

## LETTER

Mixed  $l_0/l_1$  Norm Minimization Approach to Image Colorization

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**SUMMARY** This letter proposes a new image colorization algorithm based on the sparse optimization. Introducing some assumptions, a problem of recovering a color image from a grayscale image with the small number of known color pixels is formulated as a mixed  $l_0/l_1$  norm minimization, and an iterative reweighted least squares (IRLS) algorithm is proposed. Numerical examples show that the proposed algorithm colorizes the grayscale image efficiently.

**key words:** image colorization, image restoration, sparse optimization, matrix completion

## 1. Introduction

This letter deals with digital image colorization, which is used to recover a color image from a grayscale image. Various methods have been proposed to achieve the colorization.

In [1], [2], colorization algorithms are proposed based on the texture recognition approach, where a color image is estimated using known full color images that are similar to the grayscale image to be colorized. The advantage of these algorithms is that a color image can be obtained from its grayscale image without any advance information about color, while most colorization algorithms require color data of some pixels in the grayscale image. However the performance heavily depends on known full color images and on the quality of texture recognition.

In [3], [4], colorization algorithms are provided based on the numerical optimization. These algorithms colorize a grayscale image with color data given in small regions. Because a colorization problem is ill-posed, they assume that grayscale images are converted from color images using a linear transformation and that the total variation (TV) norm of color images is small. Then the image colorization problem is formulated as the TV norm minimization problem. Various TV norm minimization techniques are proposed for image processing tasks such as denoising [5] and image inpainting [6] and inpainting colors [7]. Although the TV norm based approach can colorize an image efficiently if there are enough given color regions, it cannot recover the color of pixels far from the given color region.

This letter proposes a colorization algorithm based on

the sparse optimization, which is a numerical optimization to find a sparse vector. Candès et al. proposes the reweighted  $l_1$  minimization to obtain a sparse vector and apply to the image recover problem [8]. Motivated by their work we formulate an image colorization problem as the mixed  $l_0/l_1$  norm minimization instead of the TV norm minimization, and apply the iteratively reweighted least squares (IRLS) [9] to recover a color image. The proposed algorithm can recover the color image with small given color regions or the small number of given color pixels. We demonstrate some examples to show that the proposed colorization algorithm has a good performance.

## 2. Problem Formulation and $l_p$ Norm Based Colorization

This letter deals with the colorization problem where a color image is recovered from a grayscale image with color data given in small regions similarly with [3], [4]. Let  $I \in \mathbf{R}^{M \times N}$ ,  $I^R \in \mathbf{R}^{M \times N}$ ,  $I^G \in \mathbf{R}^{M \times N}$  and  $I^B \in \mathbf{R}^{M \times N}$  denote a grayscale intensity image and its values of red, green and blue, respectively. We assume here that a grayscale image is converted from a color image by forming a weighted sum of the values of red, green and blue as follows,

$$I = a_r I^R + a_g I^G + a_b I^B, \quad (1)$$

where  $a_r$ ,  $a_g$  and  $a_b$  are constants. Then the colorization problem considered in this letter is formulated as the following matrix completion problem,

$$\begin{aligned} \text{find} \quad & X = [X^R \ X^G \ X^B] \in \mathbf{R}^{M \times 3N} \\ \text{subject to} \quad & X^R \in \mathbf{R}^{M \times N}, \ X^G \in \mathbf{R}^{M \times N}, \ X^B \in \mathbf{R}^{M \times N}, \\ & I = a_r X^R + a_g X^G + a_b X^B, \\ & [X^R_{i,j} \ X^G_{i,j} \ X^B_{i,j}] = [I^R_{i,j} \ I^G_{i,j} \ I^B_{i,j}], \ \forall (i, j) \in I, \end{aligned} \quad (2)$$

where  $I$  denotes a given set of matrix indices, which correspond to known color pixels,  $M_{i,j}$  denotes the  $(i, j)$ -element of the matrix  $M$ , and  $X = [X^R \ X^G \ X^B]$  is a design variable. This problem is obviously ill-posed, and therefore we usually provide the additional assumption that each color value changes smoothly if the grayscale intensity value changes smoothly, that is, the neighborhood pixels have the same color when they have equal grayscale intensity values. To achieve this assumption, [4] proposes the total variation (TV) minimizing colorization, where the total variation

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norm of  $[X^R X^G X^B]$  is minimized. Several methods can be proposed based on other norms to achieve the assumption.

Now we formulate the general  $l_p$  norm minimization problem for the image colorization. Let us define  $\mathbf{x}_R \in \mathbf{R}^{MN}$ ,  $\mathbf{x}_G \in \mathbf{R}^{MN}$ ,  $\mathbf{x}_B \in \mathbf{R}^{MN}$ ,  $\mathbf{v}_R \in \mathbf{R}^{MN}$ ,  $\mathbf{v}_G \in \mathbf{R}^{MN}$ ,  $\mathbf{v}_B \in \mathbf{R}^{MN}$ ,  $\mathbf{v}_I \in \mathbf{R}^{MN}$ ,  $\mathbf{x} \in \mathbf{R}^{3MN}$  and  $\mathbf{v} \in \mathbf{R}^{3MN}$  as

$$\begin{aligned} \mathbf{x}_R &= \text{vec}(X^R), \mathbf{x}_G = \text{vec}(X^G), \mathbf{x}_B = \text{vec}(X^B), \\ \mathbf{v}_R &= \text{vec}(I^R), \mathbf{v}_G = \text{vec}(I^G), \mathbf{v}_B = \text{vec}(I^B), \\ \mathbf{v}_I &= \text{vec}(I), \mathbf{x} = [\mathbf{x}_R^T \mathbf{x}_G^T \mathbf{x}_B^T]^T \text{ and } \mathbf{v} = [\mathbf{v}_R^T \mathbf{v}_G^T \mathbf{v}_B^T]^T, \end{aligned}$$

respectively, where  $\text{vec}$  denotes the function which converts a matrix to a vector by stacking the matrix columns successively. And define  $U \in \mathbf{R}^{(M-1) \times M}$ ,  $V \in \mathbf{R}^{M(N-1) \times MN}$ ,  $\bar{U} \in \mathbf{R}^{3N(M-1) \times 3MN}$ ,  $\bar{V} \in \mathbf{R}^{3M(N-1) \times 3MN}$ ,  $D \in \mathbf{R}^{(6MN-3M-3N) \times 3MN}$  and  $C \in \mathbf{R}^{MN \times 3MN}$  as

$$U_{i,j} = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{if } i + 1 = j \\ 0, & \text{otherwise} \end{cases}, V_{i,j} = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{if } i + M = j \\ 0, & \text{otherwise} \end{cases}$$

$\bar{U} = \text{diag}(U, \dots, U)$ ,  $\bar{V} = \text{diag}(V, V, V)$ ,  $D = [\bar{U}^T \bar{V}^T]^T$  and

$$C = \begin{cases} a_r, & \text{if } i = j \\ a_g, & \text{if } i + MN = j \\ a_b, & \text{if } i + 2MN = j \\ 0, & \text{otherwise} \end{cases}$$

respectively, where  $\text{diag}(A_1, \dots, A_m)$  denotes a block diagonal matrix consisting of  $A_1, \dots, A_m$ . The vectors  $\mathbf{v}$  and  $\mathbf{x}$  correspond to  $[I^R I^G I^B]$  and  $X = [X^R X^G X^B]$ , respectively, and the matrices  $\bar{U}$  and  $\bar{V}$  denote vertical and horizontal difference operators. Then  $D\mathbf{x}$  denotes the differences between the neighbor pixels of a whole image, and the vector norm minimizing colorization is generalized as the following  $l_p$  norm minimization problem,

$$\begin{aligned} \text{Minimize } & \|D\mathbf{x}\|_p \\ \text{subject to } & \mathbf{v}_I = C\mathbf{x}, x_i = v_i, \forall i \in \bar{I}, \end{aligned} \quad (3)$$

where  $\|\cdot\|_p$  denotes the  $l_p$  norm of a vector,  $x_i$  and  $v_i$  denote the  $i$ th element of  $\mathbf{x}$  and  $\mathbf{v}$ , respectively, and  $\bar{I}$  denotes a given set of vector indices corresponding to  $\bar{I}$ . This formulation describes the TV minimization problem in the case of  $p = 1$ .

### 3. Main Results

#### 3.1 Mixed $l_0/l_1$ Norm Minimization Approach

Because the vector norm minimization approach minimizes the difference between neighbor pixels, it can correctly colorize the pixels close to the pixels given with color. However the further pixels from the given color pixels are recovered with larger errors, sometimes remain grayscale, and the colorization may be achieved incorrectly. In order to achieve the correct color recovery, this letter proposes a sparsification approach. We assume here that the neighborhood pixels

which have the nearly equal grayscale intensity value have the nearly same color, that is, the RGB values changes in proportion to the changes of the grayscale intensity value, and describe this assumption by the following equations in 2-D form,

$$\frac{X_{i,j}^R}{I_{i,j}} = \frac{X_{k,l}^R}{I_{k,l}}, \frac{X_{i,j}^G}{I_{i,j}} = \frac{X_{k,l}^G}{I_{k,l}}, \frac{X_{i,j}^B}{I_{i,j}} = \frac{X_{k,l}^B}{I_{k,l}}, \quad (4)$$

for  $(k, l) \in \{(i, j + 1), (i + 1, j)\}$ . Because  $[X^R X^G X^B]$  have to satisfy the color constraint derived from (1), there is no solution exactly satisfying (4) for all  $(i, j)$ . Therefore we consider the problem of maximizing the number of  $X_{i,j}^R$ ,  $X_{i,j}^G$  and  $X_{i,j}^B$  satisfying (4) for  $(i, j) \in \mathcal{J}$ , where  $\mathcal{J}$  is defined by

$$\mathcal{J} = \{(i, j) : |I_{i,j} - I_{i,j+1}| \leq \nu \text{ or } |I_{i,j} - I_{i+1,j}| \leq \nu\},$$

and  $\nu$  is a given constant. This problem is formulated as the following sparse optimization,

$$\begin{aligned} \text{Minimize } & \|JDG\mathbf{x}\|_0 \\ \text{subject to } & \mathbf{v}_I = C\mathbf{x}, x_i = v_i, \forall i \in \bar{I}, \end{aligned} \quad (5)$$

where  $G$  and  $J$  are given diagonal matrices defined by

$$G = \text{diag}(\bar{G}, \bar{G}, \bar{G}), \quad \bar{G}_{i,i} = \begin{cases} 1/v_{I_i}, & \text{if } v_{I_i} \neq 0 \\ 0, & \text{if } v_{I_i} = 0 \end{cases},$$

$$\text{and } J_{i,i} = \begin{cases} 1, & \text{if } i \in \bar{\mathcal{J}} \\ 0, & \text{otherwise} \end{cases},$$

respectively,  $v_{I_i}$  denotes the  $i$ th element of  $\mathbf{v}_I$ , and  $\bar{\mathcal{J}}$  denotes the set of indices of  $\mathbf{v}$  corresponding to  $\mathcal{J}$  of  $I_{i,j}$ . In (5),  $\|\cdot\|_0$  denotes the  $l_0$  norm of a vector, that is, the number of nonzero entries of the vector. Since the number of zero entries of  $JDG\mathbf{x}$  is equal to that of pixels satisfying (4), we obtain the maximum number of color pixels satisfying (4) by solving this problem. Because the pixels not in  $\mathcal{J}$  with a proper value of  $\nu$  correspond to the edge of an object in the image, this problem colorizes the pixels in the same region surrounded by its edge with the same color according to (4). Since the pixels whose gray scale intensity values equal 0 should be recovered as black pixels, the corresponding elements of  $G$  for such pixels equal 0, and hence  $\bar{G}_{i,i} = 0$  for  $v_{I_i} = 0$ .

Though (5) colorizes the pixels in  $\mathcal{J}$ , it never recovers color of the pixels not in  $\mathcal{J}$ . Therefore this letter proposes the colorization algorithm utilizing  $l_0$  norm minimization (5) for  $\mathbf{x} \in \bar{\mathcal{J}}$  and  $l_1$  norm (TV norm) minimization (3) with  $p = 1$  for  $\mathbf{x} \notin \bar{\mathcal{J}}$ . Let  $E$  denote an identity matrix with a certain size, then  $(E - J)\mathbf{x}$  denotes the pixels not in  $\bar{\mathcal{J}}$ , and we obtain the following mixed  $l_0/l_1$  norm minimizing colorization problem,

$$\begin{aligned} \text{Minimize } & \mu_0 \|JDG\mathbf{x}\|_0 + \mu_1 \|(E - J)D\mathbf{x}\|_1 \\ \text{subject to } & \mathbf{v}_I = C\mathbf{x}, x_i = v_i, \forall i \in \bar{I}, \end{aligned} \quad (6)$$

where  $\mu_0 \geq 0$  and  $\mu_1 \geq 0$  are given constants. The term of  $l_1$  norm in the objective function forces to colorize the pixels

not in  $\mathcal{J}$ , and then this problem colorizes a whole image such that the pixels in the same region have the same color.

The problem (6) is well-posed. However, since the constraints are strict, the  $l_0$  norm may not be minimized sufficiently, that is, only a few pixels satisfying (4) are obtained. In order to increase the number of pixels satisfying (4), we relax the problem using the Lagrangian relaxation and propose the following problem,

$$\begin{aligned} \text{Minimize} \quad & \mu_0 \|JDG\mathbf{x}\|_0 + \mu_1 \|(E - J)D\mathbf{x}\|_1 \\ & + \lambda_1 \|\mathbf{v}_I - C\mathbf{x}\|_2^2 + \lambda_2 \|F(\mathbf{v} - \mathbf{x})\|_2^2, \end{aligned} \quad (7)$$

where  $\lambda_1 > 0$  and  $\lambda_2 > 0$  are given constants, and  $F$  is a diagonal matrix whose diagonal matrix is defined by

$$F_{i,i} = \begin{cases} 1 & \text{if } i \in \bar{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}.$$

### 3.2 Iterative Reweighted Least Squares Algorithm

Since the problem (7) is non-convex and difficult to solve exactly, this letter applies the iteratively reweighted least squares (IRLS) [8], [9]. The IRLS provides an approximate solution of the  $l_p$  norm minimization of  $\mathbf{z} = [z_1 \ z_2 \ \dots \ z_n]^T$  by solving the following least square problem iteratively,

$$\mathbf{z}^{(k+1)} = \arg \min_{\mathbf{z}} \|W^{(k)}\mathbf{z}\|_2^2,$$

where  $\mathbf{z}^{(k+1)}$  denotes the solution at the  $(k+1)$ th iteration,  $W^{(k)}$  is the diagonal matrix of weights whose diagonal elements are defined by  $W_{i,i}^{(k)} = (|z_i^{(k)}|^{1-p/2} + \varepsilon)^{-1}$ , and  $\varepsilon > 0$  is a small constant.

Applying the IRLS to (7), the solution at the  $(k+1)$ th iteration is obtained as

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \arg \min_{\mathbf{x} \in \mathbf{R}^{3MN}} \mu_0 \|W_0^{(k)} JDG\mathbf{x}\|_2^2 + \mu_1 \|W_1^{(k)} (E - J)D\mathbf{x}\|_2^2 \\ &+ \lambda_1 \|\mathbf{v}_I - C\mathbf{x}\|_2^2 + \lambda_2 \|F(\mathbf{v} - \mathbf{x})\|_2^2. \end{aligned} \quad (8)$$

In the above equation,  $W_0^{(k)}$  and  $W_1^{(k)}$  are diagonal matrices whose diagonal elements are calculated as

$$W_{0,i,i}^{(k)} = (|p_i| + \varepsilon)^{-1}, \quad \text{and} \quad W_{1,i,i}^{(k)} = (\sqrt{q_i} + \varepsilon)^{-1}, \quad (9)$$

respectively, where  $p_i$  and  $q_i$  are the  $i$ th element of the vectors  $JDG\mathbf{x}^{(k)}$  and  $(E - J)D\mathbf{x}^{(k)}$ , respectively. The least squares problem (8) can be solved simply as

$$\mathbf{x}^{(k+1)} = \begin{bmatrix} \mu_0 W_0^{(k)} JDG \\ \mu_1 W_1^{(k)} (E - J)D \\ \lambda_1 C \\ \lambda_2 F \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \lambda_1 \mathbf{v}_I \\ \lambda_2 F\mathbf{v} \end{bmatrix}, \quad (10)$$

where  $\mathbf{0}$  denotes the zero vector of size  $6MN - 3M - 3N$ , and  $A^\dagger$  denotes the pseudoinverse of a matrix  $A$ . Finally this letter proposes the IRLS algorithm for the image colorization based on the mixed  $l_0/l_1$  norm minimizing problem as

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### Algorithm 1 IRLS algorithm for image colorization

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**Require:**  $\mathbf{v}_I, \mathbf{v}_I$  for  $i \in \bar{\mathcal{I}}, \mu_0, \mu_1, \lambda_1, \lambda_2$  and  $\varepsilon$

set  $k = 0$

set  $\mathbf{x}^{(0)} = [\mathbf{v}_I^T \ \mathbf{v}_I^T \ \mathbf{v}_I^T]^T$

set  $W_0^{(0)}$  and  $W_1^{(0)}$  as the identity matrices

**repeat**

calculate  $\mathbf{x}^{(k+1)}$  using (10)

update  $W_0^{(k+1)}$  and  $W_1^{(k+1)}$  using (9)

$k \leftarrow k + 1$

**until** termination criterion is satisfied

**Ensure:** color pixels  $\mathbf{x}^{(k+1)}$

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shown in Algorithm 1. The initial value of  $\mathbf{x}^{(k)}$  is  $[\mathbf{v}_I^T \ \mathbf{v}_I^T \ \mathbf{v}_I^T]^T$ , and  $\mathbf{x}^{(k)}$  is updated using (9)–(10).

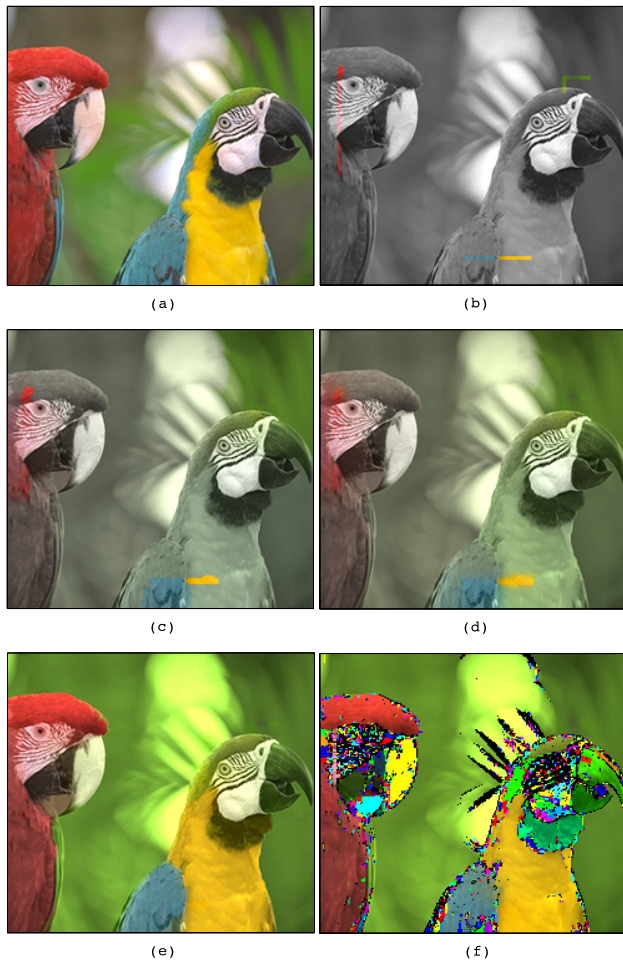
This algorithm is not guaranteed to converge, and hence we fix the number of iteration in the numerical experiments of the next section.

### 4. Numerical Examples

This section shows two kinds of examinations to demonstrate the effectiveness of the proposed algorithm. In all examples we use  $\varepsilon = 10^{-4}$ , and other parameters are chosen to give the best colorization. We utilize the termination criterion  $k = 3$ , that is, the number of iterations is 3, because almost the same results are given by more iterations. The values of the constants (1) are adopted as  $[a_r \ a_g \ a_b] = [0.29891 \ 0.58661 \ 0.11448]$ , which is usually used in the color conversion from RGB to YCbCr according to ITU-R BT.601. The grayscale intensity is represented by an integer value in the range from 0 to 255.

First we apply the algorithm to a grayscale image with some known color regions in order to investigate the effect of parameters  $\mu_0$  and  $\mu_1$ . We utilize  $\nu = 6$  and  $\lambda_1 = \lambda_2 = 100$  and examine parameters  $(\mu_0, \mu_1) = (0.1, 0.9), (0.5, 0.5), (0.9999, 0.0001)$  and  $(1, 0)$ . Figure 1 shows the results. In the case of low rate of  $\mu_0/\mu_1$ , only the pixels near the known color regions are colorized, and the pixels far from the color regions are not colorized. In the case of  $\mu_1 = 0$ , the pixels in the same region of known color pixels are correctly colorized, however, other pixels are provided with false color. Since the elements of  $J$  corresponding to these pixels are 0s, the problem of recovering them in (6) with  $\mu_1 = 0$  is ill-posed. Although they should remain grayscale, they are provided with false color due to numerical error in computing the pseudoinverse in the IRLS scheme (10). In the case of the high rate of  $\mu_0/\mu_1$  where  $\mu_1 \neq 0$ , we can see that the algorithm colorizes all pixels of the image sufficiently. Since  $\mathbf{v}_I \in [0, 255]$ , each element of the diagonal matrix  $G$  takes a small number. The value of  $l_0$  norm in (8) is affected by the squared values of the elements of  $G$ , and this causes the best value of  $\mu_0$  to be relatively greater than that of  $\mu_1$ .

Next we apply the IRLS algorithm to a grayscale image with selected 5 or 20 color pixels. The parameters are  $\mu_0 = 0.9999, \mu_1 = 0.0001, \lambda_1 = 100, \nu = 6$  and  $\lambda_2 = 100$ . The result is shown in Fig. 2. We can see that the proposed algorithm can colorize the grayscale images efficiently using



**Fig. 1** (a) Original image ( $247 \times 225$  pixels). (b) Given image. (c) Colorization result with  $\mu_0 = 0.1$  and  $\mu_1 = 0.9$ . (d) Colorization result with  $\mu_0 = 0.5$  and  $\mu_1 = 0.5$ . (e) Colorization result with  $\mu_0 = 0.9999$  and  $\mu_1 = 0.0001$ . (f) Colorization result with  $\mu_0 = 1$  and  $\mu_1 = 0$ .

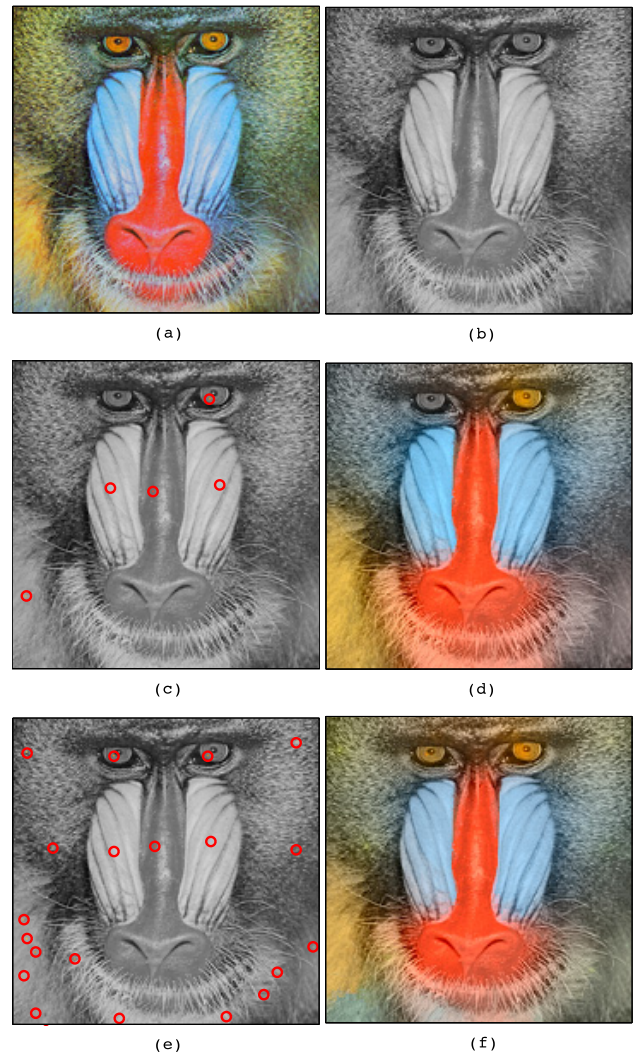
only 20 color pixels of  $180 \times 180 = 32400$  pixels.

## 5. Conclusion

This letter proposes the mixed  $l_0/l_1$  norm minimization based algorithm for image colorization. A image colorization problem is formulated as the matrix completion problem and described as the mixed  $l_0/l_1$  norm minimization problem by adding some natural assumptions. The IRLS algorithm is proposed to obtain an optimal solution approximately. Numerical examples show that the proposed algorithm can colorize all pixels of the image effectively.

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**Fig. 2** (a) Original image ( $180 \times 180$  pixels). (b) Grayscale image. (c) Given image with 5 color pixels and (d) its colorization result. (e) Given image with 20 color pixels and (f) its colorization result. Color pixels in the center of red circles are sampled from the original image.

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