the graph is bipartite domino-free. To solve these problems, they defined a partial order for the set of maximal bicliques of a bipartite domino-free graph B. They used the Hasse diagram (the Galois lattice) $G(B)$ of this partial ordered set and solved the biclique cover/partition problem by finding a minimum cut of $G(B)$. The time complexity of this algorithm is $O(n \times m)$, where n and m are the numbers of vertices and edges of the input graph, respectively.

In this paper, we define the modified Galois lattice $G_m(B)$ for a bipartite graph B. Here we do not require that B is domino-free. Next, we introduce the redundant parameter $R(B)$, and show that $R(B) = 0$ if and only if B is domino-free. Furthermore, for the input graph such that $R(B) = 1$, we show that the minimum biclique cover problem can be solved in polynomial time.

In Sect. 2, we give definitions which are necessary for our discussion. Also we define the modified Galois lattice $G_m(B)$ for a bipartite graph B. In Sect. 3, some properties of $G_m(B)$ are investigated and some lemmas related to $G_m(B)$ are proved. In Sect. 4, defining the redundant parameter $R(B)$, we prove that B is a domino-free bipartite graph if and only if $R(B) = 0$. Also, we show that if $R(B) = 1$, the minimum biclique cover problem can be solved in polynomial time.

2. Definitions

Let $B = (X_B, Y_B, E_B)$ be a bipartite graph, where $X_B = \{x_1, x_2, \ldots, x_{n_x}\}$, $Y_B = \{y_1, y_2, \ldots, y_{n_y}\}$ are the sets of vertices and $E_B \subseteq X_B \times Y_B$ is the set of edges. Let $n = n_x + n_y$ and $m = |E_B|$. Let $N_B(x) = \{y \mid (x, y) \in E_B\}$ be the set of neighbors of x in B. A subgraph $K = (X, Y, E_K)$ of B is a *biclique* if $E_K = X \times Y$, where $\emptyset \neq X \subseteq X_B$ and $\emptyset \neq Y \subseteq Y_B$.

Manuscript received April 2, 2014.
Manuscript revised July 7, 2014.

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*Preliminary version of this paper has been appeared in SIGAL workshop [1].

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DOI: 10.1587/transinf.2014FCP0019

K is an *induced* subgraph of B if $E_K = E_B \cap (X \times Y)$. A *biclique cover* of B is a set of bicliques $\{K_1, K_2, \ldots, K_s\}$ such that $E_B = \bigcup_{i=1}^{s} E_{K_i}$, and a *biclique partition* of B is a set of bicliques $\{K_1, K_2, \ldots, K_s\}$ such that $E_B = \bigcup_{i=1}^{s} E_{K_i}$ and $E_{K_i} \cap E_{K_j} = \emptyset\,(i \neq j)$. The minimum biclique cover (partition) problem is a problem of finding a minimum biclique cover (partition, respectively) for a given bipartite graph B.

A *domino* is a cycle of length six with exactly one chord that produces two $C4$'s as in Fig. 1. A bipartite graph B is *domino-free* if B has no domino as an induced subgraph. Let $\mathcal{K}_M(B)$ be the set of maximal bicliques of B. We define a partially order $<$ on $\mathcal{K}_M(B)$ as follows. For distinct bicliques $K_p, K_q \in \mathcal{K}_M(B)$, $K_p < K_q$ if and only if $Y_{K_p} \subset Y_{K_q}$. K_r and K_s are *incomparable* if neither $K_r < K_s$ nor $K_s < K_r$. Let $(\mathcal{K}_M(B), \leq)$ be the reflexive closure of the defined ordered set.

In [10], Amilhastre et al. defined a directed graph $G(B)$ for a domino-free bipartite graph B as follows. The set of vertices of $G(B)$ is $\mathcal{K}_M(B) \cup \{\top, \bot\}$, where $\top$ is the maximum element to $\mathcal{K}_M(B)$, that is, $\top > K$ for all $K \in \mathcal{K}_M(B)$ and $\bot$ is the minimum element. For two elements K_p and K_q such that $K_p < K_q$, put a directed edge (K_q, K_p) if there is no K_r such that $K_p < K_r$ and $K_r < K_q$. They call $G(B)$ as Galois lattice of B [10]. $G(B)$ is actually the Hasse diagram of the partially ordered set $(\mathcal{K}_M(B), \leq)$ [11].

In this paper, we define the modified Galois lattice $G_m(B)$ as follows. Here, we do not assume that B is domino-free. Let $X_i\,(1 \leq i \leq n_x)$ be the maximal star graph centered at x_i. Denote the set of all X_i by $X_s(B)$. Define $Y_j\,(1 \leq j \leq n_y)$ and $Y_s(B)$ in the same manner. We define the partial order on $\mathcal{K}_s(B) \equiv \mathcal{K}_M(B) \cup X_s(B) \cup Y_s(B)$ as follows: for any distinct $K_p, K_q \in \mathcal{K}_s(B)$, $K_p < K_q$ if and only if $Y_{K_p} \subseteq Y_{K_q}$ and $X_{K_p} \supseteq X_{K_q}$. Let $\mathcal{K}(B) \equiv \mathcal{K}_s(B) \cup \{\top, \bot\}$. According to this partial order on $\mathcal{K}(B)$, we construct $G_m(B)$ in the same manner as $G(B)$. Let us see an example for a bipartite graph B shown in Fig. 2. As vertices $\{x_2, x_3, x_4, y_3, y_4, y_5\}$ induces a domino, B is not domino-free. It is obvious that B has six maximal bicliques $K_1, \ldots, K_6$ such that

$$
\begin{aligned}
&X_{K_1} = \{x_1, x_4\},\ Y_{K_1} = \{y_1, y_2, y_3\},\\
&X_{K_2} = \{x_2, x_3\},\ Y_{K_2} = \{y_2, y_3, y_4\},\\
&X_{K_3} = \{x_3, x_4\},\ Y_{K_3} = \{y_2, y_3, y_5, y_6\},\\
&X_{K_4} = \{x_1, x_2, x_3, x_4\}, Y_{K_4} = \{y_2, y_3\},\\
&X_{K_5} = \{x_3\},\ Y_{K_5} = \{y_2, y_3, y_4, y_5, y_6\}\\
&X_{K_6} = \{x_4\}, \text{ and } Y_{K_6} = \{y_1, y_2, y_3, y_5, y_6\}.
\end{aligned}
$$

Then the Galois lattice $G(B)$ and the modified Galois lattice $G_m(B)$ are shown in Fig. 3 and Fig. 4, respectively. Here, we follow the conventional drawing of the Hasse diagram, that is, each edge has downward direction. Note that the Galois lattice is embedded, in some way, in the modified Galois lattice.

Amilhastre et al. [10] defined a "simplification" operation on a domino-free bipartite graph. They repeatedly apply this operation to an input bipartite graph B until no operation can be applied. The resulted graph is called as a "simplified" domino-free bipartite graph. $G_m(B)$ is coincident with $G(B)$ if B is a simplified domino-free bipartite graph.

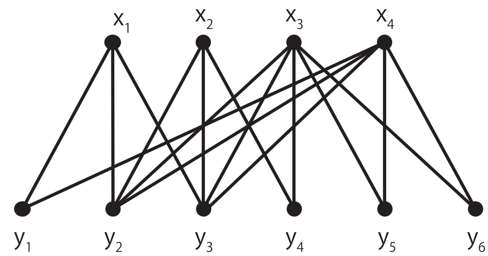

Fig. 2 A Bipartite graph B.

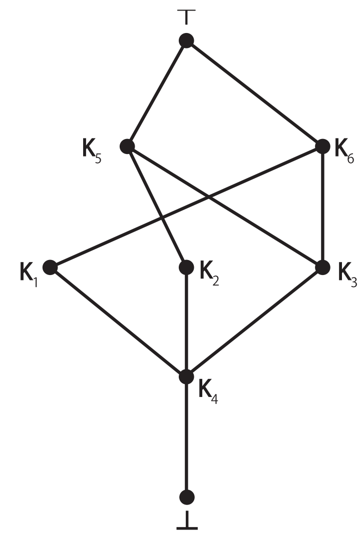

Fig. 3 The Galois lattice $G(B)$.

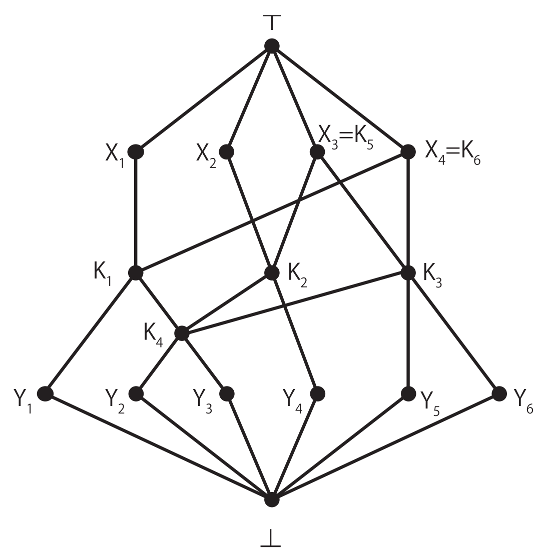

Fig. 4 The modified Galois lattice $G_m(B)$.

3. Properties of the Modified Galois Lattice

Let $K_1 = (X_{K_1}, Y_{K_1}, E_{K_1})$ and $K_2 = (X_{K_2}, Y_{K_2}, E_{K_2})$ be different bicliques of $\mathcal{K}_M(B)$. K_1 and K_2 have the following property.

Property 1: For any distinct $K_1, K_2 \in \mathcal{K}_M(B)$, $X_{K_1} \subset X_{K_2} \iff Y_{K_2} \subset Y_{K_1}$.

Proof : Let K_1 and K_2 be bicliques in $\mathcal{K}_M(B)$. Assume that $X_{K_1} \subset X_{K_2}$ and $Y_{K_2} \not\subset Y_{K_1}$. If $Y_{K_1} \subseteq Y_{K_2}$ then K_1 is not maximal. Thus $Y_{K_1} \backslash Y_{K_2} \neq \emptyset$ and $Y_{K_2} \backslash Y_{K_1} \neq \emptyset$. Then we have a biclique $K_3 = (X_{K_1}, Y_{K_1} \cup Y_{K_2}, E_{K_3})$ that properly include K_1. Therefore K_1 is not maximal. □

For two vertices $X_i \in X_s(B)$ and $Y_j \in Y_s(B)$ of $G_m(B)$, let $\mathcal{P}(i, j)$ be the set of directed paths from X_i to Y_j. Then we have the next lemma.

Lemma 1: $\mathcal{P}(i, j) \neq \emptyset \iff (x_i, y_j) \in E_B$, for all i and j.

Proof : ($\Rightarrow$) Assume that there is a directed edge from X_i to Y_j in $G_m(B)$. Then $\{y_j\} = Y_{Y_j} \subseteq Y_{X_i} = N_B(x_i)$ holds. Thus there is edge (x_i, y_j) in B. Assume that there is a path $P \in \mathcal{P}(i, j)$ from X_i to Y_j with length greater than two. Let $P = (X_i, K_{i_1}, \ldots, K_{i_s}, Y_j)$. Then $X_i > K_{i_1} > \ldots > K_{i_s} > Y_j$ and thus $X_i > Y_j$ holds. This means that in B, the center of star graph Y_j is in $Y_{X_i}(= N_B(x_i))$. Therefore, B has edge (x_i, y_j).

($\Leftarrow$) Assume that B has an edge (x_i, y_j). Then $y_j \in N_B(x_i)$, and thus, $Y_{Y_j} \subset Y_{X_i}$ and $Y_j < X_i$. Therefore, there is at least one directed path from X_i to Y_j in $G_m(B)$. □

We have the following lemmas for a vertex on a path from a vertex of $X_s(B)$ to a vertex of $Y_s(B)$ in $G_m(B)$.

Lemma 2: Let K be a vertex on a path from X_i to Y_j then $(x_i, y_j) \in E_K$.

Proof : If K is either X_i or Y_j then the lemma obviously holds. Then K is not a star graph and $X_i > K > Y_j$ holds. Therefore, in B, $Y_{X_i} \supseteq Y_K \supset Y_{Y_j} = \{y_j\}$ holds. Thus, (x_i, y_j) is an edge of K, since K is a maximal biclique. □

Lemma 3: If $(x_i, y_j) \in E_K$ for some $K \in \mathcal{K}(B) \backslash \{\top, \bot\}$ then there is a path from X_i to Y_j passing through K in $G_m(B)$.

Proof : Since $(x_i, y_j) \in E_K$, $x_i \in X_K$. Then $X_{X_i} \subseteq X_K$ and $Y_K \subseteq Y_{X_i}$. Thus $K \leq X_i$ holds. Similarly, $Y_j \leq K$ holds. From the construction of $G_m(B)$, there is a path from X_i to K and a path from K to Y_j. □

Let C be a subset of $\mathcal{K}(B) \backslash \{\top, \bot\}$. C is a *cut* of $G_m(B)$, if for all i, j, every path from X_i to Y_j on $G_m(B)$ has at least one vertex that belongs to C. That is, all paths from a vertex of $X_s(B)$ to a vertex of $Y_s(B)$ are cut by C. Obviously $\{X_1, \ldots, X_{n_x}\}$ (or also $\{Y_1, \ldots, Y_{n_y}\}$) is a cut of $G_m(B)$. A minimum cut of $G_m(B)$ is a cut with the minimum size. In Fig. 4, for example, $\{K_1, K_2, K_3\}$ is the minimum cut of $G_m(B)$.

Lemma 4: A cut of $G_m(B)$ is a biclique cover of B.

Proof : Let C be a cut of $G_m(B)$. For any $(x_i, y_j) \in E_B$, there is a path from vertex $X_i \in X_s(B)$ to vertex $Y_j \in Y_s(B)$ in $G_m(B)$ by Lemma 1. Let K be a vertex on the path and $K \in C$. From Lemma 2, K has edge (x_i, y_j). Thus, every edge (x_i, y_j) of B is contained in at least one biclique of C. □

If B is a domino-free bipartite graph, then B has the following property. (We give the proof to make the paper self-contained.)

Property 2 (Theorem 3.1 of [10]): Let B be a bipartite graph. Then B is domino-free if and only if for any distinct $K_1, K_2 \in \mathcal{K}_M(B)$ such that K_1 and K_2 have at least one common edge, one of these statements is true: (i) $X_{K_1} \subset X_{K_2}$ and $Y_{K_2} \subset Y_{K_1}$,(ii) $X_{K_2} \subset X_{K_1}$ and $Y_{K_1} \subset Y_{K_2}$.

Proof : ($\Rightarrow$) Let K_1 and K_2 be two maximal bicliques sharing a common edge $\{x, y\}$ and such that (i) and (ii) are false. From Property 1, we have $X_{K_1} \backslash X_{K_2} \neq \emptyset$, $X_{K_2} \backslash X_{K_1} \neq \emptyset$, $Y_{K_1} \backslash Y_{K_2} \neq \emptyset$ and $Y_{K_2} \backslash Y_{K_1} \neq \emptyset$. Let $x_1 \in X_{K_1} \backslash X_{K_2}$. We claim that there exists $y_2 \in Y_{K_2} \backslash Y_{K_1}$ such that $(x_1, y_2) \notin E_B$. If $Y_{K_2} \backslash Y_{K_1} \subseteq N(x_1)$ then $Y_{K_2} \subseteq N(x_1)$ since $Y_{K_1} \cap Y_{K_2} \subseteq N(x_1)$. Then K_2 is not maximal. Thus, there exists $y_2 \in Y_{K_2} \backslash Y_{K_1}$ such that $(x_1, y_2) \notin E_B$. Let $x_2 \in X_{K_2} \backslash X_{K_1}$. From similar discussion, there exist $y_1 \in Y_{K_1} \backslash Y_{K_2}$ such that $(x_2, y_1) \notin E_B$. Then, $\{x, y, x_1, y_1\}$ and $\{x, y, x_2, y_2\}$ induces two C_4's. As $(x_1, y_2) \notin E_B$ and $(x_2, y_1) \notin E_B$ holds, $\{x, y, x_1, y_1, x_2, y_2\}$ induces a domino.

($\Leftarrow$) Assume that B has a domino induced by $\{x, y, x_1, y_1, x_2, y_2\}$ with chord $\{x, y\}$. Then there is $K_1 \in \mathcal{K}_M(B)$ such that K_1 contains $C_4 = (x, y, x_1, y_1)$ and $K_2 \in \mathcal{K}_M(B)$ such that K_2 contains $C_4 = (x, y, x_2, y_2)$. Since $(x_1, y_2) \notin E_B$, $x_1 \in X_1 \backslash X_2$, so (i) is false. Similarly, we obtain that (ii) is false. □

We define *Unique Path Condition* as follows.

For all $i, j\,(1 \leq i \leq n_x, 1 \leq j \leq n_y)$

$$|\mathcal{P}(i, j)| = 1 \iff (x_i, y_j) \in E_B.$$

Lemma 5: If B is a domino-free bipartite graph then Unique Path Condition holds.

Proof : From Lemma 1, if $(x_i, y_j) \notin E_B$ then $|\mathcal{P}(i, j)| = 0$. Thus if $|\mathcal{P}(i, j)| = 1$ then $(x_i, y_j) \in E_B$. Therefore it is sufficient to prove that if $(x_i, y_j) \in E_B$ then $|\mathcal{P}(i, j)| = 1$ whenever B is a domino-free bipartite graph.

Assume that $|\mathcal{P}(i, j)| \geq 2$. Let P_1, P_2 be paths from X_i to Y_j such that $P_1 \neq P_2$. Then there are two incomparable bicliques K_1 on P_1 and K_2 on P_2. Note that neither K_1 nor K_2 is a star graph. Thus $|X_{K_1}|, |Y_{K_1}|, |X_{K_2}|, |Y_{K_2}| \geq 2$ holds. Since K_1 and K_2 are incomparable, neither $Y_{K_1} \subset Y_{K_2}$ nor $Y_{K_2} \subset Y_{K_1}$. Thus $Y_{K_1} \backslash Y_{K_2} \neq \emptyset$ and $Y_{K_2} \backslash Y_{K_1} \neq \emptyset$ hold. As K_1 and K_2 are maximal bicliques, Property 1 implies that $X_{K_1} \backslash X_{K_2} \neq \emptyset$ and $X_{K_2} \backslash X_{K_1} \neq \emptyset$. Then there exist four vertices of B, x_1, x_2, y_1 and y_2 such that $x_1 \in X_{K_1}$, $x_1 \notin X_{K_2}$, $x_2 \notin X_{K_2}$, $x_2 \in X_{K_2}$, $y_1 \in Y_{K_1}$, $y_1 \notin Y_{K_2}$, $y_2 \notin Y_{K_1}$ and $y_2 \in Y_{K_2}$. Thus, the graph induced by the set of vertices $\{x_i, x_1, x_2, y_j, y_1, y_2\}$ is a domino. This contradicts to the premise that B is a domino-free bipartite graph. Therefore, if $(x_i, y_j) \in E_B$ then $|\mathcal{P}(i, j)| = 1$. □

Also the converse of Lemma 5 holds.

Lemma 6: If Unique Path Condition holds then B is a domino-free bipartite graph.

Proof : Assume that B is not a domino-free graph. Then there is a subgraph induced by six vertices of two C_4's sharing edge (x_i, y_j). Then there are two incomparable maximal bicliques K_1 and K_2 that shares edge (x_i, y_j). Thus there is two distinct paths form X_i to Y_j in $G_m(B)$ and $|\mathcal{P}(i, j)| \geq 2$ holds. That is, if B is not a domino-free graph, then Unique Path Condition does not hold. □

Let $\mathcal{P}$ be the set of all paths from a vertex of $X_s(B)$ to a vertex of $Y_s(B)$ in $G_m(B)$, that is, $\mathcal{P} = \bigcup_{1\leq i\leq n_x, 1\leq j\leq n_y} \mathcal{P}(i, j)$. Let $P_{i,j} \in \mathcal{P}(i, j)$ be a path from X_i to Y_j. Let f be a map from $\mathcal{P}$ to E_B such that $f(P_{i,j}) \to (x_i, y_j)$. For example, in Fig. 4, a path $P = (X_2, K_2, K_4, Y_3)$ is mapped to edge (x_2, y_3), that is, $f(P) = (x_2, y_3)$.

Corollary 1: B is a domino-free bipartite graph if and only if f is bijective.

Proof : From Lemma 1, Lemma 5 and Lemma 6, the corollary holds. □

For any biqliques K_1, K_2 in B, we define a subgraph $K_{2-1} = K_2 - K_1$ such that K_{2-1} has all edges of K_2 but none of K_1, and has no singletons. We denote the edges of K_{2-1} by E_{2-1}. From Property 2, the next lemma holds.

Lemma 7: (Lemma 3.1 of [10]) Let B be a domino-free bipartite graph. Let K_1 be any maximal biclique and K_2 be any biclique in B such that $E_{K_2} \not\subset E_{K_1}$. Then K_{2-1} is a biclique.

Proof : If K_2 is a star graph, the proof is trivial. Assume that K_2 is not a star graph. Let $K_3 \in \mathcal{K}_M(B)$ such that $E_{K_2} \subseteq E_{K_3}$. By Property 2, there are two cases: (i) $X_{K_3} \subset X_{K_1}$ and (ii)$Y_{K_3} \subset Y_{K_1}$. (i) $X_{K_3} \subset X_{K_1}$ implies $X_{K_2} \subset X_{K_1}$. Then for any $x \in X_{K_2}$ and $y \in Y_{K_2} \backslash Y_{K_1}$, $(x, y) \in E_{2-1}$ and $(x, y) \notin E_{K_1}$ holds. Thus $K_{2-1} = (X_{K_2}, Y_{K_2} \backslash Y_{K_1}, E_{2-1})$ is a biclique of B. (ii) $Y_{K_3} \subset Y_{K_1}$ implies $Y_{K_2} \subset Y_{K_1}$. Then for any $x \in X_{K_2} \backslash X_{K_1}$ and $y \in Y_{K_1}$, $(x, y) \in E_{2-1}$ and $(x, y) \notin E_{K_1}$ holds. Thus $K_{2-1} = (X_{K_2} \backslash X_{K_1}, Y_{K_2}, E_{2-1})$ is a biclique of B. There is no other case. □

Theorem 1: (Theorem 3.2 of [10]) Let B be a domino-free bipartite graph. The size of a minimum biclique cover of B is equal to the size of a minimum biclique partition of B.

Proof : Let $\mathcal{S}_{\mathrm{COVER}}(B)$ be a minimum biclique cover of B and let $\mathcal{S}_{\mathrm{PARTITION}}(B)$ be a minimum biclique partition of B. Since any biclique partition of B is also a biclique cover of B, $|\mathcal{S}_{\mathrm{COVER}}(B)| \leq |\mathcal{S}_{\mathrm{PARTITION}}(B)|$ holds. Let $\mathcal{S}_{\mathrm{COVER}}(B) = \{K_1, K_2, \ldots, K_c\}$. Then $\{K_i - K_{i+1} - K_{i+2} - \cdots - K_c | 1 \leq i \leq c\}$ is a set of bicliques of B (Lemma 7) that form a biclique partition of B. Thus $|\mathcal{S}_{\mathrm{COVER}}(B)| \geq |\mathcal{S}_{\mathrm{PARTITION}}(B)|$ holds. Therefore $|\mathcal{S}_{\mathrm{COVER}}(B)| = |\mathcal{S}_{\mathrm{PARTITION}}(B)|$. □

Let $\mathcal{S}_{\mathrm{CUT}}(B)$ be a minimum cut of $G_m(B)$. The next theorem holds.

Theorem 2: Let B be a domino-free bipartite graph. Then $|\mathcal{S}_{\mathrm{CUT}}(B)| = |\mathcal{S}_{\mathrm{PARTITION}}(B)| = |\mathcal{S}_{\mathrm{COVER}}(B)|$ holds.

Proof : From Theorem 1, it is sufficient to prove that $|\mathcal{S}_{\mathrm{CUT}}(B)| = |\mathcal{S}_{\mathrm{COVER}}(B)|$. Assume that there is a path from X_i to Y_j in $G_m(B)$. Then there exists an edge (x_i, y_j) in B. As $\mathcal{S}_{\mathrm{COVER}}(B)$ covers (x_i, y_j), there exists $K \in \mathcal{S}_{\mathrm{COVER}}(B)$ such that $(x_i, y_j) \in E_K$. From Lemma 3, K is on a path from X_i to Y_j in $G_m(B)$. If B is a domino-free bipartite graph, then the path from X_i to Y_j in $G_m(B)$ is unique from Corollary 1. Thus, $\mathcal{S}_{\mathrm{COVER}}(B)$ is a cut of $G_m(B)$ and $|\mathcal{S}_{\mathrm{CUT}}(B)| \leq |\mathcal{S}_{\mathrm{COVER}}(B)|$. From Lemma 4, $|\mathcal{S}_{\mathrm{CUT}}(B)| \geq |\mathcal{S}_{\mathrm{COVER}}(B)|$ holds. Therefore, $|\mathcal{S}_{\mathrm{CUT}}(B)| = |\mathcal{S}_{\mathrm{COVER}}(B)|$. □

For a simplified domino-free bipartite graph B, Amilhastre et al. [10] showed that the size of Galois lattice $G(B)$ is $O(n + m)$. They constructed $G(B)$ in $O(n \times m)$ time. Since a minimum cut of $G(B)$ can be computed in polynomial time by using network flows techniques, the minimum cover/partition problem can be solved in polynomial time.

4. The Redundant Parameter and the Minimum Biclique Cover

We denote the degree of a vertex x in B by $d_B(x)$. We denote by $\mathcal{P}(i, *)$ the set of directed paths of $G_m(B)$ from X_i to any vertex of $Y_s(B)$, and denote by $\mathcal{P}(*, j)$ the set of directed paths of $G_m(B)$ from any vertex of $X_s(B)$ to Y_j. That is, $\mathcal{P}(i, *) = \cup_{j=1}^{n_y} \mathcal{P}(i, j)$ and $\mathcal{P}(*, j) = \cup_{i=1}^{n_x} \mathcal{P}(i, j)$.

We define $R_x(B)$ and $R_y(B)$ as follows.

$$R_x(B) \equiv \max_{1\leq i\leq n_x} (|\mathcal{P}(i, *)| - d_B(x_i)), \tag{1}$$

$$R_y(B) \equiv \max_{1\leq j\leq n_y} (|\mathcal{P}(*, j)| - d_B(y_j)). \tag{2}$$

Let $R(B) \equiv \max(R_x(B), R_y(B))$ and call it the *redundant parameter* of B. For example, for B in Fig. 2, it is easy to verify that $R(B) = 2$.

Theorem 3: B is a domino-free bipartite graph if and only if $R(B) = 0$.

Proof : Assume that B is a domino-free bipartite graph. From Corollary 1, there is a bijective map such that the unique path from X_i to Y_j is mapped to edge (x_i, y_j). Thus $|\mathcal{P}(i, *)|$ is the number of the edges incident to x_i and $|\mathcal{P}(i, *)| = d_B(x_i)$ holds for all i. Similarly, $|\mathcal{P}(*, j)| = d_B(y_j)$ holds for all j. Therefore, $R(B) = 0$ holds.

Assume that $R(B) = 0$. As $|\mathcal{P}(i, *)| \geq d_B(x_i)$, $R(B) = 0$ implies $|\mathcal{P}(i, *)| = d_B(x_i)$ for all i. From Lemma 1, there is an unique path in $\mathcal{P}(i, *)$ from X_i to each Y_j such that $(x_i, y_j) \in E_B$. Then f is a bijective map from $\mathcal{P}$ to E_B. Therefore B is a domino-free bipartite graph by Corollary 1. □

If $R(B) = 0$ then B is a domino-free bipartite graph, and any minimum cut of $G_m(B)$ defines a minimum cover/partition of B. We will show that if $R(B) = 1$, any minimum cover of B is a minimum cut of $G_m(B)$. Note that the minimum cover of B does not define the minimum partition of B, if B is not domino-free.

Theorem 4: Let B be a bipartite graph with $R(B) \leq 1$. Then any biclique cover of B is a cut of $G_m(B)$.

Proof : Assume that there is a minimum biclique cover $\mathcal{S}$ of B that is not a cut of $G_m(B)$. As $\mathcal{S}$ is not a cut, there is at least one path P that is not cut by $\mathcal{S}$ in $G_m(B)$. Let P be a

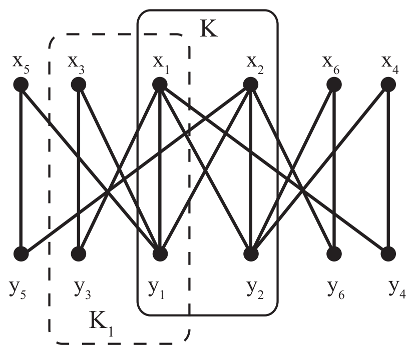

Fig. 5 K and K_1 in Case 1.

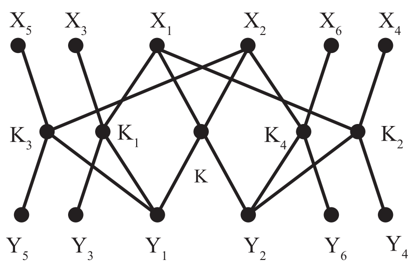

Fig. 6 The modified Galois lattice of the graph in Fig. 5 (excluding $\top$ and $\perp$).

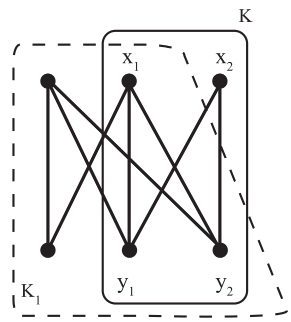

Fig. 7 K and K_1 in Case 2.

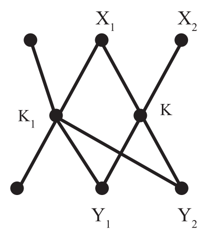

Fig. 8 The modified Galois lattice of the graph in Fig. 7 (excluding $\top$ and $\perp$).

path from X_1 to Y_1 in $G_m(B)$. Since edge (x_1, y_1) is covered by $\mathcal{S}$, if there is no vertex on P except for X_1 and Y_1, then X_1 or Y_1 is in $\mathcal{S}$. This contradicts to the assumption that P is not cut by $\mathcal{S}$. Thus there is at least one biclique K on P. Since $\mathcal{S}$ does not cut P, $K \notin \mathcal{S}$. As K is not a star graph, it has at least four vertices that induce C_4 in B. Let $x_1, x_2 \in X_K$ and $y_1, y_2 \in Y_K$ and $e_1 = (x_1, y_1)$, $e_2 = (x_1, y_2)$, $e_3 = (x_2, y_1)$ and $e_4 = (x_2, y_2)$. As $\mathcal{S}$ is a cover of B, these four edges must be covered by some bicliques K_i in $\mathcal{S}$. There are two cases that we must consider.

(Case 1) Assume that $\mathcal{S}$ has four distinct bicliques $K_1, \ldots, K_4$ such that $e_i \in E_{K_i}$ and $e_i \notin E_{K_{i'}}$ for $i \neq i'$. Then there are eight vertices such that

$$x_1, x_3 \in X_{K_1},\ y_1, y_3 \in Y_{K_1},$$
$$x_1, x_4 \in X_{K_2},\ y_2, y_4 \in Y_{K_2},$$
$$x_2, x_5 \in X_{K_3},\ y_1, y_5 \in Y_{K_3},$$
$$x_2, x_6 \in X_{K_4},\ y_2, y_6 \in Y_{K_4}.$$

See Fig. 5 and Fig. 6. Since $e_1 = (x_1, y_1) \in E_{K_1}$, K_1 is on a path P' from X_1 to Y_1 from Lemma 3. $K_1 \in \mathcal{S}$ implies $P' \neq P$. Thus the number of paths from X_1 to Y_1 is at least two. Similar discussion holds for K_2, thus the number of paths from X_1 to Y_2 is at least two. Therefore the number of paths from X_1 to Y_1 or Y_2 is at least four. From Lemma 1, there is a path from x_1 to each $y_j \in N_B(x_1)$. Thus $R(B) \geq |\mathcal{P}(1, *)| - d_B(x_1) \geq 2$ holds.

(Case 2) Assume that there is a biclique $K_1 \in \mathcal{S}$ such that K_1 has at least two edges amoung $e_i\,(i = 1 \ldots 4)$. Without loss of generality, we can assume that K_1 has e_1, e_2. (See Fig. 7 and Fig. 8.) Since $e_1 = (x_1, y_1) \in E_{K_1}$, K_1 is on a path P' from X_1 to Y_1 by Lemma 3. Thus the number of paths from X_1 to Y_1 is at least two. Since $e_2 = (x_1, y_2) \in E_K$, K is on a path P_1 from X_1 to Y_2. Since $e_2 = (x_1, y_2) \in E_{K_1}$, K_1 is on a path $P_1{}'$ from X_1 to Y_2. Thus the number of paths from X_1 to Y_2 is at least two. Therefore, there are at least four paths from X_1 to Y_1 or Y_2. From Lemma 1, there is a path from x_1 to each $y_j \in N_B(x_1)$. Thus $R(B) \geq |\mathcal{P}(1, *)| - d_B(x_1) \geq 2$ holds.

Therefore, if $R(B) \leq 1$ the assumption that $\mathcal{S}$ is not a cut of $G_m(B)$ fails. □

Theorem 4 is the best one in the sense that there is a bipartite graph B with $R(B) = 2$ for which the theorem does not hold. For example, the graph shown in Fig. 5 can be covered by $\{K_1, K_2, K_3, K_4\}$, but this set is not a cut of $G_m(B)$ (Fig. 6).

Corollary 2: Let B be a bipartite graph with $R(B) \leq 1$. Then any minimum cut of $G_m(B)$ is a minimum biclique cover of B.

Proof : Let C be a minimum cut of $G_m(B)$. From Lemma 4, C is a biclique cover of B. Let $\mathcal{S}_{\mathrm{COVER}}(B)$ be a minimum biclique cover of B. Then $|\mathcal{S}_{\mathrm{COVER}}(B)| \leq |C|$. From Theorem 4, $\mathcal{S}_{\mathrm{COVER}}(B)$ is a cut of $G_m(B)$. This implies $|\mathcal{S}_{\mathrm{COVER}}(B)| \geq |C|$. Therefore, $|\mathcal{S}_{\mathrm{COVER}}(B)| = |C|$, and thus C is a minimum biclique cover of B. □

In the rest of this paper, we investigate the size of $G_m(B)$. $G_m(B)$ could be very large if B is not domino-free. Consider the bipartite graph $B = K_{n,n} - M_n$, where $K_{n,n}$ is the complete bipartite graph with $2n$ vertices and M_n is its perfect matching. Then B has $2^n - 2$ maximal bicliques, and thus $G_m(B)$ has 2^n vertices. If $R(B) = 0$, that is, B is domino-free, then the number of edges in $G(B)$ is $O(n + m)$ [10] and also it is $O(n + m)$ in $G_m(B)$.

We will show that for a bipartite graph B with $R(B) = 1$ the number of edges in $G_m(B)$ is bounded by $2n+m$. Assume $R(B) = 1$, we have

$$\sum_{i=1}^{n_x} |\mathcal{P}(i,*)| \leq \sum_{i=1}^{n_x} (R(B) + d_B(x_i)) = n_x + m,$$
$$\sum_{j=1}^{n_y} |\mathcal{P}(*,j)| \leq \sum_{j=1}^{n_y} (R(B) + d_B(y_j)) = n_y + m.$$

Thus, the total number of paths from vertices of $X_s(B)$ to vertices of $Y_s(B)$ is at most $n+m$. Then next theorem holds.

Theorem 5: Let B be a bipartite graph with $R(B) = 1$. Then the number of edges in $G_m(B)$ is at most $2n+m$.

Proof : We replace all vertices in $G_m(B)$ that are not star graphs with bicliques as follows. Let $K \in \mathcal{K}_M(B)$ be a vertex in $G_m(B)$. Let $X_K = \{x_1, \ldots, x_s\}$, $Y_K = \{y_1, \ldots, y_t\}$ in B. Delete K and its incident edges from $G_m(B)$, and add edges $X(K) \times Y(K)$ where $X(K) = \{X_1, \ldots, X_s\}$ and $Y(K) = \{Y_1, \ldots, Y_t\}$. Note that we allow multiedges when we add edges. In this operation, the number of edges does not decrease in $G_m(B)$. The number of paths from $\top$ to $\perp$ does not change and is bounded by $2n+m$. Thus, after replacing all vertices of $\mathcal{K}_M(B)$, the total number of the edges in $G_m(B)$ is equal to the total number of the paths. Note that if we replace each multiedge with a single edge and delete $\top$ and $\perp$ and their incident edges, we obtain B. Therefore, the lemma holds. □

Gély et al. [12] gave an algorithm that outputs all maximal bicliques of an input graph $G = (U, V, E)$ in lexicographical order on U with $O((|U| + |V|)^2)$ delay. As the size of $G_m(B)$ is $O(n+m)$, $G_m(B)$ can be constructed in $O(n^3+m^3)$ time. By using network flow techniques [13], the minimum cut of $G_m(B)$ can be computed in $O(|E|\sqrt{|V|})$ for a graph $G = (V, E)$. Thus the minimum cut of $G_m(B)$ can be solved in polynomial time.

5. Conclusion

In this paper, we define the modified Galois lattice $G_m(B)$ for a bipartite graph B. We introduce the redundant parameter $R(B)$, and show that $R(B) = 0$ if and only if B is a domino-free. Furthermore, we show that the minimum biclique cover problem can be solved in polynomial time for the class of bipartite graphs B with $R(B) = 1$. This graph class properly includes the domino-free bipartite graphs.

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