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LETTER

Moment Invariants of the Weighted Image

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SUMMARY Moment invariants of a discrete image are not strictly invariant to image displacements due to quantization errors. This letter introduces a weighting function such that the pixel value is smoothly reduced to zero at the boundary of the image. Image moments of the weighted image are robust against quantization errors, and the moment invariants of the weighted image are more invariant than those of the unweighted image.

key words: *moment invariant, image moment, discrete image, invariancy*

1. Introduction

Moment invariants of a 2D image are invariant to translation, rotation, and scale changes [1], and they have been widely used in pattern recognition and image processing [2]. The moment invariants are strictly invariant only when the coordinates are continuous. When dealing with a discrete image where the coordinates are discrete, the invariant property is comprised due to quantization errors [3]. Especially for a high-order moment, a small quantization error around the boundary of the image causes a large error in image moments, because the pixel value around the boundary of the image has a significant contribution to the image moment. Then invariancy of the moment invariants is apparently degraded. Several experimental results using artificial and natural images were presented in [4].

This letter introduces a weighting function such that the pixel value is smoothly reduced to zero at the boundary of the image. The weighting function decreases the contribution of the boundary image to the image moment. So the image moments of the weighted image are robust against quantization errors, and the moment invariants of the weighted image are more invariant than those of the unweighted image.

2. Image Moments and Moment Invariants

Let $f(x, y)$ be a continuous image of size $N \times N$, where the origin is chosen as the center of the image such that $-N/2 \leq x, y \leq N/2$. Then the image moment of order (p, q) is defined as

$$M_{pq} = \int_{-N/2}^{N/2}\int_{-N/2}^{N/2} x^p y^q f(x, y) dxdy. \tag{1}$$

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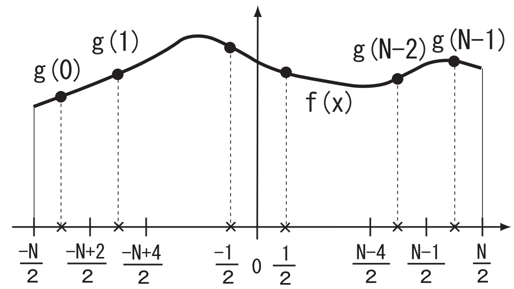


Fig. 1 Relation between $f(x)$ and $g(n)$.

We sample the continuous image $f(x, y)$ with the sampling interval of 1 to generate the $N \times N$ discrete image

$$g(n, m) = f(n - (N-1)/2,\ m - (N-1)/2) \quad (n, m = 0, 1, \cdots, N-1).$$

Figure 1 illustrates the relation between a 1-D continuous image $f(x)$ and the discrete image $g(n)$. The image moment of order (p, q) for the discrete image $g(n, m)$ is defined by

$$M_{pq}^{\mathrm{d}} = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(n - (N-1)/2)^p \cdot (m - (N-1)/2)^q g(n, m). \tag{2}$$

We here call M_{pq} and M_{pq}^{d} as the continuous and discrete moments, respectively.

The following quantities computed from M_{pq} are called the invariant moments, and they are strictly invariant to translation, rotation, and scale changes of the 2D image:

$$\phi(0) = \eta_{20} + \eta_{02} \tag{3a}$$
$$\phi(1) = (\eta_{20} - \eta_{02})^2 + 4\eta_{11}^2 \tag{3b}$$
$$\phi(2) = (\eta_{30} - 3\eta_{12})^2 + (3\eta_{21} - \eta_{03})^2 \tag{3c}$$
$$\phi(3) = (\eta_{30} + \eta_{12})^2 + (\eta_{21} + \eta_{03})^2 \tag{3d}$$
$$\phi(4) = (\eta_{30} - 3\eta_{12})(\eta_{30} + \eta_{12})[(\eta_{30} + \eta_{12})^2 - 3(\eta_{21} + \eta_{03})^2] + (3\eta_{21} - \eta_{03})(\eta_{21} + \eta_{03})[3(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2] \tag{3e}$$
$$\phi(5) = (\eta_{20} - \eta_{02})[(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2] + 4\eta_{11}(\eta_{30} + \eta_{12})(\eta_{21} + \eta_{03}) \tag{3f}$$

with

$$\eta_{pq} = \mu_{pq}/\mu_{00}^{(p+q)/2+1} \tag{4}$$

$$\mu_{pq} = \int_{-N/2}^{N/2}\int_{-N/2}^{N/2}(x - M_{10}/M_{00})^p \cdot (y - M_{01}/M_{00})^q f(x,y)dxdy. \quad (5)$$

However, when dealing with a discrete image, we approximately compute the value of μ_{pq} by replacing the integral by the summation and the continuous moment M_{pq} by the discrete moment M^{d}_{pq}. Then the approximately computed invariant moments do not have the perfect invariant property.

3. Weighting Function

We see from Eq. (2) that the pixel value around the boundary of the image has a significant contribution to the discrete moment. Therefore a small quantization error caused around the boundary of the discrete image may yield a large error in image moments. We thus introduce a weighting function such that the pixel value is smoothly reduced to zero at the boundary of the image. This decreases the contribution of the boundary image to the image moment. Here we use the following circular weighting function that is continuous up to the first derivative:

$$w(r) = \begin{cases} 1 - 2(r/M)^2 & (0 \le r < M/2) \\ 2(1 - r/M)^2 & (M/2 \le r < M) \\ 0 & (M \le |x|) \end{cases}, \quad (6)$$

where $M = N/2$, and $r = \sqrt{x^2 + y^2}$ which is the Euclidean distance from the origin. When dealing with a discrete image, we compute the weight value at (n,m) by putting $r = \sqrt{n^2 + m^2}$.

4. Results

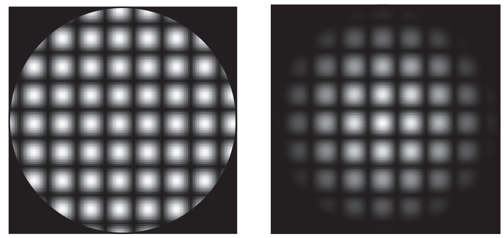
Consider the following continuous image of size 128×128:

$$f(x,y) = \frac{1}{4}\left(\cos\frac{\pi(x+2)}{8} + 1\right)\left(\cos\frac{\pi(y+2)}{8} + 1\right) \quad (-64 \le x, y \le 64). \quad (7)$$

The corresponding discrete image of size 128×128 is given by

$$g(n,m) = \frac{1}{4}\left(\cos\frac{\pi(n-61.5)}{8} + 1\right)\left(\cos\frac{\pi(m-61.5)}{8} + 1\right) \quad (n,m = 0,1,\cdots,127). \quad (8)$$

Figure 2 shows the original image g and the weighted images wg, where we put $g = 0$ outside the circle of radius 64. We computed the continuous moment M_{pq} for each of f and wf and the discrete moment M^{d}_{pq} for each of g and wg, and then evaluated the relative error between M_{pq} and M^{d}_{pq} by $e_{pq} = |M^{\mathrm{d}}_{pq} - M_{pq}|\ /M_{pq}$. Tables 1 and 2 summarize the results for the unweighted and weighted images, respectively. We see that the relative error is decreased by applying the weighting function on the discrete image. We also see that the relative error of a high-order moment is not necessarily larger than that of a low-order moment, because a regular pattern characteristic of the artificial image amplifies the error in a specific moment. This phenomenon occurs also for other frequencies and phases, and occurs in the other artificial image used in [4]. Unfortunately we cannot say which characteristics of an image increase the error between continuous and discrete moments, because it is quite difficult to compute the true value of the continuous moment even for the artificial image.



(a) (b)

Fig. 2 Unweighted and weighted artificial images: (a) g, (b) wg.

Table 1 Relative error for unweighted image.

p	q	M_{pq}	M^{d}_{pq}	e (%)
0	0	$3.1741 \cdot 10^3$	$3.1809 \cdot 10^3$	0.22
0	1	$-8.6897 \cdot 10^2$	$-8.8576 \cdot 10^2$	1.93
0	2	$3.2121 \cdot 10^6$	$3.2261 \cdot 10^6$	0.43
0	3	$-5.1645 \cdot 10^6$	$-5.2633 \cdot 10^6$	1.91
1	1	$-7.8928 \cdot 10^3$	$-8.0861 \cdot 10^3$	2.45
1	2	$-1.2188 \cdot 10^6$	$-1.2497 \cdot 10^6$	2.53
1	3	$-1.8781 \cdot 10^7$	$-1.9169 \cdot 10^7$	2.09
2	2	$2.1939 \cdot 10^9$	$2.2039 \cdot 10^9$	0.46
2	3	$2.5023 \cdot 10^9$	$2.5057 \cdot 10^9$	0.14
3	3	$-5.2990 \cdot 10^{10}$	$-5.4301 \cdot 10^{10}$	2.47

Table 2 Relative error for weighted image.

p	q	M_{pq}	M^{d}_{pq}	e (%)
0	0	$9.2356 \cdot 10^2$	$9.2356 \cdot 10^2$	0.00
0	1	$-7.9425 \cdot 10^{-1}$	$-7.9574 \cdot 10^{-1}$	0.19
0	2	$4.1274 \cdot 10^5$	$4.1274 \cdot 10^5$	0.00
0	3	$2.9601 \cdot 10^4$	$2.9597 \cdot 10^4$	0.01
1	1	$-1.3865 \cdot 10^1$	$-1.3874 \cdot 10^1$	0.07
1	2	$4.5887 \cdot 10^3$	$4.5895 \cdot 10^3$	0.02
1	3	$4.2997 \cdot 10^4$	$4.2979 \cdot 10^4$	0.04
2	2	$1.5228 \cdot 10^8$	$1.5228 \cdot 10^8$	0.00
2	3	$-8.5958 \cdot 10^6$	$-8.5988 \cdot 10^6$	0.03
3	3	$1.8444 \cdot 10^8$	$1.8439 \cdot 10^8$	0.03

We rotated the continuous image $f(x,y)$ from 0° to 45° by 2.25° to generate 21 continuous images, and sampled them to yield the discrete versions, that is, $g^{(i)}(n,m)$ $(i = 0,1,\cdots,20)$. We computed the normalized standard deviation of the invariant moment defined by:

$$\Delta\phi(n) = \frac{\sqrt{\frac{1}{R}\sum_{i=0}^{R-1}\left(\phi_i(n) - \phi_{\mathrm{mean}}(n)\right)^2}}{\phi_{\mathrm{mean}}(n)} \quad (9)$$

Table 3 $\phi_i(n)$ and $\Delta\phi(n)$ for unweighted artificial image.

i	$\phi_i(0)$	$\phi_i(1)$	$\phi_i(2)$	$\phi_i(3)$	$\phi_i(4)$	$\phi_i(5)$
0	$6.3762 \cdot 10^{-1}$	$2.4011 \cdot 10^{-6}$	$5.0028 \cdot 10^{-4}$	$1.0631 \cdot 10^{-6}$	$-2.4517 \cdot 10^{-11}$	$1.6473 \cdot 10^{-9}$
1	$6.3762 \cdot 10^{-1}$	$2.6661 \cdot 10^{-6}$	$4.9706 \cdot 10^{-4}$	$1.0629 \cdot 10^{-6}$	$-2.4430 \cdot 10^{-11}$	$1.7355 \cdot 10^{-9}$
2	$6.3762 \cdot 10^{-1}$	$3.0581 \cdot 10^{-6}$	$4.8668 \cdot 10^{-4}$	$1.0645 \cdot 10^{-6}$	$-2.4228 \cdot 10^{-11}$	$1.8613 \cdot 10^{-9}$
3	$6.3763 \cdot 10^{-1}$	$3.0227 \cdot 10^{-6}$	$4.7460 \cdot 10^{-4}$	$1.0698 \cdot 10^{-6}$	$-2.4105 \cdot 10^{-11}$	$1.8593 \cdot 10^{-9}$
4	$6.3762 \cdot 10^{-1}$	$2.8071 \cdot 10^{-6}$	$4.7389 \cdot 10^{-4}$	$1.0745 \cdot 10^{-6}$	$-2.4241 \cdot 10^{-11}$	$1.8002 \cdot 10^{-9}$
5	$6.3762 \cdot 10^{-1}$	$2.8326 \cdot 10^{-6}$	$4.8691 \cdot 10^{-4}$	$1.0729 \cdot 10^{-6}$	$-2.4521 \cdot 10^{-11}$	$1.8051 \cdot 10^{-9}$
6	$6.3762 \cdot 10^{-1}$	$2.8789 \cdot 10^{-6}$	$4.9857 \cdot 10^{-4}$	$1.0663 \cdot 10^{-6}$	$-2.4582 \cdot 10^{-11}$	$1.8092 \cdot 10^{-9}$
7	$6.3762 \cdot 10^{-1}$	$2.6870 \cdot 10^{-6}$	$4.9838 \cdot 10^{-4}$	$1.0608 \cdot 10^{-6}$	$-2.4375 \cdot 10^{-11}$	$1.7384 \cdot 10^{-9}$
8	$6.3762 \cdot 10^{-1}$	$2.5580 \cdot 10^{-6}$	$4.9544 \cdot 10^{-4}$	$1.0600 \cdot 10^{-6}$	$-2.4286 \cdot 10^{-11}$	$1.6953 \cdot 10^{-9}$
9	$6.3763 \cdot 10^{-1}$	$2.7010 \cdot 10^{-6}$	$4.9822 \cdot 10^{-4}$	$1.0640 \cdot 10^{-6}$	$-2.4495 \cdot 10^{-11}$	$1.7486 \cdot 10^{-9}$
10	$6.3762 \cdot 10^{-1}$	$2.7772 \cdot 10^{-6}$	$4.9681 \cdot 10^{-4}$	$1.0706 \cdot 10^{-6}$	$-2.4684 \cdot 10^{-11}$	$1.7821 \cdot 10^{-9}$
11	$6.3762 \cdot 10^{-1}$	$2.6016 \cdot 10^{-6}$	$4.8435 \cdot 10^{-4}$	$1.0753 \cdot 10^{-6}$	$-2.4539 \cdot 10^{-11}$	$1.7319 \cdot 10^{-9}$
12	$6.3762 \cdot 10^{-1}$	$2.6254 \cdot 10^{-6}$	$4.7464 \cdot 10^{-4}$	$1.0740 \cdot 10^{-6}$	$-2.4247 \cdot 10^{-11}$	$1.7392 \cdot 10^{-9}$
13	$6.3762 \cdot 10^{-1}$	$3.0326 \cdot 10^{-6}$	$4.8052 \cdot 10^{-4}$	$1.0685 \cdot 10^{-6}$	$-2.4210 \cdot 10^{-11}$	$1.8539 \cdot 10^{-9}$
14	$6.3762 \cdot 10^{-1}$	$3.1153 \cdot 10^{-6}$	$4.9344 \cdot 10^{-4}$	$1.0658 \cdot 10^{-6}$	$-2.4441 \cdot 10^{-11}$	$1.8804 \cdot 10^{-9}$
15	$6.3763 \cdot 10^{-1}$	$2.6682 \cdot 10^{-6}$	$4.9925 \cdot 10^{-4}$	$1.0676 \cdot 10^{-6}$	$-2.4647 \cdot 10^{-11}$	$1.7416 \cdot 10^{-9}$
16	$6.3763 \cdot 10^{-1}$	$2.4409 \cdot 10^{-6}$	$4.9957 \cdot 10^{-4}$	$1.0681 \cdot 10^{-6}$	$-2.4672 \cdot 10^{-11}$	$1.6675 \cdot 10^{-9}$
17	$6.3763 \cdot 10^{-1}$	$2.8402 \cdot 10^{-6}$	$4.9935 \cdot 10^{-4}$	$1.0648 \cdot 10^{-6}$	$-2.4551 \cdot 10^{-11}$	$1.7940 \cdot 10^{-9}$
18	$6.3762 \cdot 10^{-1}$	$3.1941 \cdot 10^{-6}$	$4.9178 \cdot 10^{-4}$	$1.0626 \cdot 10^{-6}$	$-2.4291 \cdot 10^{-11}$	$1.8982 \cdot 10^{-9}$
19	$6.3762 \cdot 10^{-1}$	$2.8829 \cdot 10^{-6}$	$4.7524 \cdot 10^{-4}$	$1.0650 \cdot 10^{-6}$	$-2.3960 \cdot 10^{-11}$	$1.8082 \cdot 10^{-9}$
20	$6.3761 \cdot 10^{-1}$	$2.5884 \cdot 10^{-6}$	$4.6578 \cdot 10^{-4}$	$1.0672 \cdot 10^{-6}$	$-2.3795 \cdot 10^{-11}$	$1.7170 \cdot 10^{-9}$
(%)	$\Delta\phi(0) = 0.000$	$\Delta\phi(1) = 7.660$	$\Delta\phi(2) = 2.180$	$\Delta\phi(3) = 0.408$	$\Delta\phi(4) = 0.939$	$\Delta\phi(5) = 3.818$

Table 4 $\phi_i(n)$ and $\Delta\phi(n)$ for weighted artificial image.

i	$\phi_i(0)$	$\phi_i(1)$	$\phi_i(2)$	$\phi_i(3)$	$\phi_i(4)$	$\phi_i(5)$
0	$9.6778 \cdot 10^{-1}$	$1.0582 \cdot 10^{-9}$	$5.5975 \cdot 10^{-6}$	$2.0792 \cdot 10^{-6}$	−7.0932e-12	$6.7638 \cdot 10^{-11}$
1	$9.6778 \cdot 10^{-1}$	$1.0574 \cdot 10^{-9}$	$5.5977 \cdot 10^{-6}$	$2.0793 \cdot 10^{-6}$	−7.0937e-12	$6.7613 \cdot 10^{-11}$
2	$9.6778 \cdot 10^{-1}$	$1.0567 \cdot 10^{-9}$	$5.5980 \cdot 10^{-6}$	$2.0794 \cdot 10^{-6}$	−7.0945e-12	$6.7595 \cdot 10^{-11}$
3	$9.6778 \cdot 10^{-1}$	$1.0582 \cdot 10^{-9}$	$5.5978 \cdot 10^{-6}$	$2.0795 \cdot 10^{-6}$	−7.0946e-12	$6.7645 \cdot 10^{-11}$
4	$9.6778 \cdot 10^{-1}$	$1.0602 \cdot 10^{-9}$	$5.5970 \cdot 10^{-6}$	$2.0795 \cdot 10^{-6}$	−7.0942e-12	$6.7708 \cdot 10^{-11}$
5	$9.6778 \cdot 10^{-1}$	$1.0593 \cdot 10^{-9}$	$5.5965 \cdot 10^{-6}$	$2.0794 \cdot 10^{-6}$	−7.0937e-12	$6.7679 \cdot 10^{-11}$
6	$9.6778 \cdot 10^{-1}$	$1.0562 \cdot 10^{-9}$	$5.5968 \cdot 10^{-6}$	$2.0793 \cdot 10^{-6}$	−7.0935e-12	$6.7579 \cdot 10^{-11}$
7	$9.6778 \cdot 10^{-1}$	$1.0550 \cdot 10^{-9}$	$5.5973 \cdot 10^{-6}$	$2.0793 \cdot 10^{-6}$	−7.0934e-12	$6.7537 \cdot 10^{-11}$
8	$9.6778 \cdot 10^{-1}$	$1.0569 \cdot 10^{-9}$	$5.5975 \cdot 10^{-6}$	$2.0792 \cdot 10^{-6}$	−7.0932e-12	$6.7596 \cdot 10^{-11}$
9	$9.6778 \cdot 10^{-1}$	$1.0582 \cdot 10^{-9}$	$5.5973 \cdot 10^{-6}$	$2.0793 \cdot 10^{-6}$	−7.0934e-12	$6.7639 \cdot 10^{-11}$
10	$9.6778 \cdot 10^{-1}$	$1.0562 \cdot 10^{-9}$	$5.5974 \cdot 10^{-6}$	$2.0795 \cdot 10^{-6}$	−7.0946e-12	$6.7582 \cdot 10^{-11}$
11	$9.6778 \cdot 10^{-1}$	$1.0542 \cdot 10^{-9}$	$5.5977 \cdot 10^{-6}$	$2.0797 \cdot 10^{-6}$	−7.0961e-12	$6.7527 \cdot 10^{-11}$
12	$9.6778 \cdot 10^{-1}$	$1.0564 \cdot 10^{-9}$	$5.5977 \cdot 10^{-6}$	$2.0798 \cdot 10^{-6}$	−7.0966e-12	$6.7599 \cdot 10^{-11}$
13	$9.6778 \cdot 10^{-1}$	$1.0603 \cdot 10^{-9}$	$5.5973 \cdot 10^{-6}$	$2.0798 \cdot 10^{-6}$	−7.0963e-12	$6.7724 \cdot 10^{-11}$
14	$9.6778 \cdot 10^{-1}$	$1.0603 \cdot 10^{-9}$	$5.5971 \cdot 10^{-6}$	$2.0799 \cdot 10^{-6}$	−7.0963e-12	$6.7725 \cdot 10^{-11}$
15	$9.6778 \cdot 10^{-1}$	$1.0570 \cdot 10^{-9}$	$5.5974 \cdot 10^{-6}$	$2.0799 \cdot 10^{-6}$	−7.0969e-12	$6.7622 \cdot 10^{-11}$
16	$9.6778 \cdot 10^{-1}$	$1.0567 \cdot 10^{-9}$	$5.5976 \cdot 10^{-6}$	$2.0800 \cdot 10^{-6}$	−7.0972e-12	$6.7613 \cdot 10^{-11}$
17	$9.6778 \cdot 10^{-1}$	$1.0609 \cdot 10^{-9}$	$5.5977 \cdot 10^{-6}$	$2.0798 \cdot 10^{-6}$	−7.0964e-12	$6.7741 \cdot 10^{-11}$
18	$9.6778 \cdot 10^{-1}$	$1.0634 \cdot 10^{-9}$	$5.5981 \cdot 10^{-6}$	$2.0795 \cdot 10^{-6}$	−7.0952e-12	$6.7813 \cdot 10^{-11}$
19	$9.6778 \cdot 10^{-1}$	$1.0610 \cdot 10^{-9}$	$5.5989 \cdot 10^{-6}$	$2.0793 \cdot 10^{-6}$	−7.0946e-12	$6.7729 \cdot 10^{-11}$
20	$9.6778 \cdot 10^{-1}$	$1.0588 \cdot 10^{-9}$	$5.5994 \cdot 10^{-6}$	$2.0792 \cdot 10^{-6}$	−7.0944e-12	$6.7656 \cdot 10^{-11}$
(%)	$\Delta\phi(0) = 0.000$	$\Delta\phi(1) = 0.208$	$\Delta\phi(2) = 0.011$	$\Delta\phi(3) = 0.013$	$\Delta\phi(4) = 0.019$	$\Delta\phi(5) = 0.105$

$$\phi_{\text{mean}}(n) = \frac{1}{R} \sum_{j=0}^{R-1} \phi_j(n), \tag{10}$$

where $\phi_i(n)$ is the invariant moment of $g^{(i)}$, and $R = 21$. Similarly we computed the normalized standard deviation for the weighted image wg. The results are summarized in Tables 3 and 4. We see that the invariant moments of the weighted image are more invariant than those of the unweighted image.

We interpolated a discrete natural image by the sinc interpolation, rotated the continuous image from 0° to 85° by 5° to generate 18 images, and sampled them to obtain the discrete versions. Similarly we derived the discrete versions of the corresponding weighted images. Figure 3 shows the unweighted and weighted images. Tables 5 and 6 summarizes the results for weighted and unweighted images, respectively, where the normalized standard deviations are

Table 5 $\phi_i(n)$ and $\Delta\phi(n)$ for unweighted natural image.

i	$\phi_i(0)$	$\phi_i(1)$	$\phi_i(2)$	$\phi_i(3)$	$\phi_i(4)$	$\phi_i(5)$
0	$3.2739 \cdot 10^{-1}$	$4.3821 \cdot 10^{-4}$	$1.1390 \cdot 10^{-4}$	$1.6564 \cdot 10^{-4}$	$-1.1168 \cdot 10^{-8}$	$-3.4462 \cdot 10^{-6}$
1	$3.2738 \cdot 10^{-1}$	$4.3906 \cdot 10^{-4}$	$1.1435 \cdot 10^{-4}$	$1.6557 \cdot 10^{-4}$	$-1.1168 \cdot 10^{-8}$	$-3.4484 \cdot 10^{-6}$
2	$3.2738 \cdot 10^{-1}$	$4.3771 \cdot 10^{-4}$	$1.1372 \cdot 10^{-4}$	$1.6564 \cdot 10^{-4}$	$-1.1168 \cdot 10^{-8}$	$-3.4444 \cdot 10^{-6}$
3	$3.2738 \cdot 10^{-1}$	$4.4009 \cdot 10^{-4}$	$1.1430 \cdot 10^{-4}$	$1.6560 \cdot 10^{-4}$	$-1.1167 \cdot 10^{-8}$	$-3.4530 \cdot 10^{-6}$
4	$3.2738 \cdot 10^{-1}$	$4.3846 \cdot 10^{-4}$	$1.1426 \cdot 10^{-4}$	$1.6563 \cdot 10^{-4}$	$-1.1194 \cdot 10^{-8}$	$-3.4473 \cdot 10^{-6}$
5	$3.2738 \cdot 10^{-1}$	$4.3983 \cdot 10^{-4}$	$1.1420 \cdot 10^{-4}$	$1.6558 \cdot 10^{-4}$	$-1.1190 \cdot 10^{-8}$	$-3.4512 \cdot 10^{-6}$
6	$3.2738 \cdot 10^{-1}$	$4.3974 \cdot 10^{-4}$	$1.1468 \cdot 10^{-4}$	$1.6557 \cdot 10^{-4}$	$-1.1237 \cdot 10^{-8}$	$-3.4513 \cdot 10^{-6}$
7	$3.2738 \cdot 10^{-1}$	$4.4121 \cdot 10^{-4}$	$1.1466 \cdot 10^{-4}$	$1.6557 \cdot 10^{-4}$	$-1.1214 \cdot 10^{-8}$	$-3.4567 \cdot 10^{-6}$
8	$3.2738 \cdot 10^{-1}$	$4.3884 \cdot 10^{-4}$	$1.1458 \cdot 10^{-4}$	$1.6558 \cdot 10^{-4}$	$-1.1218 \cdot 10^{-8}$	$-3.4479 \cdot 10^{-6}$
9	$3.2739 \cdot 10^{-1}$	$4.3885 \cdot 10^{-4}$	$1.1389 \cdot 10^{-4}$	$1.6563 \cdot 10^{-4}$	$-1.1160 \cdot 10^{-8}$	$-3.4484 \cdot 10^{-6}$
10	$3.2738 \cdot 10^{-1}$	$4.3975 \cdot 10^{-4}$	$1.1407 \cdot 10^{-4}$	$1.6558 \cdot 10^{-4}$	$-1.1171 \cdot 10^{-8}$	$-3.4514 \cdot 10^{-6}$
11	$3.2739 \cdot 10^{-1}$	$4.3591 \cdot 10^{-4}$	$1.1340 \cdot 10^{-4}$	$1.6573 \cdot 10^{-4}$	$-1.1087 \cdot 10^{-8}$	$-3.4396 \cdot 10^{-6}$
12	$3.2738 \cdot 10^{-1}$	$4.3939 \cdot 10^{-4}$	$1.1472 \cdot 10^{-4}$	$1.6559 \cdot 10^{-4}$	$-1.1209 \cdot 10^{-8}$	$-3.4501 \cdot 10^{-6}$
13	$3.2738 \cdot 10^{-1}$	$4.3859 \cdot 10^{-4}$	$1.1418 \cdot 10^{-4}$	$1.6559 \cdot 10^{-4}$	$-1.1221 \cdot 10^{-8}$	$-3.4471 \cdot 10^{-6}$
14	$3.2738 \cdot 10^{-1}$	$4.4071 \cdot 10^{-4}$	$1.1436 \cdot 10^{-4}$	$1.6558 \cdot 10^{-4}$	$-1.1225 \cdot 10^{-8}$	$-3.4546 \cdot 10^{-6}$
15	$3.2738 \cdot 10^{-1}$	$4.4046 \cdot 10^{-4}$	$1.1507 \cdot 10^{-4}$	$1.6554 \cdot 10^{-4}$	$-1.1266 \cdot 10^{-8}$	$-3.4530 \cdot 10^{-6}$
16	$3.2738 \cdot 10^{-1}$	$4.3829 \cdot 10^{-4}$	$1.1389 \cdot 10^{-4}$	$1.6562 \cdot 10^{-4}$	$-1.1194 \cdot 10^{-8}$	$-3.4465 \cdot 10^{-6}$
17	$3.2738 \cdot 10^{-1}$	$4.3986 \cdot 10^{-4}$	$1.1434 \cdot 10^{-4}$	$1.6554 \cdot 10^{-4}$	$-1.1248 \cdot 10^{-8}$	$-3.4506 \cdot 10^{-6}$
(%)	$\Delta\phi(0) = 0.001$	$\Delta\phi(1) = 0.275$	$\Delta\phi(2) = 0.345$	$\Delta\phi(3) = 0.026$	$\Delta\phi(4) = 0.357$	$\Delta\phi(5) = 0.113$

Table 6 $\phi_i(n)$ and $\Delta\phi(n)$ for weighted natural image.

i	$\phi_i(0)$	$\phi_i(1)$	$\phi_i(2)$	$\phi_i(3)$	$\phi_i(4)$	$\phi_i(5)$
0	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9479 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
1	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
2	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8413 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
3	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
4	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6993 \cdot 10^{-6}$
5	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1374 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
6	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
7	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
8	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1374 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
9	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6993 \cdot 10^{-6}$
10	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
11	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
12	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
13	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
14	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6993 \cdot 10^{-6}$
15	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6993 \cdot 10^{-6}$
16	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8413 \cdot 10^{-4}$	$7.1374 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6994 \cdot 10^{-6}$
17	$4.9589 \cdot 10^{-1}$	$1.1176 \cdot 10^{-3}$	$5.8414 \cdot 10^{-4}$	$7.1375 \cdot 10^{-4}$	$-3.9478 \cdot 10^{-7}$	$-6.6993 \cdot 10^{-6}$
(%)	$\Delta\phi(0) = 0.0000$	$\Delta\phi(1) = 0.0002$	$\Delta\phi(2) = 0.0003$	$\Delta\phi(3) = 0.0002$	$\Delta\phi(4) = 0.0003$	$\Delta\phi(5) = 0.0004$

(a) (b)

Fig. 3 Unweighted and weighted natural images: (a) g, (b) wg.

computed by Eqs. (9) and (10) with $R = 18$. We see that a similar result is obtained in this case.

5. Conclusion

We have shown that invariancy of moment invariants of a discrete image can be improved by applying the weighting function such that the pixel value is smoothly reduced to zero at the boundary of the image. The method's effectiveness has been shown through numerical simulations using several images.

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