PAPER

The Spanning Connectivity of the Burnt Pancake Graphs

Cherng CHIN[†,††*a)], ***Nonmember***, **Tien-Hsiung WENG**[†††], ***Member***, **Lih-Hsing HSU**[†††], ***and*** **Shang-Chia CHIOU**[††], ***Nonmembers***

SUMMARY Let u and v be any two distinct vertices of an undirected graph G, which is k-connected. For $1 \leq w \leq k$, a w-container $C(u, v)$ of a k-connected graph G is a set of w-disjoint paths joining u and v. A w-container $C(u, v)$ of G is a w^*-container if it contains all the vertices of G. A graph G is w^*-connected if there exists a w^*-container between any two distinct vertices. Let $\kappa(G)$ be the connectivity of G. A graph G is *super spanning connected* if G is i^*-connected for $1 \leq i \leq \kappa(G)$. In this paper, we prove that the n-dimensional burnt pancake graph B_n is super spanning connected if and only if $n \neq 2$.
key words: *interconnection networks, Hamiltonian cycles, Hamiltonian connected, container*

1. Introduction

The architecture of an *interconnection network* is usually represented as a *graph* where the vertices represent the processor and the edges represent the links between processors. For the graph definitions and notations, we follow [12]. Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. We use $Nbd_G(u)$ to denote the set $\{v \mid (u, v) \in E(G)\}$. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is $|Nbd_G(u)|$. We use $\delta(G)$ to denote $\min\{\deg_G(u) \mid u \in V(G)\}$. A graph is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. A path is a sequence of adjacent vertices written as $\langle v_0, v_1, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except for the possibly that $v_0 = v_m$. We also write the path $\langle v_0, P, v_m\rangle$, where $P = \langle v_0, v_1, \ldots, v_m\rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v denoted by $d(u, v)$ is the length of the shortest path of G joining u and v. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *hamiltonian cycle* is a cycle of length $V(G)$. A *hamiltonian path* is a path of length $V(G) - 1$.

Connectivity is an important issue for interconnection networks. The *connectivity* of a graph G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. Assume that G is a k-connected graph. It follows from Menger's Theorem that there are k *internally vertex-disjoint* (abbreviated as *disjoint*) *paths* joining any two distinct vertices u and v [22]. A k-container $C(u, v)$ of G is a set of k disjoint paths joining u to v. In this paper, we discuss another type of container, called spanning container. A *spanning k-container*, (abbreviated as *k^*-container*), $C(u, v)$ is a k-container such that it contains all vertices of G. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices. In particular, a graph G is 1^*-connected if and only if it is hamiltonian connected, and a graph G is 2^*-connected if and only if it is hamiltonian. All 1^*-connected graphs except K_1 and K_2 are 2^*-connected. Thus, we define the *spanning connectivity* of a graph G, $\kappa^*(G)$, to be the largest integer k such that G is w^*-connected for all $1 \leq w \leq k$ if G is a 1^*-connected graph. Obviously, spanning connectivity is a hybrid concept of hamiltonicity and connectivity. A graph G is *super spanning connected* if $\kappa^*(G) = \kappa(G)$. Obviously, the complete graph K_n is super spanning connected if $n \geq 2$.

A lot of interconnection networks are proved to be super spanning connected [16], [19], [24]. The spanning connectivity for general graphs are discussed in [17], [18]. The corresponding concept of spanning connectivity in bipartite graphs is spanning laceability. A lot of interconnection networks are proved to be super spanning laceable [2], [3], [11], [15], [16], [19], [21], [23], [24]. The burnt pancake graphs B_n was proposed by Gates and Papadimitriou [6]. Since then, many interesting properties of the burnt pancake graphs have been studied [5], [9], [13], [14]. In particular, the burnt pancake graph can be used for genome analysis [7]. In this paper, we prove that the n-dimensional burnt pancake graph B_n is super spanning connected if and only if $n \neq 2$.

2. The Burnt Pancake Graph and Its Properties

Let n be a positive integer. We use $\langle n\rangle$ to denote the set $\{1, 2, \ldots, n\}$. To save space, the negative sign may be placed on the top of an expression. Thus, $\bar{u}_1 = -u_1$. We use $[n]$ to denote the set $\langle n\rangle \cup \{\bar{i} \mid i \in \langle n\rangle\}$. A *signed permutation* of $\langle n\rangle$ is an n-permutation $u_1u_2 \ldots u_n$ of $[n]$ such that $|u_1||u_2| \ldots |u_n|$,

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[†]The author is with the Department of Computer Science and Communication Engineering, Providence University, Taichung, Taiwan 43301, R.O.C.
[††]The authors are with the Graduate School of Design, National Yunlin University of Science and Technology, Taiwan, R.O.C.
[†††]The authors are with the Department of Computer Science and Information Engineering, Providence University, Taichung, Taiwan 43301, R.O.C.
[*]Corresponding author
a) E-mail: cchin@pu.edu.tw
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taking the absolute value of each element, forms a permutation of $\langle n\rangle$. For example, $1\bar{3}265\bar{4}$ is a signed permutation of $\langle 6\rangle$. We will use bold face to denote any signed permutation of $\langle n\rangle$. Hence, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ denote a sequence of signed permutation of $\langle n\rangle$. Let $\mathbf{u} = u_1u_2\ldots u_n$ be a signed permutation of $\langle n\rangle$. We use $(\mathbf{u})_i$ to denote the i-th component u_i of $\mathbf{u}$. For $1 \le i \le n$, the i-th prefix reversal of $\mathbf{u}$, denoted by $(\mathbf{u})^i$, is the signed permutation $\mathbf{v} = v_1v_2\ldots v_n$ with $v_j = -u_{i-j+1}$ for $1 \le j \le i$ and $v_j = u_j$ if otherwise. For example, $(1\bar{3}265\bar{4})^4 = \bar{6}\bar{2}3\bar{1}\bar{5}4$. Thus, $((\mathbf{u})^i)^i = \mathbf{u}$. The *n-dimensional burnt pancake graph* B_n is a graph containing all the signed permutation of $\langle n\rangle$. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent in B_n if and only if $\mathbf{v} = (\mathbf{u})^i$. The burnt pancake graph B_1, B_2, and B_3 are shown in Fig. 1.

Obviously, B_n is an n-regular graph with $2^n n!$ vertices. We will use $B_n^{\{i\}}$ to denote the i-th subgraph of B_n induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_n = i$. Obviously, B_n can be decomposed into $2n$ vertex disjoint subgraphs $B_n^{\{i\}}$ for every $i \in [n]$ such that each $B_n^{\{i\}}$ is isomorphic to B_{n-1}. Thus, the burnt pancake graph can be constructed recursively. Let $H \subseteq [n]$, we use B_n^H to denote the subgraph of B_n induced by $\cup_{i\in H} V(B_n^{\{i\}})$. For $1 \le i, j \le n$ and $i \ne j$, we use $E^{i,j}$ to denote the set of edges between $B_n^{\{i\}}$ and $B_n^{\{j\}}$.

It is easy to check the following Lemma.

Lemma 1. *Assume that $n \ge 2$. Then $|E^{i,j}| = 2^{n-2}(n-2)!$ if $1 \le |i| \ne |j| \le n$. Moreover, $|E^{i,\bar{i}}| = 0$ for any i with $1 \le i \le n$.*

The following Theorem is proved in [14].

Theorem 1. *B_n is 1^*-connected if $n \ne 2$, and B_n is 2^*-connected if $n \ge 2$.*

Lemma 2. *Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of B_n with $d(\mathbf{u}, \mathbf{v}) \le 2$. Then $(\mathbf{u})_1 \ne (\mathbf{v})_1$. Furthermore, $\{|((\mathbf{u})^i)_1| \mid 1 \le i \le n\} = \langle n\rangle$.*

Lemma 3. *Let $n \ge 4$ and $i_1, i_2, \ldots, i_m$ be an m-permutation of $[n]$ such that $i_k \ne -i_{k+1}$ for $1 \le k < m$. Let H denote the set $\{i_1, i_2, \ldots, i_m\}$. Then there is a hamiltonian path of B_n^H joining any vertex $\mathbf{u} \in V(B_n^{\{i_1\}})$ to any other vertex $\mathbf{v} \in V(B_n^{\{i_m\}})$.*

Proof. We set $\mathbf{x_1} = \mathbf{u}$ and $\mathbf{y_m} = \mathbf{v}$. By Theorem 1, this lemma holds for $m = 1$. Assume that $m \ge 2$. By Lemma 1, we choose $(\mathbf{y_j}, \mathbf{x_{j+1}}) \in E^{i_j,i_{j+1}}$ with $\mathbf{y_j} \ne \mathbf{x_j}$ and $\mathbf{y_m} \ne \mathbf{x_m}$ for every $1 \le j \le m-1$. By Theorem 1, there is a hamiltonian path Q_j of $B_n^{\{i_j\}}$ joining $\mathbf{x_j}$ to $\mathbf{y_j}$ for every $1 \le j \le m$. The path $\langle \mathbf{x_1}, Q_1, \mathbf{y_1}, \mathbf{x_2}, Q_2, \mathbf{y_2}, \ldots, \mathbf{x_m}, Q_m, \mathbf{y_m}\rangle$ forms a desired path. □

Let I be a subset of $[n]$. We use $D(I)$ to denote $|\{j \mid j \in \langle n\rangle$ such that $\{j, \bar{j}\} \subset I\}|$. We have the following lemma.

Lemma 4. *Suppose that I be a subset of $[n]$ with $D(I) \ge 2$. Then there exists a hamiltonian path of B_n^I joining any vertex $\mathbf{u} \in V(B_n^{\{i\}})$ to any vertex $\mathbf{v} \in V(B_n^{\{j\}})$ with $\{i, j\} \subset I$ and $|i| \ne |j|$.*

Proof. The proof follows from Lemma 3 if we can construct the required permutation of elements in I. We only use several examples to illustrate that such permutation exists. Let $I = \{1, 2, \bar{1}, \bar{2}\}$. Suppose that $i = 1$ and $j = 2$. Then the corresponding sequence can be $1, \bar{2}, \bar{1}, 2$. Let $I = \{1, 2, 3, \bar{1}, \bar{2}\}$. Suppose that $i = 1$ and $j = 3$. Then the corresponding sequence can be $1, \bar{2}, \bar{1}, 2, 3$. □

Similarly, we have the following lemma.

Lemma 5. *Suppose that I is a subset of $[n]$ with $|I| \ge 5$. Then there exists a hamiltonian path of B_n^I joining any vertex $\mathbf{u} \in V(B_n^{\{i\}})$ to any vertex $\mathbf{v} \in V(B_n^{\{j\}})$ with $\{i, j\} \subset I$ and $i \ne j$.*

Lemma 6. *Let $n \ge 4$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices in $B_n^{\{t\}}$ for some $t \in [n]$. Suppose that B_{n-1} is k^*-connected. Then there is a $(k+1)^*$-container of B_n between $\mathbf{u}$ and $\mathbf{v}$.*

Proof. By assumption, there is a k^*-container $\{Q_1, Q_2, \ldots, Q_k\}$ of $B_n^{\{t\}}$ joining $\mathbf{u}$ to $\mathbf{v}$. We need to find a $(k+1)^*$-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.

Suppose that $(\mathbf{u})_1 = (\mathbf{v})_1 = p$. Thus, $(\mathbf{u})^n$ and $(\mathbf{v})^n$ are two distinct vertices in $B_n^{\{\bar{p}\}}$. By Lemma 1, there is a hamiltonian path Q of $B_n^{\{\bar{p}\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We write Q as $\langle (\mathbf{u})^n, Q', \mathbf{y}, \mathbf{z}, (\mathbf{v})^n\rangle$. By Lemma 2, $(\mathbf{y})_1 \ne (\mathbf{z})_1$, $(\mathbf{y})_1 \ne \bar{t}$, and $(\mathbf{z})_1 \ne \bar{t}$. Since $n \ge 4$, $|[n] - \{t, \bar{p}\}| \ge 6$. By Lemma 5, there exists a hamiltonian path R of $B_n^{[n]-\{t,\bar{p}\}}$ joining $(\mathbf{y})^n$ to $(\mathbf{z})^n$. We set Q_{k+1} as $\langle \mathbf{u}, (\mathbf{u})^n, Q', \mathbf{y}, (\mathbf{y})^n, R, (\mathbf{z})^n, \mathbf{z}, (\mathbf{v})^n, \mathbf{v}\rangle$. Obviously, $\{Q_1, Q_2, \ldots, Q_{k+1}\}$ forms a $(k+1)^*$-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$. See Fig. 2 (a) for illustration.

Suppose that $(\mathbf{u})_1 \ne (\mathbf{v})_1$. Thus, $(\mathbf{u})^n$ and $(\mathbf{v})^n$ are in different subgraphs. Obviously, $|[n] - \{t\}| \ge 7$. By Lemma 5, there is a hamiltonian path Q of $B_n^{[n]-\{t\}}$ joining $(\mathbf{u})^n$ to $(\mathbf{v})^n$. We set Q_{k+1} as $\langle \mathbf{u}, (\mathbf{u})^n, Q, (\mathbf{v})^n, \mathbf{v}\rangle$. Obviously, $\{Q_1, Q_2, \ldots, Q_{k+1}\}$ forms a $(k+1)^*$-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$. See Fig. 2 (b) for illustration.

Thus, the lemma is proved. □

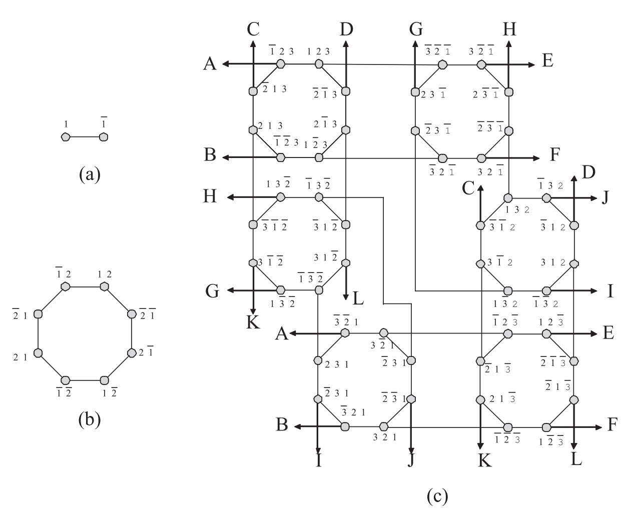

Fig. 1 The burnt pancake graphs (a) B_1, (b) B_2, and (c) B_3.

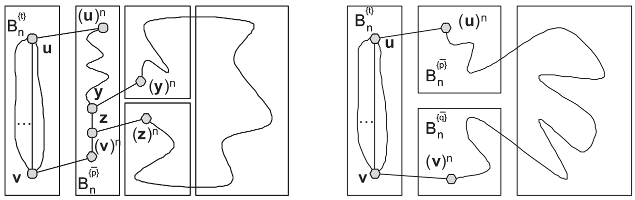

Fig. 2 Illustration for Lemma 6.

3. Basic Lemmas

The following lemma is a well-known result that gives a necessary and sufficient condition for a system of distinct representative.

Lemma 7. *[8] Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ *be a collection of sets. There exists* $\{x_1, x_2, \ldots, x_m\}$ *such that* $x_i \in A_i$ *for* $1 \le i \le m$ *and* $x_i \ne x_j$ *if* $i \ne j$ *if and only if* $|\cup_{i\in J} A_i| \ge |J|$ *for all* $J \subseteq \{1, 2, \ldots, m\}$.

Lemma 8. *Assume that n and k are positive integers with* $n \ge 4$ *and* $3 \le k \le n-1$. *Let s and t be two different elements in* $[n]$. *Let* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ *be k different vertices in* B_n *with* $(\mathbf{x_i})_1 = \bar{s}$, $(\mathbf{x_i})_n \ne t$ *for* $1 \le i \le k$, $(\mathbf{x_i})_n \ne (\mathbf{x_1})_n$ *for* $2 \le i \le k$, *and* $|(\mathbf{x_i})_n| \ne |(\mathbf{x_j})_n|$ *for* $2 \le i \ne j \le k$. *Let* $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}$ *be k different vertices in* B_n *with* $(\mathbf{y_i})_1 = \bar{t}$, $(\mathbf{y_i})_n \ne s$ *for* $1 \le i \le k$, $(\mathbf{y_i})_n \ne (\mathbf{y_1})_n$ *for* $2 \le i \le k$, *and* $|(\mathbf{y_i})_n| \ne |(\mathbf{y_j})_n|$ *for* $2 \le i \ne j \le k$. *Suppose that* $\{(\mathbf{x_i})_n \mid 1 \le i \le k\} \ne \{(\mathbf{y_i})_n \mid 1 \le i \le k\}$. *Then there exist k disjoint paths* $P_1, P_2, \ldots, P_k$ *such that (1)* P_i *joining* $\mathbf{x_i}$ *to* $\mathbf{y_{\pi(i)}}$ *for some permutation* π *from the set* $\{1, 2, \ldots, k\}$ *into itself and (2)* $\cup_{i=1}^{k} P_i$ *spans* $B_n^{[n]-\{s,t\}}$.

Proof. Since $(\mathbf{x_i})_1 = \bar{s}$, $(\mathbf{x_i})^n \in V(B_n^{\{s\}})$ and $\mathbf{x_i} \notin V(B_n^{\{s\}})$. Since $(\mathbf{x_i})_n \ne (\mathbf{x_1})_n$ for $2 \le i \le k$, and $|(\mathbf{x_i})_n| \ne |(\mathbf{x_j})_n|$ for $2 \le i \ne j \le k$, $\mathbf{x_i} \notin V(B_n^{\{t\}})$. Similarly, $\mathbf{y_i} \notin V(B_n^{\{s,t\}})$ and $(\mathbf{y_i})^n \in V(B_n^{\{t\}})$. Let I be the set $\{(\mathbf{x_i})_n \mid (\mathbf{x_i})_n = (\mathbf{y_j})_n$ for some $1 \le i, j \le k\}$. We can reorder the indices of $\{1, 2, \ldots, k\}$ so that $(\mathbf{x_i})_n = (\mathbf{y_i})_n$ for $1 \le i \le |I|$. By Lemma 3, there exists a hamiltonian path P_i of $B_n^{\{(\mathbf{x_i})_n\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for $1 \le i \le |I|$. Since $\{(\mathbf{x_i})_n \mid 1 \le i \le k\} \ne \{(\mathbf{y_i})_n \mid 1 \le i \le k\}$, $|I| < k$.

For $|I| + 1 \le i \le k$, let $A_i = \{\mathbf{y_j} \mid |I| + 1 \le j \le k$ with $(\mathbf{x_i})_n \ne -(\mathbf{y_j})_n\}$. Obviously, $|A_i| \ge k - 1 - |I|$. Thus, $|\cup_{i\in J} A_i| \ge k - 1 - |I| \ge |J|$ if $\emptyset \ne J \subset \{|I|+1, |I|+2, \ldots, k\}$. Since $\{(\mathbf{x_i})_n \mid 1 \le i \le k\} \ne \{(\mathbf{y_i})_n \mid 1 \le i \le k\}$, $|\cup_{i=|I|+1}^{k} A_i| = k - |I|$. By Lemma 7, there exists $\{\mathbf{y_i} \mid |I| + 1 \le i \le k\}$ such that $(\mathbf{x_i})_n \ne -(\mathbf{y_i})_n$ and $\mathbf{y_i} \ne \mathbf{y_j}$ for $i \ne j$. Let X be $\{(\mathbf{x_i})_n \mid 1 \le i \le k-1\} \cup \{(\mathbf{y_i})_n \mid 1 \le i \le k-1\} \cup \{s, t\}$.

Suppose that $D([n] - X) \ne 1$. By Lemma 3, there exists a hamiltonian path P_i of $B_n^{\{(\mathbf{x_i})_n, (\mathbf{y_i})_n\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for $|I| + 1 \le i \le k-1$. By Lemma 4 and Lemma 5, there exists a hamiltonian path P_k of $B_n^{[n]-X}$ joining $\mathbf{x_k}$ to $\mathbf{y_k}$. Obviously, $\{P_1, P_2, \ldots, P_k\}$ forms a set of the required paths. See Fig. 3 (a) for illustration.

Suppose that $D([n] - X) = 1$. We claim that $|I| < k - 1$. Suppose not. Then $\{(\mathbf{x_k})_n, -(\mathbf{x_k})_n, (\mathbf{y_k})_n, -(\mathbf{y_k})_n\} \subset ([n] - X)$ and $D([n] - X) \ge 2$. We get a contradiction. Let m be the only positive integer such that $\{m, \bar{m}\} \subset [n] - X$. Then there exists a hamiltonian path P_i of $B_n^{\{(\mathbf{x_i})_n, (\mathbf{y_i})_n\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for $|I| + 1 \le i \le k-2$. Moreover, there exists a hamiltonian path P_{k-1} of $B_n^{\{(\mathbf{x_{k-1}})_n, m, (\mathbf{y_{k-1}})_n\}}$ joining $\mathbf{x_{k-1}}$ to $\mathbf{y_{k-1}}$. Furthermore, there exists a hamiltonian path P_k of $B_n^{([n]-X)-\{m\}}$ joining $\mathbf{x_k}$ to $\mathbf{y_k}$. Obviously, $\{P_1, P_2, \ldots, P_k\}$ forms a set of the required paths. See Fig. 3 (b) for illustration. The lemma is proved. □

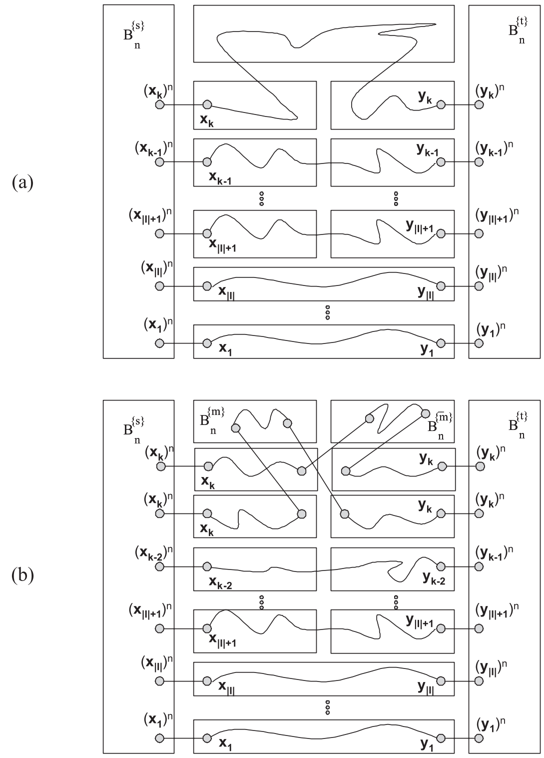

Fig. 3 Illustration for Lemma 8.

Lemma 9. *Let* $n \ge 4$ *and k be any positive integer with* $3 \le k \le n-1$. *Let* $\mathbf{u}$ *be any vertex in* $B_n^{\{s\}}$ *and* $\mathbf{v}$ *be any vertex in* $B_n^{\{t\}}$ *such that* $s \ne t$. *Suppose that* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ *are k vertices in* B_n *with* $(\mathbf{x_i})_1 = \bar{s}$ *and* $(\mathbf{x_i})_n \ne t$ *for* $1 \le i \le k$; *and* $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}$ *are k vertices in* B_n *with* $(\mathbf{y_i})_1 = \bar{t}$ *and* $(\mathbf{y_i})_n \ne s$ *for* $1 \le i \le k$. *Suppose that there exists a permutation* π *on* $\{1, 2, \ldots, k\}$ *and k disjoint paths,* $P_1, P_2, \ldots, P_k$, *such that* P_i *is a path joining* $\mathbf{x_i}$ *to* $\mathbf{y_{\pi(i)}}$ *for* $1 \le i \le k$ *and* $\cup_{i=1}^{k} P_i$ *spans* $B_n^{[n]-\{s,t\}}$. *Moreover, there are k internal disjoint paths,* $S_1, S_2, \ldots, S_k$, *of* $B_n^{\{s\}}$ *such that* S_i *is a path joining* $\mathbf{u}$ *to* $(\mathbf{x_i})^n$ *and* $\cup_{i=1}^{k} S_i$ *spans* $B_n^{\{s\}}$. *Furthermore, there are k internal disjoint paths,* $T_1, T_2, \ldots, T_k$, *of* $B_n^{\{t\}}$ *such that* T_i *is a path joining* $\mathbf{v}$ *to* $(\mathbf{y_i})^n$ *and* $\cup_{i=1}^{k} T_i$ *spans* $B_n^{\{t\}}$. *Then there exists a* k^**-container of* B_n *joining* $\mathbf{u}$ *to* $\mathbf{v}$. *Moreover, this* k^**-container does not contain the edge* $(\mathbf{u}, \mathbf{v})$ *if* $(\mathbf{u}, \mathbf{v}) \in E(B_n)$.

Proof. We set Q_i as $\langle \mathbf{u}, S_i, (\mathbf{x_i})^n, \mathbf{x_i}, P_i, \mathbf{y_{\pi(i)}}, (\mathbf{y_{\pi(i)}})^n, T_{\pi(i)}, \mathbf{v} \rangle$ for $1 \le i \le k$. Obviously, $\{Q_1, Q_2, \ldots, Q_k\}$ forms the required k^*-container between $\mathbf{u}$ and $\mathbf{v}$. See Fig. 4 for illustration. □

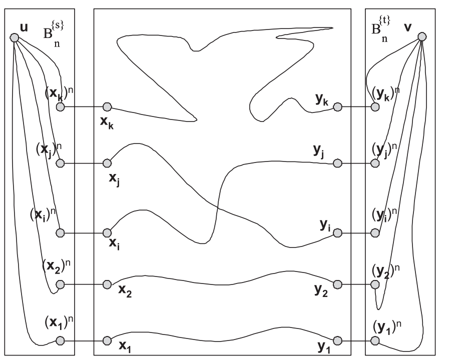

Fig. 4 Illustration for Lemma 9.

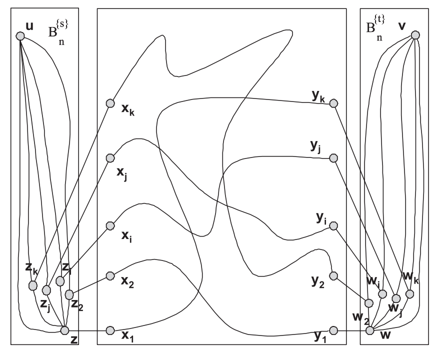

Fig. 5 Illustration for Lemma 10.

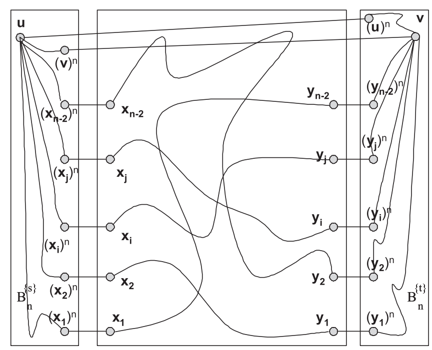

Fig. 6 Illustration for Lemma 11.

Lemma 10. *Let $n \geq 4$ and k be any positive integer with $3 \leq k \leq n-1$. Let $\mathbf{u}$ be any vertex in $B_n^{\{s\}}$ and $\mathbf{v}$ be any vertex in $B_n^{\{t\}}$ such that $s \neq t$. Suppose that B_{n-1} is k^*-connected. Then there is a k^*-container of B_n between $\mathbf{u}$ and $\mathbf{v}$. Moreover, this k^*-container does not contain the edge $(\mathbf{u},\mathbf{v})$ if $(\mathbf{u},\mathbf{v}) \in E(B_n)$.*

Proof. Let r be any element in $[n] - \{s, \bar{s}, t, \bar{t}\}$. Suppose that $s \neq \bar{t}$. We set $\mathbf{z}$ to be a vertex with $(\mathbf{z})_1 = r$, $(\mathbf{z})_2 = \bar{t}$, $(\mathbf{z})_n = s$, and $\mathbf{z} \neq \mathbf{u}$; and set $\mathbf{w}$ to be a vertex with $(\mathbf{w})_1 = r$, $(\mathbf{w})_2 = \bar{s}$, $(\mathbf{w})_n = t$, and $\mathbf{w} \neq \mathbf{v}$. Suppose $s = \bar{t}$. We set $\mathbf{z}$ to be a vertex with $(\mathbf{z})_1 = r$, $(\mathbf{z})_n = s$, and $\mathbf{z} \neq \mathbf{u}$; and set $\mathbf{w}$ to be a vertex with $(\mathbf{w})_1 = r$, $(\mathbf{w})_n = t$, and $\mathbf{w} \neq \mathbf{v}$.

Thus, $\mathbf{z} \in V(B_n^{\{s\}})$ and $\mathbf{w} \in V(B_n^{\{t\}})$. By assumption, there exists a k^*-container of $B_n^{\{s\}}$, $\{R_1, R_2, \ldots, R_k\}$, joining $\mathbf{u}$ to $\mathbf{z}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{z_i}, \mathbf{z} \rangle$. (Note that $\mathbf{z_i} = \mathbf{u}$ if $l(R_i) = 1$.) Obviously, $((\mathbf{z_i})^n)_1 \notin \{s,t\}$ for $1 \leq i \leq k$. By Lemma 2, $|(\mathbf{z_i})_1| \neq |(\mathbf{z_j})_1|$ for $1 \leq i \neq j \leq k$. Again, there exists a k^*-container of $B_n^{\{t\}}$, $\{H_1, H_2, \ldots, H_k\}$, joining $\mathbf{w}$ to $\mathbf{v}$. We write $H_i = \langle \mathbf{w}, \mathbf{w_i}, H_i', \mathbf{v} \rangle$. (Note that $\mathbf{w_i} = \mathbf{v}$ if $l(H_i) = 1$.) Again, $((\mathbf{w_i})^n)_1 \notin \{s,t\}$ for $1 \leq i \leq k$, and $|(\mathbf{w_i})_1| \neq |(\mathbf{w_j})_1|$ for $1 \leq i \neq j \leq k$. We can reorder the indices of $\{1, 2, \ldots, k\}$ so that $\{(\mathbf{z_i})_1 \mid 2 \leq i \leq k\} \neq \{(\mathbf{w_i})_1 \mid 2 \leq i \leq k\}$.

Let $\mathbf{x_1} = (\mathbf{z})^n$ and $\mathbf{x_i} = (\mathbf{z_i})^n$ for $2 \leq i \leq k$. Similarly, let $\mathbf{y_1} = (\mathbf{w})^n$ and $\mathbf{y_i} = (\mathbf{w_i})^n$ for $2 \leq i \leq k$. Obviously, $(\mathbf{x_i})_1 = \bar{s}$, $(\mathbf{x_i})_n \neq t$ for $1 \leq i \leq k$, $(\mathbf{x_i})_n \neq (\mathbf{x_1})_n$ for $2 \leq i \leq k$, and $|(\mathbf{x_i})_n| \neq |(\mathbf{x_j})_n|$ for $2 \leq i \neq j \leq k$. Moreover, $(\mathbf{y_i})_1 = \bar{t}$, $(\mathbf{y_i})_n \neq s$ for $1 \leq i \leq k$, $(\mathbf{y_i})_n \neq (\mathbf{y_1})_n$ for $2 \leq i \leq k$, and $|(\mathbf{y_i})_n| \neq |(\mathbf{y_j})_n|$ for $2 \leq i \neq j \leq k$. Furthermore, $(\mathbf{x_1})_n = (\mathbf{y_1})_n = \bar{r}$, and $\{(\mathbf{x_i})_n \mid 1 \leq i \leq k\} \neq \{(\mathbf{y_i})_n \mid 1 \leq i \leq k\}$. By Lemma 8, there exist a permutation π and k disjoint paths, $P_1, P_2, \ldots, P_k$, such that P_i joining $\mathbf{x_i}$ to $\mathbf{y_{\pi(i)}}$ and $\cup_{i=1}^{k} P_i$ spans $B_n^{[n]-\{s,t\}}$. By assumption, there exist k internal disjoint paths, $S_1, S_2, \ldots, S_k$, such that S_i is a path joining $\mathbf{u}$ to $(\mathbf{x_i})^n$ for $1 \leq i \leq k$ and $\cup_{i=1}^{k} S_i$ spans $B_n^{\{s\}}$. Similarly, there exist k internal disjoint paths, $T_1, T_2, \ldots, T_k$, such that T_i is a path joining $(\mathbf{y_i})^n$ to $\mathbf{v}$ for $1 \leq i \leq k$ and $\cup_{i=1}^{k} T_i$ spans $B_n^{\{t\}}$. By Lemma 9, the required k^*-container between $\mathbf{u}$ and $\mathbf{v}$ exists. See Fig. 5 for illustration. The lemma is proved. □

Lemma 11. *Assume that $n \geq 4$. Let $\mathbf{u}$ be any vertex in $B_n^{\{s\}}$ with $(\mathbf{u})^n \in V(B_n^{\{t\}})$ and $\mathbf{v}$ be any vertex in $B_n^{\{t\}}$ with $\mathbf{v} \neq (\mathbf{u})^n$ and $(\mathbf{v})^n \in V(B_n^{\{s\}})$. Suppose that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n-2}}$ are $(n-2)$ vertices in B_n with $(\mathbf{x_i})_1 = \bar{s}$ and $(\mathbf{x_i})_n \neq t$ for $1 \leq i \leq n-2$; and $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_{n-2}}$ are $(n-2)$ vertices in B_n with $(\mathbf{y_i})_1 = \bar{t}$ and $(\mathbf{y_i})_n \neq s$ for $1 \leq i \leq n-2$. Suppose that there exist a permutation π on $\{1, 2, \ldots, n-2\}$ and $(n-2)$ disjoint paths, $P_1, P_2, \ldots, P_{n-2}$, such that P_i is a path joining $\mathbf{x_i}$ to $\mathbf{y_{\pi(i)}}$ for $1 \leq i \leq n-2$ and $\cup_{i=1}^{n-2} P_i$ spans $B_n^{[n]-\{s,t\}}$. Moreover, there are $(n-1)$ internal disjoint paths, $S_1, S_2, \ldots, S_{n-1}$, of $B_n^{\{s\}}$ such that S_i is a path joining $\mathbf{u}$ to $(\mathbf{x_i})^n$ for $1 \leq i \leq n-2$, S_{n-1} is a path joining $\mathbf{u}$ to $(\mathbf{v})^n$, and $\cup_{i=1}^{n-1} S_i$ spans $B_n^{\{s\}}$. Furthermore, there are $(n-1)$ internal disjoint paths, $T_1, T_2, \ldots, T_{n-1}$, of $B_n^{\{t\}}$ such that T_i is a path joining $(\mathbf{y_i})^n$ to $\mathbf{v}$ for $1 \leq i \leq n-2$, T_{n-1} is a path joining $(\mathbf{u})^n$ to $\mathbf{v}$, and $\cup_{i=1}^{n-1} T_i$ spans $B_n^{\{t\}}$. Then there exists an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.*

Proof. Let $Q_i = \langle \mathbf{u}, S_i, (\mathbf{x_i})^n, \mathbf{x_i}, P_i, \mathbf{y_{\pi(i)}}, (\mathbf{y_{\pi(i)}})^n, T_{\pi(i)}, \mathbf{v} \rangle$ for $1 \leq i \leq n-2$, $Q_{n-1} = \langle \mathbf{u}, (\mathbf{u})^n, T_{n-1}, \mathbf{v} \rangle$, and $Q_n = \langle \mathbf{u}, S_{n-1}, (\mathbf{v})^n, \mathbf{v} \rangle$. Then $\{Q_1, Q_2, \ldots, Q_n\}$ forms an n^*-container between $\mathbf{u}$ and $\mathbf{v}$. See Fig. 6 for illustration. The lemma is proved. □

Lemma 12. *Assume that $n \geq 4$. Let $\mathbf{u}$ be any vertex in $B_n^{\{s\}}$ with $(\mathbf{u})^n \in V(B_n^{\{t\}})$ and $\mathbf{v}$ be any vertex in $B_n^{\{t\}}$ with $\mathbf{v} \neq (\mathbf{u})^n$ and $(\mathbf{v})^n \in V(B_n^{\{s\}})$. Suppose that B_{n-1} is $(n-1)^*$-connected. Then there is an n^*-container of B_n between $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Obviously, $|s| \neq |t|$. By assumption, there exists an $(n-1)^*$-container of $B_n^{\{s\}}$, $\{R_1, R_2, \ldots, R_{n-1}\}$, joining $\mathbf{u}$ to $(\mathbf{v})^n$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{z_i}, (\mathbf{v})^n \rangle$. (Note that $\mathbf{z_i} = \mathbf{u}$ if $l(R_i) = 1$.) Obviously, $(\mathbf{z_i})_1 \notin \{s,t\}$ for $1 \leq i \leq n-1$. By

Lemma 2, $|(\mathbf{z_i})_1| \neq |(\mathbf{z_j})_1|$ for $1 \leq i \neq j \leq n-1$. Again, there exists an $(n-1)^*$-container of $B_n^{\{t\}}$, $\{H_1, H_2, \ldots, H_{n-1}\}$, joining $(\mathbf{u})^n$ to $\mathbf{v}$. We write $H_i = \langle (\mathbf{u})^n, \mathbf{w_i}, H_i', \mathbf{v}\rangle$. (Note that $\mathbf{w_i} = \mathbf{v}$ if $l(H_i) = 1$.) Again, $(\mathbf{w_i})_1 \notin \{s, t\}$ for $1 \leq i \leq n-1$, and $|(\mathbf{w_i})_1| \neq |(\mathbf{w_j})_1|$ for $1 \leq i \neq j \leq n-1$. We can reorder the indices of $\{1, 2, \ldots, n-1\}$ so that $\{(\mathbf{z_i})_1 \mid 1 \leq i \leq n-2\} \neq \{(\mathbf{w_i})_1 \mid 1 \leq i \leq n-2\}$.

Let $\mathbf{x_i} = (\mathbf{z_i})^n$ for $1 \leq i \leq n-2$ and $\mathbf{y_i} = (\mathbf{w_i})^n$ for $1 \leq i \leq n-2$. Obviously, $(\mathbf{x_i})_1 = s$, $(\mathbf{x_i})_n \neq t$ for $1 \leq i \leq n-2$, $|(\mathbf{x_i})_n| \neq |(\mathbf{x_j})_n|$ for $1 \leq i \neq j \leq n-2$. Moreover, $(\mathbf{y_i})_1 = t$, $(\mathbf{y_i})_n \neq s$ for $1 \leq i \leq n-2$, and $|(\mathbf{y_i})_n| \neq |(\mathbf{y_j})_n|$ for $1 \leq i \neq j \leq n-2$. Furthermore, $\{(\mathbf{x_i})_n \mid 1 \leq i \leq n-2\} \neq \{(\mathbf{y_i})_n \mid 1 \leq i \leq n-2\}$. By Lemma 8, there exist a permutation π on $\{1, 2, \ldots, n-2\}$ and $(n-2)$ disjoint paths, $P_1, P_2, \ldots, P_{n-2}$, such that P_i joining $\mathbf{x_i}$ to $\mathbf{y_{\pi(i)}}$ and $\cup_{i=1}^{n-2} P_i$ spans $B_n^{[n]-\{s,t\}}$. By assumption, there exist $(n-1)$ internal disjoint paths, $S_1, S_2, \ldots, S_{n-1}$, such that S_i is a path joining $\mathbf{u}$ to $(\mathbf{x_i})^n$ for $1 \leq i \leq n-2$, S_{n-1} is a path joining $\mathbf{u}$ to $(\mathbf{v})^n$, and $\cup_{i=1}^{n-1} S_i$ spans $B_n^{\{s\}}$. Similarly, there exist $(n-1)$ internal disjoint paths, $T_1, T_2, \ldots, T_{n-1}$, such that T_i is a path joining $(\mathbf{y_i})^n$ to $\mathbf{v}$ for $1 \leq i \leq n-2$, T_{n-1} is a path joining $(\mathbf{u})^n$ to $\mathbf{v}$, and $\cup_{i=1}^{n-1} T_i$ spans $B_n^{\{t\}}$. By Lemma 11, there exists an n^*-container between $\mathbf{u}$ and $\mathbf{v}$. See Fig. 7 for illustration. The lemma is proved. □

Lemma 13. *Assume that $n \geq 4$. Let $\mathbf{u}$ be any vertex in $B_n^{\{s\}}$ with $(\mathbf{u})^n \in V(B_n^{\{t\}})$ and $\mathbf{v}$ be any vertex in $B_n^{\{t\}}$ with $\mathbf{v} \neq (\mathbf{u})^n$ and $(\mathbf{v})^n \notin V(B_n^{\{s\}})$. Suppose that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_{n-1}}$ are $(n-1)$ vertices in B_n with $(\mathbf{x_i})_1 = \bar{s}$ and $(\mathbf{x_i})_n \neq t$ for $1 \leq i \leq n-1$; and $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_{n-1}}$ are $(n-1)$ vertices in B_n with $(\mathbf{y_i})_1 = \bar{t}$ and $(\mathbf{y_i})_n \neq s$ for $1 \leq i \leq n-1$. Moreover, $(\mathbf{v})^n = \mathbf{y_{n-1}}$. Suppose that there exist a permutation π on $\{1, 2, \ldots, n-1\}$ and $(n-1)$ disjoint paths, $P_1, P_2, \ldots, P_{n-1}$, such that P_i is a path joining $\mathbf{x_i}$ to $\mathbf{y_{\pi(i)}}$ for $1 \leq i \leq n-1$ and $\cup_{i=1}^{n-1} P_i$ spans $B_n^{[n]-\{s,t\}}$. Moreover, there are $(n-1)$ internal disjoint paths, $S_1, S_2, \ldots, S_{n-1}$, of $B_n^{\{s\}}$ such that S_i is a path joining $\mathbf{u}$ to $(\mathbf{x_i})^n$ for $1 \leq i \leq n-1$ and $\cup_{i=1}^{n-1} S_i$ spans $B_n^{\{s\}}$. Furthermore, there are $(n-1)$ internal disjoint paths, $T_1, T_2, \ldots, T_{n-1}$, of $B_n^{\{t\}}$ such that T_i is a path joining $\mathbf{v}$ to $(\mathbf{y_i})^n$ for $1 \leq i \leq n-2$, T_{n-1} is a path joining $(\mathbf{u})^n$ to $\mathbf{v}$, and $\cup_{i=1}^{n-1} T_i$ spans $B_n^{\{t\}}$. Then there exists an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.*

Proof. Without loss of generality, we assume that $\pi(n-1) = n-1$. Let $Q_i = \langle \mathbf{u}, S_i, (\mathbf{x_i})^n, \mathbf{x_i}, P_i, \mathbf{y_{\pi(i)}}, (\mathbf{y_{\pi(i)}})^n, T_{\pi(i)}, \mathbf{v}\rangle$ for $1 \leq i \leq n-2$, $Q_{n-1} = \langle \mathbf{u}, S_{n-1}, (\mathbf{x_{n-1}})^n, \mathbf{x_{n-1}}, P_{n-1}, \mathbf{y_{n-1}} = (\mathbf{v})^n, \mathbf{v}\rangle$, and $Q_n = \langle \mathbf{u}, (\mathbf{u})^n, T_{n-1}, \mathbf{v}\rangle$. Then $\{Q_1, Q_2, \ldots, Q_n\}$ forms an n^*-container between $\mathbf{u}$ and $\mathbf{v}$. See Fig. 8 for illustration. The lemma is proved. □

Lemma 14. *Assume that $n \geq 4$. Let $\mathbf{u}$ be any vertex in $B_n^{\{s\}}$ with $(\mathbf{u})^n \in V(B_n^{\{t\}})$ and $\mathbf{v}$ be any vertex in $B_n^{\{t\}}$ with $\mathbf{v} \neq (\mathbf{u})^n$ and $(\mathbf{v})^n \notin V(B_n^{\{s\}})$. Suppose that B_{n-1} is $(n-1)^*$-connected. Then there is an n^*-container of B_n between $\mathbf{u}$ and $\mathbf{v}$.*

Proof. Since $(\mathbf{u})^n \in V(B_n^{\{t\}})$, $|s| \neq |t|$ and $(\mathbf{u})_1 = \bar{t}$. We set $\mathbf{z}$ be the vertex with $(\mathbf{z})_1 = t$ and $(\mathbf{z})_i = (\mathbf{u})_i$ for $2 \leq i \leq n$. Thus, $\mathbf{z} \in V(B_n^{\{s\}})$.

By assumption, there exists an $(n-1)^*$-container of $B_n^{\{s\}}$, $\{R_1, R_2, \ldots, R_{n-1}\}$, joining $\mathbf{u}$ to $\mathbf{z}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{z_i}, \mathbf{z}\rangle$. (Note that $\mathbf{z_i} = \mathbf{u}$ if $l(R_i) = 1$.) Obviously, $(\mathbf{z_i})_1 \notin \{s, t\}$ for $1 \leq i \leq n-1$. By Lemma 2, $|(\mathbf{z_i})_1| \neq |(\mathbf{z_j})_1|$ for $1 \leq i \neq j \leq n-1$. Again, there exists an $(n-1)^*$-container of $B_n^{\{t\}}$, $\{H_1, H_2, \ldots, H_{n-1}\}$, joining $(\mathbf{u})^n$ to $\mathbf{v}$. We write $H_i = \langle (\mathbf{u})^n, \mathbf{w_i}, H_i', \mathbf{v}\rangle$. (Note that $\mathbf{w_i} = \mathbf{v}$ if $l(H_i) = 1$.) Again, $(\mathbf{w_i})_1 \notin \{s, t\}$ for $1 \leq i \leq n-1$, and $|(\mathbf{w_i})_1| \neq |(\mathbf{w_j})_1|$ for $1 \leq i \neq j \leq n-1$. We can reorder the indices of $\{1, 2, \ldots, n-1\}$ so that $\{(\mathbf{z_i})_1 \mid 2 \leq i \leq n-1\} \neq \{(\mathbf{w_i})_1 \mid 2 \leq i \leq n-1\}$.

Let $\mathbf{x_1} = (\mathbf{z})^n$ and $\mathbf{x_i} = (\mathbf{z_i})^n$ for $2 \leq i \leq n-1$. Similarly, $\mathbf{y_1} = (\mathbf{v})^n$ and $\mathbf{y_i} = (\mathbf{w_i})^n$ for $2 \leq i \leq n-1$. Obviously, $(\mathbf{x_i})_1 = s$, $(\mathbf{x_i})_n \neq t$ for $1 \leq i \leq n-1$, $(\mathbf{x_i})_n \neq (\mathbf{x_1})_n$ for $2 \leq i \leq n-1$, and $|(\mathbf{x_i})_n| \neq |(\mathbf{x_j})_n|$ for $2 \leq i \neq j \leq n-1$. Moreover, $(\mathbf{y_i})_1 = t$, $(\mathbf{y_i})_n \neq s$ for $1 \leq i \leq n-1$, $(\mathbf{y_i})_n \neq (\mathbf{y_1})_n$ for $2 \leq i \leq n-1$, and $|(\mathbf{y_i})_n| \neq |(\mathbf{y_j})_n|$ for $2 \leq i \neq j \leq n-1$. Furthermore, $(\mathbf{x_1})_n = (\mathbf{y_1})_n = \bar{t}$, and $\{(\mathbf{x_i})_n \mid 1 \leq i \leq n-1\} \neq \{(\mathbf{y_i})_n \mid 1 \leq i \leq n-1\}$.

By Lemma 8, there exist a permutation π on $\{1, 2, \ldots, n-1\}$ and $(n-1)$ disjoint paths, $P_1, P_2, \ldots, P_{n-1}$, such that P_i joining $\mathbf{x_i}$ to $\mathbf{y_{\pi(i)}}$ and $\cup_{i=1}^{n} P_i$ spans $B_n^{[n]-\{s,t\}}$. By assumption, there exist $(n-1)$ internal disjoint paths, $S_1, S_2, \ldots, S_{n-1}$, such that S_i is a path joining $\mathbf{u}$ to $(\mathbf{x_i})^n$ for $1 \leq i \leq n-1$ and $\cup_{i=1}^{n-1} S_i$ spans $B_n^{\{s\}}$. Similarly, there exist $(n-1)$ internal disjoint paths, $T_1, T_2, \ldots, T_{n-1}$, such that T_i is a path joining $(\mathbf{y_i})^n$ to $\mathbf{v}$ for $1 \leq i \leq n-1$ and $\cup_{i=1}^{n-1} T_i$ spans $B_n^{\{t\}}$. By Lemma 13, there exists an n^*-container between $\mathbf{u}$ and $\mathbf{v}$. See Fig. 9 for illustration. The lemma is proved. □

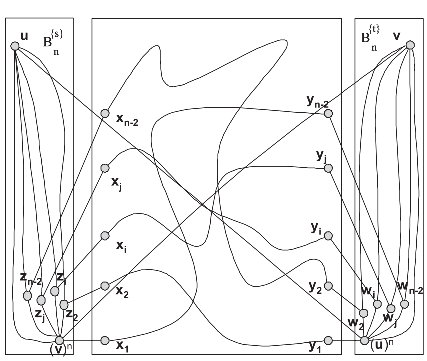

Fig. 7 Illustration for Lemma 12.

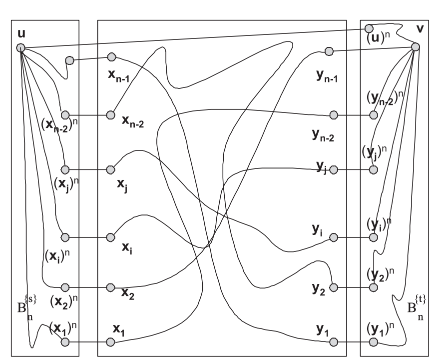

Fig. 8 Illustration for Lemma 13.

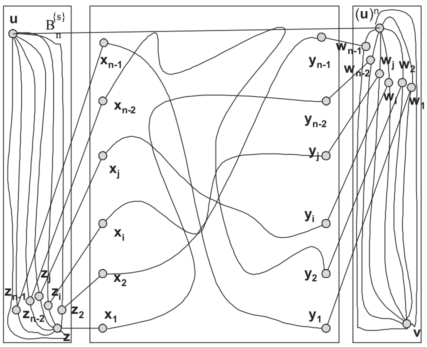

Fig. 9 Illustration for Lemma 14.

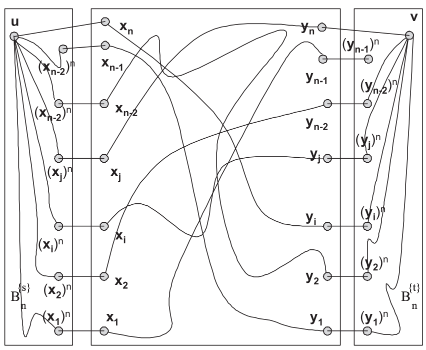

Fig. 10 Illustration for Lemma 15.

Lemma 15. *Assume that* $n \geq 4$. *Let* $\mathbf{u}$ *be a vertex in* $B_n^{\{s\}}$ *and* $\mathbf{v}$ *be a vertex in* $B_n^{\{t\}}$ *with* $s \neq t$, $(\mathbf{u})^n \notin V(B_n^{\{t\}})$, *and* $(\mathbf{v})^n \notin V(B_n^{\{s\}})$. *Suppose that* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ *are n vertices in* B_n *with* $(\mathbf{x_i})_1 = \bar{s}$, $(\mathbf{x_i})_n \neq t$ *for* $1 \leq i \leq n$; *and* $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_n}$ *are n vertices in* B_n *with* $(\mathbf{y_i})_1 = \bar{t}$, $(\mathbf{y_i})_n \neq s$ *for* $1 \leq i \leq n$. *Moreover,* $(\mathbf{u})^n = \mathbf{x_n}$ *and* $(\mathbf{v})^n = \mathbf{y_n}$. *Suppose that there exist a permutation on* $\{1, 2, \ldots, n\}$ *and n disjoint paths,* $P_1, P_2, \ldots, P_n$, *such that* P_i *is a path joining* $\mathbf{x_i}$ *to* $\mathbf{y_{\pi(i)}}$ *for* $1 \leq i \leq n$ *and* $\cup_{i=1}^n P_i$ *spans* $B_n^{[n]-\{s,t\}}$. *Moreover, there are* $(n-1)$ *internal disjoint paths,* $S_1, S_2, \ldots, S_{n-1}$, *of* $B_n^{\{s\}}$ *such that* S_i *is a path joining* $\mathbf{u}$ *to* $(\mathbf{x_i})^n$ *for* $1 \leq i \leq n-1$ *and* $\cup_{i=1}^{n-1} S_i$ *spans* $B_n^{\{s\}}$. *Furthermore, there are* $(n-1)$ *internal disjoint paths,* $T_1, T_2, \ldots, T_{n-1}$, *of* $B_n^{\{t\}}$ *such that* T_i *is a path joining* $\mathbf{v}$ *to* $(\mathbf{y_i})^n$ *for* $1 \leq i \leq n-1$, *and* $\cup_{i=1}^{n-1} T_i$ *spans* $B_n^{\{t\}}$. *Then there exists an* n^**-container of* B_n *joining* $\mathbf{u}$ *to* $\mathbf{v}$.

Proof. The proof is similar as that of Lemma 13. We just illustrate the proof in Fig. 10. □

Lemma 16. *Assume that n is a positive integer with* $n \geq 4$. *Let s and t be two different elements in* $[n]$. *Let* $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ *be n different vertices in* B_n *with* $(\mathbf{x_i})_1 = \bar{s}$, $(\mathbf{x_i})_n \neq \bar{t}$ *for* $1 \leq i \leq n$, *and* $(\mathbf{x_i})_n \neq (\mathbf{x_j})_n$ *for* $1 \leq i \neq j \leq n$. *Let* $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_n}$ *be n different vertices in* B_n *with* $(\mathbf{y_i})_1 = \bar{t}$, $(\mathbf{y_i})_n \neq \bar{s}$ *for* $1 \leq i \leq n$, *and* $(\mathbf{y_i})_n \neq (\mathbf{y_j})_n$ *for* $1 \leq i \neq j \leq n$. *Suppose that* $\{(\mathbf{x_i})_n \mid 1 \leq i \leq n\} \neq \{(\mathbf{y_i})_n \mid 1 \leq i \leq n\}$. *Moreover,* $\{|(\mathbf{x_i})_n| \mid 1 \leq i \leq n\} = \langle n \rangle - \{|s|\}$ *and* $\{|(\mathbf{y_i})_n| \mid 1 \leq i \leq n\} = \langle n \rangle - \{|t|\}$. *Then there exist a permutation* π *on* $\{1, 2, \ldots, n\}$ *and n disjoint paths* $P_1, P_2, \ldots, P_n$ *such that* P_i

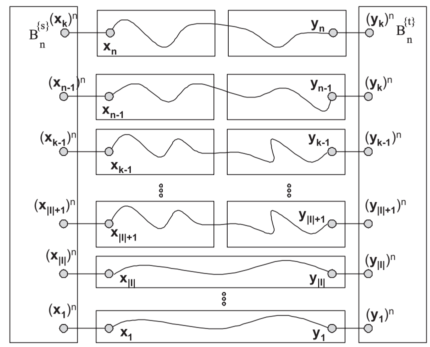

Fig. 11 Illustration for Lemma 16.

joining $\mathbf{x_i}$ *to* $\mathbf{y_{\pi(i)}}$ *for* $1 \leq i \leq n$ *and* $\cup_{i=1}^n P_i$ *spans* $B_n^{[n]-\{s,t\}}$.

Proof. Obviously, $\mathbf{x_i} \notin V(B_n^{\{s,t\}})$ and $(\mathbf{x_i})^n \in V(B_n^{\{s\}})$. Similarly, $\mathbf{y_i} \notin V(B_n^{\{s,t\}})$ and $(\mathbf{y_i})^n \in V(B_n^{\{t\}})$. Let I be the set $\{(\mathbf{x_i})_n \mid (\mathbf{x_i})_n = (\mathbf{y_j})_n$ for some $1 \leq i, j \leq k\}$. We can reorder the indices of $\{1, 2, \ldots, n\}$ so that $(\mathbf{x_i})_n = (\mathbf{y_i})_n$ for $1 \leq i \leq |I|$. By Theorem 1, there exists a hamiltonian path P_i of $B_n^{\{(\mathbf{x_i})_n\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for $1 \leq i \leq |I|$.

For $|I| + 1 \leq i \leq n$, let $A_i = \{\mathbf{y_j} \mid |I| + 1 \leq j \leq n$ with $(\mathbf{x_i})_n \neq -(\mathbf{y_j})_n\}$. By Lemma 7, there exists $\{\mathbf{y_i} \mid |I|+1 \leq i \leq n\}$ such that $(\mathbf{x_i})_n \neq -(\mathbf{y_i})_n$ and $\mathbf{y_i} \neq \mathbf{y_j}$ for $i \neq j$. Let X be $\{(\mathbf{x_i})_n \mid 1 \leq i \leq n-1\} \cup \{(\mathbf{y_i})_n \mid 1 \leq i \leq n-1\} \cup \{s, t\}$. Moreover, $\{|(\mathbf{x_i})_n| \mid 1 \leq i \leq n\} = \langle n \rangle - \{|s|\}$ and $\{|(\mathbf{y_i})_n| \mid 1 \leq i \leq n\} = \langle n \rangle - \{|t|\}$. Obviously, $D([n] - X) = 0$. By Lemma 3, there exists a hamiltonian path P_i of $B_n^{\{(\mathbf{x_i})_n, (\mathbf{y_i})_n\}}$ joining $\mathbf{x_i}$ to $\mathbf{y_i}$ for $|I| + 1 \leq i \leq n-1$. Again, there exists a hamiltonian path P_n of $B_n^{[n]-X}$ joining $\mathbf{x_n}$ to $\mathbf{y_n}$. Obviously, $\{P_1, P_2, \ldots, P_n\}$ forms a set of the required paths. See Fig. 11 for illustration. □

Lemma 17. *Assume that* $n \geq 4$. *Let* $\mathbf{u}$ *be a vertex in* $B_n^{\{s\}}$ *and* $\mathbf{v}$ *be a vertex in* $B_n^{\{t\}}$ *with* $s \neq t$, $(\mathbf{u})^n \notin V(B_n^{\{t\}})$, *and* $(\mathbf{v})^n \notin V(B_n^{\{s\}})$. *Suppose that* B_{n-1} *is* $(n-1)^*$*-connected. Then there is an* n^**-container of* B_n *between* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. Suppose that $s \neq \bar{t}$. We set $\mathbf{z}$ be a vertex with $(\mathbf{z})_1 = t$, $(\mathbf{z})_2 = (\mathbf{u})_1$, and $(\mathbf{z})_n = s$; and set $\mathbf{w}$ be a vertex with $(\mathbf{w})_1 = s$, $(\mathbf{w})_2 = (\mathbf{v})_1$, and $(\mathbf{w})_n = t$. Suppose $s = \bar{t}$. Since $n \geq 4$, there exists an element r in $\langle n \rangle - \{|s|, |(\mathbf{u})_1|, |(\mathbf{v})_1|\}$. We set $\mathbf{z}$ be a vertex with $(\mathbf{z})_1 = r$, $(\mathbf{z})_2 = (\mathbf{u})_1$, and $(\mathbf{z})_n = s$; and set $\mathbf{w}$ be a vertex with $(\mathbf{w})_1 = \bar{r}$, $(\mathbf{w})_2 = (\mathbf{v})_1$, and $(\mathbf{w})_n = t$.

By assumption, there exists an $(n-1)^*$-container of $B_n^{\{s\}}$, $\{R_1, R_2, \ldots, R_{n-1}\}$, joining $\mathbf{u}$ to $\mathbf{z}$. We write $R_i = \langle \mathbf{u}, R_i', \mathbf{z_i}, \mathbf{z} \rangle$. (Note that $\mathbf{z_i} = \mathbf{u}$ if $l(R_i) = 1$.) Obviously, $(\mathbf{z_i})_1 \notin \{s, t\}$ for $1 \leq i \leq n-1$. By Lemma 2, $|(\mathbf{z_i})_1| \neq |(\mathbf{z_j})_1|$ for $1 \leq i \neq j \leq n-1$. Again, there exists an $(n-1)^*$-container of $B_n^{\{t\}}$, $\{H_1, H_2, \ldots, H_{n-1}\}$, joining $\mathbf{w}$ to $\mathbf{v}$. We write $H_i = \langle \mathbf{w}, \mathbf{w_i}, H_i', \mathbf{v} \rangle$. (Note that $\mathbf{w_i} = \mathbf{v}$ if $l(H_i) = 1$.) Again, $(\mathbf{w_i})_1 \notin \{s, t\}$ for $1 \leq i \leq n-1$, and $|(\mathbf{w_i})_1| \neq |(\mathbf{w_j})_1|$ for $1 \leq i \neq j \leq n-1$. We can reorder the indices of $\{1, 2, \ldots, n-1\}$ so that $\{|(\mathbf{z_i})_1| \mid 2 \leq i \leq n-1\} = \{|(\mathbf{w_i})_1| \mid 2 \leq i \leq n-1\} = \langle n \rangle - \{|s|, |t|\}$ if $s \neq \bar{t}$; and $\{|(\mathbf{z_i})_1| \mid 2 \leq i \leq n-1\} = \{|(\mathbf{w_i})_1| \mid 2 \leq i \leq n-1\} = \langle n \rangle - \{|s|, |r|\}$ if otherwise.

Let $\mathbf{x_1} = (\mathbf{z})^n$, $\mathbf{x_i} = (\mathbf{z_i})^n$ for $2 \leq i \leq n-1$ and $\mathbf{x_n} = (\mathbf{u})^n$.

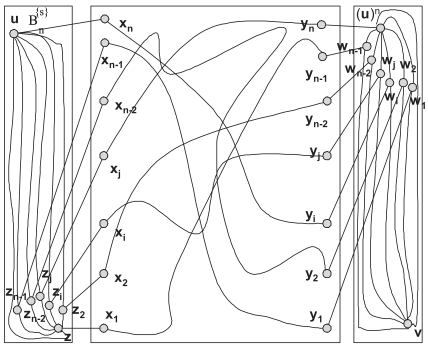

Fig. 12 Illustration for Lemma 17.

Similarly, $\mathbf{y_1} = (\mathbf{w})^n$, $\mathbf{y_i} = (\mathbf{w_i})^n$ for $2 \le i \le n-1$ and $\mathbf{y_n} = (\mathbf{v})^n$. Obviously, $(\mathbf{x_i})_1 = s$, $(\mathbf{x_i})_n \ne t$ for $1 \le i \le n$, $(\mathbf{x_i})_n \ne (\mathbf{x_1})_n$ for $2 \le i \le n$, and $|(\mathbf{x_i})_n| \ne |(\mathbf{x_j})_n|$ for $2 \le i \ne j \le n$. Moreover, $(\mathbf{y_i})_1 = t$, $(\mathbf{y_i})_n \ne s$ for $1 \le i \le n$, $(\mathbf{y_i})_n \ne (\mathbf{y_1})_n$ for $2 \le i \le n-1$, and $|(\mathbf{y_i})_n| \ne |(\mathbf{y_j})_n|$ for $2 \le i \ne j \le n-1$. Note that $(\mathbf{x_1})_n = \bar{t}$ and $(\mathbf{y_1})_n = \bar{s}$ if $s \ne t$; and $(\mathbf{x_1})_n = \bar{r}$ and $(\mathbf{y_1})_n = r$ if $s = t$. Thus, $\{(\mathbf{x_i})_n \mid 1 \le i \le n\} \ne \{(\mathbf{y_i})_n \mid 1 \le i \le n\}$. Moreover, $\{|(\mathbf{x_i})_n| \mid 1 \le i \le n\} = \langle n\rangle - \{|s|\}$ and $\{|(\mathbf{y_i})_n| \mid 1 \le i \le n\} = \langle n\rangle - \{|t|\}$. By Lemma 16, there exist a permutation π on $\{1, 2, \ldots, n\}$ and n disjoint paths, $P_1, P_2, \ldots, P_n$, such that P_i joining $\mathbf{x_i}$ to $\mathbf{y}_{\pi(\mathbf{i})}$ and $\cup_{i=1}^n P_i$ spans $B_n^{[n]-\{s,t\}}$. By assumption, there exist $(n-1)$ internal disjoint paths, $S_1, S_2, \ldots, S_{n-1}$, such that S_i is a path joining $\mathbf{u}$ to $(\mathbf{x_i})^n$ for $1 \le i \le n-1$ and $\cup_{i=1}^{n-1} S_i$ spans $B_n^{\{s\}}$. Similarly, there exist $(n-1)$ internal disjoint paths, $T_1, T_2, \ldots, T_{n-1}$, such that T_i is a path joining $(\mathbf{y_i})^n$ to $\mathbf{v}$ for $1 \le i \le n-1$ and $\cup_{i=1}^{n-1} T_i$ spans $B_n^{\{t\}}$. By Lemma 15, there exists an n^*-container between $\mathbf{u}$ and $\mathbf{v}$. See Fig. 12 for illustration. The lemma is proved. □

4. Main Result

Theorem 2. *B_n is n^*-connected for any positive integer n.*

Proof. We prove this theorem by induction. It is easy to see that B_1 is 1^*-connected and B_2 is 2^*-connected. Since the B_3 is vertex transitive, by brute force, we have checked that B_3 is 3^*-connected. We list the result in the appendix.

Assume that B_k is k^*-connected for every $3 \le k \le n-1$. Let $\mathbf{u}$ and $\mathbf{v}$ be any two distinct vertices of B_n with $\mathbf{u} \in V(B_n^{\{s\}})$ and $\mathbf{v} \in V(B_n^{\{t\}})$. We need to find an n^*-container between $\mathbf{u}$ and $\mathbf{v}$ of B_n. Suppose that $s = t$. By induction, there exists an $(n-1)^*$-container of $B_n^{\{s\}}$ joining $\mathbf{u}$ to $\mathbf{v}$. By Lemma 6, there is an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$. Thus, we assume that $s \ne t$.

Case 1: $(\mathbf{u})^n \in V(B_n^{\{t\}})$ and $(\mathbf{v})^n \in V(B_n^{\{s\}})$.

Suppose that $\mathbf{u} = (\mathbf{v})^n$. By Lemma 10, there is an $(n-1)^*$-container $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ of B_n joining $\mathbf{u}$ to $\mathbf{v}$ not using the edge $(\mathbf{u}, \mathbf{v})$. We set Q_n as $\langle \mathbf{u}, \mathbf{v}\rangle$. Then $\{Q_1, Q_2, \ldots, Q_n\}$ forms an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$. Suppose that $\mathbf{u} \ne (\mathbf{v})^n$. By Lemma 12, there is an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.

Case 2: $((\mathbf{u})^n \in V(B_n^{\{t\}})$ and $(\mathbf{v})^n \notin V(B_n^{\{s\}}))$ or $((\mathbf{u})^n \notin V(B_n^{\{t\}})$ and $(\mathbf{v})^n \in V(B_n^{\{s\}}))$. Without loss of generality, we assume that $(\mathbf{u})^n \in V(B_n^{\{t\}})$ and $(\mathbf{v})^n \notin V(B_n^{\{s\}})$. By Lemma 14, there is an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.

Case 3: $(\mathbf{u})^n \notin V(B_n^{\{t\}})$ and $(\mathbf{v})^n \notin V(B_n^{\{s\}})$. By Lemma 17, there is an n^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.

The theorem is proved. □

Theorem 3. *B_n is super spanning connected if and only if $n \ne 2$.*

Proof. We prove this theorem by induction. Obviously, this theorem is true for B_1. Since P_2 is isomorphic to a cycle with eight vertices, B_2 is not 1^*-connected. Thus, B_2 is not super spanning connected. By Theorem 1 and Theorem 2, this theorem holds on B_3. Assume that B_k is super spanning connected for every $3 \le k \le n-1$. By Theorem 1 and Theorem 2, B_n is k^*-connected for any $k \in \{1, 2, n\}$. Thus, we still need to construct a k^*-container of B_n between any two distinct vertices $\mathbf{u} \in V(B_n^{\{s\}})$ and $\mathbf{v} \in V(B_n^{\{t\}})$ for every $3 \le k \le n-1$.

Suppose that $s = t$. By induction, B_{n-1} is $(k-1)^*$-connected. By Lemma 6, there is a k^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$. Suppose that $s \ne t$. By induction, B_{n-1} is k^*-connected. By Lemma 10, there is a k^*-container of B_n joining $\mathbf{u}$ to $\mathbf{v}$.

Hence, the theorem is proved. □

5. Conclusion

Graph containers do exist in engineering designed information and telecommunication networks and in biological neural systems. See [1], [12] and their references. The study of w-container and their w^*-versions plays a pivotal role in design and implementation of parallel routing and efficient information transmission in large-scale network system. In biological informatics and neuroinformatics, the existence and structure of a w^*-container signifies the cascade effect in signal transduction system and the reaction in a metabolic pathway. Recently, there are a lot studies on w^*-container [2], [3], [11], [16], [19], [21], [24]. In this paper, we prove that the burnt pancake graph B_n is super connected for $n \ne 2$.

Assume that G is k^*-connected.We may also define the k^*-connected distance between any two vertices u and v, denoted by $d_k^s(u, v)$, which is the minimum length among all k^*- containers between u and v. The k^*-diameter of G, denote by $D_k^s(G)$, is $\max\{d_k^s(u, v) \mid u$ and v are two different vertices of $G\}$. In particular, we are intrigued in $D^s_{\kappa(G)}(G)$ and $D_2^s(G)$. There are some studies on the k^*-diameter of some interconnection networks [4], [20]. Later, we will study $D_n^s(B_n)$ and $D_2^s(B_n)$.

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Appendix

3^*-container from 123 to $\bar{1}23$

$P_1 = \langle 123, \bar1 23\rangle$

$P_2 = \langle 123, \bar2\bar1 3, 2\bar1 3, 1\bar2 3, \bar1\bar2 3, 213, \bar3\bar1\bar2, 13\bar2, \bar1 3\bar2, \bar3 1\bar2, 31\bar2, \bar1\bar3\bar2, 231, \bar2 31, \bar3 21, 321, \bar2\bar3 1, 2\bar3 1, 3\bar2 1, \bar3\bar2 1, \bar1 23\rangle$

$P_3 = \langle 123, \bar3\bar2\bar1, 3\bar2\bar1, 2\bar3\bar1, \bar2\bar3\bar1, 32\bar1, \bar3 2\bar1, \bar2 3\bar1, 23\bar1, 1\bar3\bar2, 3\bar1\bar2, 21\bar3, \bar1\bar2\bar3, 1\bar2\bar3, 2\bar1\bar3, \bar2\bar1\bar3, 12\bar3, \bar1 2\bar3, \bar2 1\bar3, 3\bar1 2, 1\bar3 2, \bar1\bar3 2, 312, \bar3 12, \bar1 32, 132, \bar3\bar1 2, \bar2 13, \bar1 23\rangle$

3^*-container from 123 to $\bar{2}\bar{1}3$

$P_1 = \langle 123, \bar2\bar1 3\rangle$

$P_2 = \langle 123, \bar1 23, \bar2 13, 213, \bar1\bar2 3, 1\bar2 3, 2\bar1 3, \bar2\bar1 3\rangle$

$P_3 = \langle 123, \bar3\bar2\bar1, 23\bar1, 1\bar3\bar2, 3\bar1\bar2, \bar3\bar1\bar2, 13\bar2, \bar1 3\bar2, \bar3 1\bar2, 31\bar2, \bar1\bar3\bar2, 231, \bar3\bar2 1, 3\bar2 1, 2\bar3 1, \bar2\bar3 1, 321, \bar3 21, \bar2 31, \bar1\bar3 2, 312, \bar2\bar1\bar3, 2\bar1\bar3, 1\bar2\bar3, \bar1\bar2\bar3, 21\bar3, \bar2 1\bar3, \bar1 2\bar3, 12\bar3, 3\bar2\bar1, 2\bar3\bar1, \bar2\bar3\bar1, 32\bar1, \bar3 2\bar1, \bar2 3\bar1, 1\bar3 2, 3\bar1 2, \bar3\bar1 2, 132, \bar1 32, \bar3 12, \bar2\bar1 3\rangle$

3^*-container from 123 to $2\bar{1}3$

$P_1 = \langle 123, \bar2\bar1 3, 2\bar1 3\rangle$

$P_2 = \langle 123, \bar3\bar2\bar1, 3\bar2\bar1, 2\bar3\bar1, \bar2\bar3\bar1, 32\bar1, \bar3 2\bar1, \bar2 3\bar1, 23\bar1, 1\bar3\bar2, 3\bar1\bar2, \bar3\bar1\bar2, 13\bar2, \bar1 3\bar2, \bar3 1\bar2, 2\bar1 3\rangle$

$P_3 = \langle 123, \bar1 23, \bar3\bar2 1, 231, \bar1\bar3\bar2, 31\bar2, 2\bar1\bar3, 1\bar2\bar3, \bar1\bar2\bar3, 21\bar3, \bar2 1\bar3, 3\bar1 2, 1\bar3 2, \bar1\bar3 2, \bar2 31, \bar3 21, 321, \bar2\bar3 1, 2\bar3 1, 3\bar2 1, \bar1 2\bar3, 12\bar3, \bar2\bar1\bar3, 312, \bar3 12, \bar1 32, 132, \bar3\bar1 2, \bar2 13, 213, \bar1\bar2 3, 1\bar2 3, 2\bar1 3\rangle$

3^*-container from 123 to $1\bar{2}3$

$P_1 = \langle 123, \bar1 23, \bar2 13, 213, \bar1\bar2 3, 1\bar2 3\rangle$

$P_2 = \langle 123, \bar2\bar1 3, 2\bar1 3, 1\bar2 3\rangle$

$P_3 = \langle 123, \bar3\bar2\bar1, 3\bar2\bar1, 2\bar3\bar1, \bar2\bar3\bar1, 132, \bar3\bar1 2, 3\bar1 2, 1\bar3 2, \bar2 3\bar1, 23\bar1, 1\bar3\bar2, 3\bar1\bar2, \bar3\bar1\bar2, 13\bar2, \bar1 3\bar2, \bar3 1\bar2, 31\bar2, \bar1\bar3\bar2, 231, \bar3\bar2 1, 3\bar2 1, 2\bar3 1, \bar2\bar3 1, \bar1 32, \bar3 12, 312, \bar1\bar3 2, \bar2 31, \bar3 21, 321, \bar1\bar2\bar3, 21\bar3, \bar2 1\bar3, \bar1 2\bar3, 12\bar3, \bar2\bar1\bar3, 2\bar1\bar3, 1\bar2\bar3, 32\bar1, \bar3 2\bar1, 1\bar2 3\rangle$

3^*-container from 123 to $\bar{1}\bar{2}3$

$P_1 = \langle 123, \bar1 23, \bar2 13, 213, \bar1\bar2 3\rangle$

$P_2 = \langle 123, \bar2\bar1 3, 2\bar1 3, 1\bar2\bar2\bar1 3, \bar1\bar2 3\rangle$

$P_3 = \langle 123, \bar3\bar2\bar1, 3\bar2\bar1, 2\bar3\bar1, \bar2\bar3\bar1, 32\bar1, \bar3 2\bar1, \bar2 3\bar1, 23\bar1, 1\bar3\bar2, \bar1\bar3\bar2, 231, \bar3\bar2 1, 3\bar2 1, 2\bar3 1, \bar2\bar3 1, 321, \bar1\bar2\bar3, 1\bar2\bar3, 2\bar1\bar3, 31\bar2, \bar3 1\bar2, \bar1 3\bar2, 13\bar2, \bar3\bar1\bar2, 3\bar1\bar2, 21\bar3, \bar2 1\bar3, \bar1 2\bar3, 12\bar3, \bar2\bar1\bar3, 312, \bar3 12, \bar1 32, 132, \bar3\bar1 2, 3\bar1 2, 1\bar3 2, \bar1\bar3 2, \bar2 31, \bar3 21, \bar1\bar2 3\rangle$

3^*-container from 123 to 213

$P_1 = \langle 123, \bar1 23, \bar2 13, 213\rangle$

$P_2 = \langle 123, \bar2\bar1 3, 2\bar1 3, 1\bar2\bar2\bar1 3, \bar1\bar2 3, 213\rangle$

$P_3 = \langle 123, \bar3\bar2\bar1, 3\bar2\bar1, 2\bar3\bar1, 13\bar2, \bar1 3\bar2, \bar3 1\bar2, 31\bar2, \bar1\bar3\bar2, 1\bar3\bar2, 23\bar1, \bar2 3\bar1, \bar3 2\bar1, 32\bar1, \bar2\bar3\bar1, 132, \bar3\bar1 2, 3\bar1 2, 1\bar3 2, \bar1\bar3 2, 312, \bar3 12, \bar1 32, \bar2\bar3 1, 2\bar3 1, 3\bar2 1, \bar3\bar2 1, 231, \bar2 31, \bar3 21, 321, \bar1\bar2\bar3, 1\bar2\bar3, 2\bar1\bar3, \bar2\bar1\bar3, 12\bar3, \bar1 2\bar3, \bar2 1\bar3, 21\bar3, 3\bar1\bar2, \bar3\bar1\bar2, 213\rangle$

3^*-container from 123 to $\bar{2}13$

$P_1 = \langle 123, \bar{1}23, \bar{2}13 \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1},$
$1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, \bar{3}1\bar{2}, 2\bar{1}3, 1\bar{2}3, \bar{1}\bar{2}3, 213, \bar{2}13 \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, 312, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{2}1\bar{3}, \bar{1}2\bar{3},$
$12\bar{3}, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2},$
$\bar{1}3\bar{2}, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1,$
$\bar{1}32, 132, \bar{3}\bar{1}2, \bar{2}13 \rangle$

3^*-container from 123 to $\bar{3}\bar{2}\bar{1}$

$P_1 = \langle 123, \bar{3}\bar{2}\bar{1} \rangle$

$P_2 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, 32\bar{1},$
$\bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, \bar{3}\bar{2}\bar{1} \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3},$
$312, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1},$
$\bar{3}\bar{2}\bar{1} \rangle$

3^*-container from 123 to $3\bar{2}\bar{1}$

$P_1 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, 1\bar{2}\bar{2}\bar{1}3, \bar{1}\bar{2}3, 213, \bar{2}13, \bar{3}\bar{1}2, 132,$
$\bar{1}32, \bar{3}12, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3},$
$3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, \bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1,$
$\bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1} \rangle$

$P_3 = \langle 123, \bar{1}23, \bar{3}\bar{2}1, 231, \bar{1}\bar{3}\bar{2}, 31\bar{2}, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2},$
$\bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1},$
$3\bar{2}\bar{1} \rangle$

3^*-container from 123 to $2\bar{3}\bar{1}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$2\bar{3}1, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3},$
$\bar{2}1\bar{3}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2},$
$\bar{3}\bar{1}\bar{2}, 13\bar{2}, 2\bar{3}\bar{1} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2,$
$312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1} \rangle$

3^*-container from 123 to $\bar{2}\bar{3}\bar{1}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312,$
$\bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{2}\bar{3}\bar{1} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2},$
$23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1} \rangle$

3^*-container from 123 to $32\bar{1}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3},$
$\bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132,$
$\bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3},$
$1\bar{2}\bar{3}, 32\bar{1} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1} \rangle$

3^*-container from 123 to $\bar{3}2\bar{1}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, \bar{3}2\bar{1} \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1} \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3},$
$\bar{2}1\bar{3}, \bar{1}2\bar{3}, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2},$
$13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, \bar{1}\bar{3}2, 1\bar{3}2,$
$3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{1}32, \bar{3}12, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, 3\bar{2}\bar{1},$
$2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1} \rangle$

3^*-container from 123 to $\bar{2}3\bar{1}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2},$
$\bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1, 321,$
$\bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1},$
$\bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1} \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1} \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}, 21\bar{3},$
$\bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, 1\bar{3}2, \bar{2}3\bar{1} \rangle$

3^*-container from 123 to $23\bar{1}$

$P_1 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, 1\bar{2}3, \bar{1}\bar{2}3, 213, \bar{2}1\bar{3}, \bar{3}\bar{1}2, 132,$
$\bar{1}32, \bar{3}12, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3},$
$3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, \bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1,$
$\bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1} \rangle$

$P_3 = \langle 123, \bar{1}23, \bar{3}\bar{2}1, 231, \bar{1}\bar{3}\bar{2}, 31\bar{2}, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2},$
$\bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, 23\bar{1} \rangle$

3^*-container from 123 to $13\bar{2}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2},$
$23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2,$
$\bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2},$
$13\bar{2} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, 13\bar{2} \rangle$

3^*-container from 123 to $\bar{1}3\bar{2}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2} \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3},$
$\bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2,$
$1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2},$
$\bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2} \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231,$
$\bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{1}3\bar{2} \rangle$

3^*-container from 123 to $\bar{3}1\bar{2}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2} \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2,$
$3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3},$
$\bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2},$
$\bar{1}\bar{3}\bar{2}, 31\bar{2}, \bar{3}1\bar{2} \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231,$
$\bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{1}3\bar{2}, \bar{3}1\bar{2} \rangle$

3^*-container from 123 to $31\bar{2}$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}\rangle$
$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231,$
$\bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2},$
$31\bar{2}\rangle$
$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3},$
$\bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2,$
$1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}\rangle$

3^*-container from 123 to $\bar{1}\bar{3}\bar{2}$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1},$
$\bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3},$
$\bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2},$
$\bar{1}\bar{3}\bar{2}\rangle$
$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{1}\bar{3}\bar{2}\rangle$
$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}\rangle$

3^*-container from 123 to $1\bar{3}\bar{2}$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2},$
$\bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}\rangle$
$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}\rangle$
$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}, 21\bar{3},$
$\bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, 1\bar{3}2, \bar{2}3\bar{1}, \bar{3}2\bar{1},$
$32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{1}2\bar{3}, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1, 2\bar{3}1, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2},$
$1\bar{3}\bar{2}\rangle$

3^*-container from 123 to $3\bar{1}\bar{2}$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}\rangle$
$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}\rangle$
$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, \bar{2}31, \bar{3}21, 321,$
$\bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132,$
$\bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}, 3\bar{1}\bar{2}\rangle$

3^*-container from 123 to $\bar{3}\bar{1}\bar{2}$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{3}\bar{1}\bar{2}\rangle$
$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2},$
$\bar{3}\bar{1}\bar{2}\rangle$
$P_3 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, 1\bar{2}3, \bar{1}\bar{2}3, \bar{3}21, 321, \bar{2}\bar{3}1, 2\bar{3}1,$
$3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{1}\bar{3}2, 1\bar{3}2, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1},$
$\bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 3\bar{1}2, \bar{3}\bar{1}2, 132,$
$\bar{1}32, \bar{3}12, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, 3\bar{1}\bar{2},$
$\bar{3}\bar{1}\bar{2}\rangle$

3^*-container from 123 to 132
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1},$
$1\bar{2}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 231, \bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{2}\bar{3}1,$
$\bar{1}32, 132\rangle$
$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132\rangle$
$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3},$
$\bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132\rangle$

3^*-container from 123 to $\bar{1}32$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312,$
$\bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{1}32\rangle$
$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1},$
$1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21, 231,$
$\bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{2}\bar{3}1, \bar{1}32\rangle$
$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32\rangle$

3^*-container from 123 to $\bar{3}12$
$P_1 = \langle 123, \bar{2}\bar{1}3, \bar{3}12\rangle$
$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1},$
$1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, \bar{3}1\bar{2}, 2\bar{1}3, 1\bar{2}3, \bar{1}\bar{2}3,$
$213, \bar{2}13, \bar{3}\bar{1}2, 132, \bar{1}32, \bar{3}12\rangle$
$P_3 = \langle 123, \bar{1}23, \bar{3}\bar{2}1, 231, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3},$
$21\bar{3}, \bar{2}1\bar{3}, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, \bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1,$
$2\bar{3}1, 3\bar{2}1, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{3}12\rangle$

3^*-container from 123 to 312
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312\rangle$
$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, 312\rangle$
$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1},$
$1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31,$
$231, \bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{2}\bar{3}1, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2,$
$1\bar{3}2, \bar{1}\bar{3}2, 312\rangle$

3^*-container from 123 to $\bar{1}\bar{3}2$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312,$
$\bar{1}\bar{3}2\rangle$
$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, \bar{1}\bar{3}2\rangle$
$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3},$
$32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, 2\bar{3}\bar{1},$
$\bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2\rangle$

3^*-container from 123 to $1\bar{3}2$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2},$
$13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1},$
$\bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2\rangle$
$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231,$
$\bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2\rangle$
$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312,$
$\bar{1}\bar{3}2, 1\bar{3}2\rangle$

3^*-container from 123 to $3\bar{1}2$
$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2},$
$\bar{1}\bar{3}\bar{2}, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1},$
$32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, 3\bar{1}2\rangle$
$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2\rangle$
$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1,$
$\bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3},$
$\bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3}, 3\bar{1}2\rangle$

3^*-container from 123 to $\bar{3}\bar{1}2$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, \bar{3}\bar{1}2 \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2 \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, 1\bar{2}3, \bar{1}\bar{2}3, 213, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 3\bar{1}2, \bar{3}\bar{1}2 \rangle$

3^*-container from 123 to $\bar{3}\bar{2}1$

$P_1 = \langle 123, \bar{1}23, \bar{3}\bar{2}1 \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, \bar{3}1\bar{2}, 2\bar{1}3, 1\bar{2}3, \bar{1}\bar{2}3, 213, \bar{2}13, \bar{3}\bar{1}2, 132, \bar{1}32, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1 \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, 312, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1 \rangle$

3^*-container from 123 to $3\bar{2}1$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1 \rangle$

$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 3\bar{2}1 \rangle$

$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1 \rangle$

3^*-container from 123 to $2\bar{3}1$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1 \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1 \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1 \rangle$

3^*-container from 123 to $\bar{2}\bar{3}1$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1 \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1 \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1 \rangle$

3^*-container from 123 to 321

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21, 321 \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 321 \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{2}\bar{3}\bar{1}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 321 \rangle$

3^*-container from 123 to $\bar{3}21$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, \bar{3}21 \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21 \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, 1\bar{3}2, \bar{2}3\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, 32\bar{1}, \bar{3}2\bar{1}, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21 \rangle$

3^*-container from 123 to $\bar{2}31$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31 \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31 \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, \bar{2}31 \rangle$

3^*-container from 123 to 231

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 231 \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231 \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 321, \bar{3}21, \bar{2}31, 231 \rangle$

3^*-container from 123 to $\bar{1}2\bar{3}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{1}2\bar{3} \rangle$

3^*-container from 123 to $12\bar{3}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3}, 12\bar{3} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 12\bar{3} \rangle$

3^*-container from 123 to $\bar{2}\bar{1}\bar{3}$

$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3} \rangle$

$P_2 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}, 32\bar{1}, \bar{3}2\bar{1}, \bar{2}3\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3} \rangle$

$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3} \rangle$

3^*-container from 123 to $2\bar{1}\bar{3}$

$$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}\rangle$$
$$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}, 2\bar{1}\bar{3}\rangle$$
$$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}\rangle$$

3^*-container from 123 to $1\bar{2}\bar{3}$

$$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}\rangle$$
$$P_2 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 23\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, 2\bar{3}1, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}, \bar{1}\bar{2}\bar{3}, 1\bar{2}\bar{3}\rangle$$
$$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 3\bar{2}\bar{1}, 2\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}, \bar{1}2\bar{3}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}\rangle$$

3^*-container from 123 to $\bar{1}\bar{2}\bar{3}$

$$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, 1\bar{3}2, 3\bar{1}2, \bar{3}\bar{1}2, 132, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}, \bar{1}\bar{2}\bar{3}\rangle$$
$$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}\rangle$$
$$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, \bar{2}\bar{3}1, 2\bar{3}1, 3\bar{2}1, \bar{3}\bar{2}1, 231, \bar{2}31, \bar{3}21, 321, \bar{1}\bar{2}\bar{3}\rangle$$

3^*-container from 123 to $21\bar{3}$

$$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}\rangle$$
$$P_2 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, 1\bar{3}\bar{2}, \bar{1}\bar{3}\bar{2}, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3}, \bar{2}1\bar{3}, 21\bar{3}\rangle$$
$$P_3 = \langle 123, \bar{2}\bar{1}3, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, 1\bar{3}2, \bar{2}3\bar{1}, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 312, \bar{1}\bar{3}2, \bar{2}31, \bar{3}21, 321, \bar{2}\bar{3}1, 2\bar{3}1, \bar{1}3\bar{2}, 13\bar{2}, \bar{3}\bar{1}\bar{2}, 3\bar{1}\bar{2}, 21\bar{3}\rangle$$

3^*-container from 123 to $\bar{2}1\bar{3}$

$$P_1 = \langle 123, \bar{1}23, \bar{2}13, 213, \bar{1}\bar{2}3, 1\bar{2}3, \bar{3}2\bar{1}, 32\bar{1}, \bar{2}\bar{3}\bar{1}, 2\bar{3}\bar{1}, 3\bar{2}\bar{1}, 12\bar{3}, \bar{2}\bar{1}\bar{3}, 2\bar{1}\bar{3}, 1\bar{2}\bar{3}, \bar{1}\bar{2}\bar{3}, 21\bar{3}, \bar{2}1\bar{3}\rangle$$
$$P_2 = \langle 123, \bar{2}\bar{1}3, 2\bar{1}3, \bar{3}1\bar{2}, 31\bar{2}, \bar{1}\bar{3}\bar{2}, 1\bar{3}\bar{2}, 3\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, 13\bar{2}, \bar{1}3\bar{2}, 2\bar{3}1, \bar{2}\bar{3}1, 321, \bar{3}21, \bar{2}31, 231, \bar{3}\bar{2}1, 3\bar{2}1, \bar{1}2\bar{3}, \bar{2}1\bar{3}\rangle$$
$$P_3 = \langle 123, \bar{3}\bar{2}\bar{1}, 23\bar{1}, \bar{2}3\bar{1}, 1\bar{3}2, \bar{1}\bar{3}2, 312, \bar{3}12, \bar{1}32, 132, \bar{3}\bar{1}2, 3\bar{1}2, \bar{2}1\bar{3}\rangle$$

Cherng Chin is a Ph.D. student of Graduate School of Design at National Yunlin University of Science and Technology and a senior lecturer at the Department of Computer Science and Communication Engineering at Providence University, both in Taiwan, R.O.C. He received his M.S. in Computer Science from PACE University, New York, U.S.A. His current research focuses on Graph Theory, Information Design, Shape Grammars, Computaional Design, and Generative Design of Culture Symbol.

Tien-Hsiung Weng is an asssociate professor at the Department of Computer Science and Information Engineering at Providence University, Taiwan, ROC. He received his Ph.D. in Computer Science from University of Houston, Texas, USA. His current research focuses on graph theory, parallel programming model, performance measurement, and compiler analysis for code improvement and parallel programming.

Lih-Hsing Hsu received his BS degree in mathematics from Chung Yuan Christian University, Taiwan, Republic of China, in 1975, and his Ph.D. degree in mathematics from the State University of New York at Stony Brook in 1981. From 1981 to 1985, he was associate professor at Department of Applied Mathematics at National Chiao Tung University in Taiwan. In 1985, he was a professor in National Chiao Tung University. In 1985 to 1988, he was chairman of Department of Applied Mathematics at National Chiao Tung University. After 1988, he joined with Department of Computer and Information Science of National Chiao Tung University. In 2004, he is retied from National Chiao Tung University by holding a title as honorary scholar of National Chiao Tung University. Currently, he is the Chairman of Department of Computer Science and Information Engineering, Providence University, Shalu, TaiChung Hsien, Republic of China. His research interests include interconnection networks, algorithms, graph theory, and VLSI layout.

Shang-Chia Chiou is a Professor at Graduate School of Design at National Yunlin University of Science and Technology, Taiwan, ROC. He received his Ph.D. in Architecture, Carnegie Mellon University, U.S.A. His current research focuses on Computational Design, Shape Grammars, Computer Aided Design, Architectural Design and Cultural Heritage.