

## LETTER

## Smaller Bound of Superconcentrator

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**SUMMARY** A Superconcentrator is a directed acyclic graph with specific properties. The existence of linear-sized superconcentrator has been proved in [4]. Since then, the size has been decreased significantly. The best known size is  $28N$  which is proved by U. Schöning in [8]. Our work follows their construction and proves a smaller size superconcentrator.

**key words:** expander, superconcentrator, theoretical computer science

## 1. Introduction

A superconcentrator is a class of expanders to meet some constraints. An  $N$ -superconcentrator  $T_N$  is a directed acyclic graph with  $N$  input nodes  $X$  and  $N$  output nodes  $Y$ . For any subset  $S$  of  $X$  and  $T$  of  $Y$  where  $|S| = |T|$ , there are  $|S|$  vertex disjoint paths from  $S$  to  $T$ . Superconcentrator supports many applications in computer science. For instance, a communication network can be viewed as a superconcentrator such that every group contacts others through nonintersecting paths.

It has been acknowledged that there exists linear size superconcentrator where the number of edge in superconcentrator grows linearly to its node number. Gaber and Galil [4] first present a superconcentrator with  $O(N)$  edges. Following their work, the size bound became smaller and smaller. It has been successfully improved to  $39N$  in [6],  $38.5N$  in [3],  $36N$  in [2],  $34.2N$  in [7], and  $33N$ . To our best knowledge, the smallest bound it has ever been proved is attributed to [8]  $28N$ . In this paper, we successfully decrease it by  $0.5864N$ . Our method is non-constructive. We modify the construction in [1] at the cost of some additional directed edges. However, we succeed in reducing the size of recursive  $N/2$ -superconcentrator to a number little smaller than it.

## 2. Construction

Here we construct an infinite family of superconcentrator with  $27.4136N$  edges. Let  $T_n$  be the  $n$ -superconcentrator.

2.1 The Construction of  $T_n$ 

First, we define some parameters which will be determined

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later. Let  $r$  and  $\lambda$  be in  $[0, 1]$ . Let  $X$  and  $Y$  be the input set and output set of  $T_n$ . We denote  $X$  and  $Y$  by  $\{x_1, \dots, x_n\}$  and  $\{y_1, \dots, y_n\}$  respectively. Similarly we define  $X'$  and  $Y'$ . However, the size of  $X'$  and  $Y'$  is little smaller than  $n$ . Let  $X' = \{x'_1, \dots, x'_{\lambda n}\}$  and  $Y' = \{y'_1, \dots, y'_{\lambda n}\}$ .

The edges directed from  $X$  to  $X'$  form a bipartite graph  $G = (n, d, \lambda n)$  with  $(\alpha, \beta)$  expanding. Similarly we do the same operation on edges between  $Y$  and  $Y'$ . Arcs  $(x_i, y_i)$  are in  $T_n$  where  $i \in \{1, \dots, rn\}$ . In addition, for each  $i \in \{1, \dots, \lambda n/2\}$ , the arcs  $(x'_i, x'_{i+\lambda n/2})$ ,  $(y'_{i+\lambda n/2}, y'_i)$ ,  $(x'_{i+\lambda n/2}, y'_i)$  and  $(x'_i, y'_{i+\lambda n/2})$  are in  $T_n$ . Finally, we recursively construct  $T_{\lambda n/2}$  between node set  $X'' = \{x'_1, \dots, x'_{\lambda n/2}\}$  and  $Y'' = \{y'_1, \dots, y'_{\lambda n/2}\}$ . Figure 1 illustrates such construction.

Note that the directed edges from  $X$  to  $Y$  reduce the number of vertex-disjoint paths passing through  $X'$  and  $Y'$ . One of the merit is to narrow the size of  $X''$  so as to decrease the size of  $T_n$ . However, its side effect is the increasing edges between  $X$  and  $Y$ . We will determine the parameter  $\lambda$  and  $r$  later.

## 2.2 Existence of Bipartite Graph

**Definition 2.1:** An  $(n, d, \lambda n)$  graph is a bipartite graph with  $n$  vertices on left side and  $\lambda n$  vertices on right side. Each vertex on left side has degree  $d$  and each vertex on right side has degree  $d/\lambda$ . We define  $N_G(S)$  the neighbor vertex of vertex set  $S$  in graph  $G$ .

An  $(n, d, \lambda n)$  graph is an expander  $(\alpha, \beta)$  if every subset on the left side of size at most  $\alpha n$  has more than  $\lambda \beta n$  neighbors on the right side. The existence of such graph  $G$  is confirmed by following inequality in [7].

$$d > \frac{h(\alpha) + h(\beta)\lambda}{h(\alpha) - \beta h\left(\frac{\alpha}{\beta}\right)}. \quad (1)$$

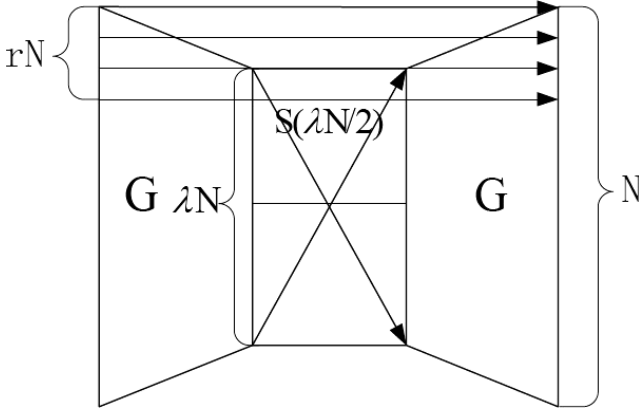
**Theorem 2.2:** For an integer  $n$ ,  $0 < \alpha \leq \beta < 1$ ,  $\lambda > 0$ , there exists an integer  $d$  such that  $G = (n, d, \lambda n)$  is an expander  $(\alpha, \beta)$ . For large  $n$ , the expander exists if inequality (1) holds.

Here  $h$  is an entropy function under  $\log_2$  base such that

$$h(a) = -a \log(a) - (1 - a) \log(1 - a). \quad (2)$$

Based on it, the following expander exists.

**Theorem 2.3:** Let  $G = (n, \lambda n, d)$  be the bipartite graph describe above. We require that graph  $G$  own following properties.  $X$  and  $Y$  are its parts where  $|X| = n$  and  $|Y| = \lambda n$ . Let  $S$  be a subset of  $X$ . If  $|S| = \alpha n$ , we have that:



**Fig. 1** It is a graph of  $n$ -superconcentrator.  $G$  is a bipartite graph with parameters  $(n, d, \lambda n)$ .  $S(\lambda n/2)$  is a  $\lambda n/2$ -superconcentrator constructed by recursion.

1. if  $\alpha < \lambda/4$ , then  $|N_G(S)| \geq 2\alpha n$ .
2. if  $\lambda/4 \leq \alpha \leq \lambda/2$ , then  $|N_G(S)| \geq \alpha n + \lambda n/4$ .
3. if  $\lambda/2 < \alpha \leq 1 - r/2$ , then  $|N_G(S)| \geq \alpha n + (\lambda - \alpha)n/2$ .

Note that  $\lambda \geq 1 - r/2$ . However, we should pick up  $\lambda$  strictly larger than  $1 - r/2$  so as to construct an expander with few edges. we claim that such graph  $G$  exists by Theorem 2.2. Here, we replace  $\beta$  with  $2\alpha/\lambda$ ,  $\alpha/\lambda + 1/4$  and  $(\lambda + \alpha)/2\lambda$  respectively. Then we get following inequalities

$$d > \frac{h(\alpha) + h\left(\frac{2\alpha}{\lambda}\right)\lambda}{h(\alpha) - h\left(\frac{2\alpha}{\lambda}\right)\frac{2\alpha}{\lambda}} \quad (\alpha \in (0, \lambda/4])$$

$$d > \frac{h(\alpha) + h\left(\frac{\alpha}{\lambda} + \frac{1}{4}\right)\lambda}{h(\alpha) - h\left(\frac{\alpha}{\lambda} + \frac{1}{4}\right)\frac{\alpha}{\lambda}} \quad (\alpha \in (\lambda/4, \lambda/2])$$

$$d > \frac{h(\alpha) + h\left(\frac{\alpha}{2\lambda} + \frac{1}{2}\right)\lambda}{h(\alpha) - h\left(\frac{2\alpha}{\lambda}\right)\left(\frac{\alpha}{2\lambda} + \frac{1}{2}\right)} \quad (\alpha \in (\lambda/2, 1 - r/2])$$

In our construction, we put  $\lambda = 0.9734$  and another parameter  $r = 0.1246$ . A simple calculation shows that  $d = 6$  can satisfy above three inequalities. Now we can compute the total size of  $T_n$  by following equation:

$$|T_n| = (2d + r + 2\lambda)n \left(1 - \frac{\lambda}{2}\right) + O(1) \quad (3)$$

We obtain our final result  $|T_n| = 27.4136n + O(1)$ . We leave the proof of our construction to next section.

**Theorem 2.4:**  $T_n$  is an  $n$ -superconcentrator with  $27.4136n + O(1)$  edges where  $X$  and  $Y$  are input set and output set.

### 3. Proof

The proof in this paper is of some similarity to proof in [1]. Specifically, we rely on lemma 3.2 which is similar to Lemma 4.3 in [1]. The proof technique is quite analogous. However, we use  $T_{\lambda n/2}$  as the recursive superconcentrator. The node number in two side of this expander  $(n, d, \lambda n)$  is

different. Therefore, not all the proof in [1] works. We must modify some discussions. All we do is to ensure that similar result is hold under new parameter.

#### 3.1 Main Theorem

As in [1], we use induction in our proof. That is, we have a  $\lambda n/2$ -superconcentrator. We want to show the existence of  $n$ -superconcentrator. First, we denote the graph between  $X(Y)$  and  $X'(Y')$  by  $\Lambda_X(\Lambda_Y)$  resp).  $S$  is any subset of  $X$ , and  $T$  is any subset of  $Y$  where  $|S| = |T|$ . We show that there exists  $|S|$  vertex disjoint paths in  $T_n$ . Our proof depends on the following theorem which is a modification of Lemma 4.1 in [1]. We denote  $(1 - r/2)$  by  $\gamma$  in the following discussion.

**Theorem 3.1:**  $S$  is any subset of  $X$ , and  $T$  is any subset of  $Y$  where  $|S| = |T| \leq \gamma n$ . There exist matchings  $M_S^* \subset \Lambda_X$  and  $M_T^* \subset \Lambda_Y$ , such that  $M_S^*$  and  $M_T^*$  contain  $|S|$  edges and satisfy following property:

1.  $M_S^*$  saturates  $S$  and  $M_T^*$  saturates  $T$ .
2. Let  $\iota$  be an arbitrary integer in  $\{1, \dots, \gamma n/2\}$ . If  $M_S^*$  covers  $x'_\iota$  and  $x'_{\iota+\lambda n/2}$ ,  $M_T^*$  covers at least one of  $y'_\iota$  and  $y'_{\iota+\lambda n/2}$  and vice versa.

Assume the correctness of theorem 3.1, we show that it will induce the correctness of our construction.  $S$  is any subset of  $X$  and  $T$  is any subset of  $Y$  where  $|S| = |T|$ . If  $|S| > \gamma n = (1 - r/2)n$ ,  $S$  and  $T$  will have collision in subscript set of  $\{1, \dots, rn\}$ . Since there are directed edge between  $x_i$  and  $y_i$  for  $i \leq rn$ , we can eliminate those pairs of  $x_i$  and  $y_i$  from  $S$  and  $T$ . The size of remaining set  $S$  is at most  $(1 - r/2)n$ , and so is  $T$ . Now we can apply theorem 3.1 to  $S$  and  $T$ .

For each pair  $(x'_\iota, x'_{\iota+\lambda n/2})$  where  $\iota \in \{1, \dots, \gamma n/2\}$ , we divide it into two cases:

1. If  $M_S^*$  covers both  $x'_\iota$  and  $x'_{\iota+\lambda n/2}$ , we have that  $M_T^*$  covers at least one of  $y'_\iota$  and  $y'_{\iota+\lambda n/2}$ . If  $y'_\iota$  is covered, we derive a directed path from  $x'_{\iota+\lambda n/2}$  to  $y'_\iota$ . Then we eliminate this two node from matching set  $M_S^*$  and  $M_T^*$ . Otherwise,  $y'_{\iota+\lambda n/2}$  must be covered, we drive a directed path  $(x'_{\iota+\lambda n/2} y'_\iota y'_{\iota+\lambda n/2})$  which connects  $x'_{\iota+\lambda n/2}$  and  $y'_{\iota+\lambda n/2}$ . Similarly, we remove both nodes.
2. If only  $x'_{\iota+\lambda n/2}$  is covered by  $M_S^*$ , we map it to  $x'_\iota$  by arcs  $(x'_{\iota+\lambda n/2}, x'_\iota)$ .

The remaining node in  $M_S^*$  belongs to node set  $\{x'_1, \dots, x'_{\gamma n/2}\}$ . (include mappings). We do same operation on set  $T$  according to matchings  $M_T^*$ . Therefore, the remaining node in  $M_T^*$  belongs to  $\{y'_1, \dots, y'_{\gamma n/2}\}$ . Since there is a  $\lambda n/2$ -superconcentrator ( $\lambda \geq \gamma$ ) between  $X''$  and  $Y''$ , and both remaining node set has equal size, we complete our proof.

#### 3.2 Proof of Theorem 3.1

The proof of theorem 3.1 depends on the following lemma which appears in [1] with some changes.

**Lemma 3.2:** Let  $S$  and  $T$  be the subset of  $X$  and  $Y$  respec-

tively where  $|S| = |T| \leq \gamma n$ . There exist matchings  $M_S$  and  $M_T$  and a subset  $I$  of  $\{1, \dots, \lambda n\}$  have following properties.

1. Each edge in  $M_S$  is incident with a vertex in  $S$ , and each edge in  $M_T$  is incident with a vertex in  $T$ .
2. Let  $X'_I$  denote the subset of  $X'$  of the form  $\{x'_i : i \in I\}$ .  $Y'_I$  denote the subset of  $Y'$  of the form  $\{y'_i : i \in I\}$ . Then  $M_S$  saturates  $X'_I$  and  $M_T$  saturates  $Y'_I$ .
3. Let  $\alpha$  be number between 0 and  $\gamma$  such that  $|S| = |T| = \alpha n$ . If  $\alpha \in [\lambda/4, \lambda/2]$ ,  $|I| \geq \alpha n - \lambda n/4$ . If  $\alpha$  is larger than  $\lambda/2$  and smaller than  $\gamma$ ,  $|I| \geq \alpha n - (\lambda - \alpha)n/2$ .

We have shown the existence of bipartite graph  $G$  which satisfies Theorem 2.3. Next we prove that such construction owns the properties of Lemma 3.2.

Let  $U' = \{u'_1, \dots, u'_{\lambda n}\}$  and  $V' = \{v'_1, \dots, v'_{\lambda n}\}$ .  $G'$  is the following graph on  $S \cup U' \cup V' \cup T$ .  $G'[S \cup U']$  is isomorphic to  $\Lambda_X[S \cup X']$  such that  $(x, u'_i)$  is an edge iff  $(x, x'_i)$  is an edge in  $T_n$ . Similarly we define for  $G'[V' \cup T]$ .  $G'[U' \cup V']$  is a matching such that  $(u'_i, v'_i)$  is an edge iff  $i = i'$ . By Menger's Theorem, the maximum size of  $I$  is the minimum size of vertex set  $C$  which separates  $S$  and  $T$ . Suppose  $|C \cap S| = \alpha n$ ,  $|C \cap U'| = \beta n$ ,  $|C \cap V'| = \gamma n$ ,  $|C \cap T| = \delta n$ .

1. If  $\alpha \in [\lambda/4, \lambda/2]$ , we assume that both  $a$  and  $d$  is no larger than  $\alpha + \lambda/4$  or  $|C|$  is enough big. Since  $C$  is a cut-set,  $2\alpha - a - d + \lambda/2 - \lambda$  is at most  $b + c$ . Therefore,  $|C| = a + b + c + d \geq 2(\alpha - \lambda/4)$ . (The detailed discussion is same as in [1].)
2. If  $\alpha \in (\lambda/2, 1 - r/2]$ , we assume that both  $a$  and  $d$  is no larger than  $\alpha - (\lambda - \alpha)/2$  or  $|C|$  is enough big. Since  $C$  is a cut-set,  $2\alpha - a - d + (\lambda - \alpha) - \lambda$  is at most  $b + c$ . Therefore,  $|C| = a + b + c + d \geq \alpha \geq (3\alpha - \lambda)/2$ . (The detailed discussion is same as in [1].)

Now we use Lemma 3.2 to complete the proof of Theorem 3.1. We prove that there exists matchings  $M_S^* \subseteq \Lambda_X$  and  $M_T^* \subseteq \Lambda_X$  satisfying following two properties:

1.  $M_S^*$  saturates  $S$  and  $M_T^*$  saturates  $T$ .
2. Let  $i$  be any arbitrary integer in  $\{1, \dots, \gamma n/2\}$  such that neither  $i$  nor  $i + \lambda n/2$  in  $I$ . Then  $M_S^*$  covers at most one of  $x'_i$  and  $x'_{i+\lambda n/2}$  and  $M_T^*$  covers at most one of  $y'_i$  and  $y'_{i+\lambda n/2}$ .

These two properties imply Theorem 3.1. Now let  $M_S$ ,  $M_T$  and  $I$  satisfy Lemma 3.2.  $X'_I$  and  $Y'_I$  are as in 2 of Lemma 3.2. We show the existence of such  $M_S^*$  and  $M_T^*$ . Let  $\tilde{\Lambda}_X$  be the graph formed from  $\Lambda_X[S \cup X']$  by identifying  $x'_i$  with  $x'_{i+\lambda n/2}$  iff neither  $x'_i$  nor  $x'_{i+\lambda n/2}$  is in  $I$ . Similarly, we define graph  $\tilde{\Lambda}_Y$ . To finish our proof, it needs to prove there exist matchings in  $M_S^*$  and  $M_T^*$  that saturate both  $S$  and  $X'_I$  simultaneously, and,  $T$  and  $Y'_I$  simultaneously.

A There are matchings in  $\tilde{\Lambda}_X$  and  $\tilde{\Lambda}_Y$  saturating  $S$  and  $T$  respectively. There also are matchings, possibly different, in  $\tilde{\Lambda}_X$  and  $\tilde{\Lambda}_Y$  saturating  $X'_I$  and  $Y'_I$ .

Since  $M_S$ ,  $M_T$  and  $I$  satisfy Lemma 3.2, there exist matchings in  $\tilde{\Lambda}_X$  and  $\tilde{\Lambda}_Y$  saturates  $X'_I$  and  $Y'_I$  respectively. All we need to do is to show the first part of A. By Hall's theorem,

it needs to show that for any subset  $S_0$  of  $S$  and  $T_0$  of  $T$ , we have  $|\mathcal{N}_{\tilde{\Lambda}_X}(S_0)| \geq |S_0|$  and  $|\mathcal{N}_{\tilde{\Lambda}_Y}(T_0)| \geq |T_0|$ . Consider the following inequality:

$$|\mathcal{N}_{\tilde{\Lambda}_X}(S_0)| \geq |\mathcal{N}_{\Lambda_X}(S_0) \cap X'_I| + |\mathcal{N}_{\Lambda_X}(S_0) \cap (X' - X'_I)|/2$$

We divide the proof into two cases.

Case 1:  $\alpha \leq \lambda/2$  where  $|S| = \alpha n$ . If  $S_0$  is size less than  $\lambda/4$ , by Theorem 2.3

$$\begin{aligned} |\mathcal{N}_{\tilde{\Lambda}_X}(S_0)| &\geq |\mathcal{N}_{\Lambda_X}(S_0)|/2 + |\mathcal{N}_{\Lambda_X}(S_0) \cap X'_I|/2 \\ &\geq (2|S_0|)/2 \\ &= |S_0| \end{aligned}$$

Then, We turn to the case that  $|S_0|$  is larger than  $\lambda/4$ . According to Theorem 2.3,  $\mathcal{N}_{\Lambda_X}(S_0)$  has size at least  $|S_0| + \lambda n/4$ . Moreover, by Lemma 3.2, since  $|I|$  is at least  $(\alpha - \lambda/4)n$ ,  $|\mathcal{N}_{\Lambda_X}(S_0) \cap X'_I| \geq |S_0| - \lambda n/4$ . Thus we have:

$$\begin{aligned} |\mathcal{N}_{\tilde{\Lambda}_X}(S_0)| &\geq |\mathcal{N}_{\Lambda_X}(S_0)|/2 + |\mathcal{N}_{\Lambda_X}(S_0) \cap X'_I|/2 \\ &\geq (|S_0| + \lambda n/4)/2 + (|S_0| - \lambda n/4)/2 \\ &= |S_0| \end{aligned}$$

Similarly, we come to the same conclusion for  $T_0$ .

Case 2:  $\alpha \in (\lambda/2, 1 - r/2]$  where  $|S| = \alpha n$ . Then we assume that  $|S_0| \geq (\lambda - \alpha)n/2$ . By Theorem 2.3,  $\mathcal{N}_{\Lambda_X}(S_0)$  has size at least  $|S_0| + (\lambda - \alpha)n/2$ . Moreover, by Lemma 3.2,  $|I|$  is at least  $\alpha n - (\lambda - \alpha)n/2$ . So  $|\mathcal{N}_{\Lambda_X}(S_0) \cap X'_I| \geq |S_0| - (\lambda - \alpha)n/2$ . Now we have

$$\begin{aligned} |\mathcal{N}_{\tilde{\Lambda}_X}(S_0)| &\geq |\mathcal{N}_{\Lambda_X}(S_0)|/2 + |\mathcal{N}_{\Lambda_X}(S_0) \cap X'_I|/2 \\ &\geq (|S_0| + (\lambda - \alpha)n/2)/2 \\ &\quad + (|S_0| - (\lambda - \alpha)n/2)/2 \\ &= |S_0| \end{aligned}$$

Similarly, we come to the same conclusion for  $T_0$ . We have completed the proof of A which implies the existence of  $M_S^*$  and  $M_T^*$  in Theorem 3.1.

#### 4. Conclusion

In this paper, we show the existence of superconcentrator of density 27.4136 by a non-constructive proof. The proof combines the ideas from [1], [3] and [7] to generate a smaller-sized superconcentrator. Since the lower bound is  $(5 - o(1))N$  [5], the gap is still large to be narrowed.

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