PAPER

On Minimum Feedback Vertex Sets in Bipartite Graphs and Degree-Constraint Graphs*

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SUMMARY We consider the minimum feedback vertex set problem for some bipartite graphs and degree-constrained graphs. We show that the problem is linear time solvable for bipartite permutation graphs and NP-hard for grid intersection graphs. We also show that the problem is solvable in $O(n^2 \log^6 n)$ time for n-vertex graphs with maximum degree at most three.

key words: *3-regular graph, bipartite permutation graph, feedback vertex set, grid intersection graph, nonseparating independent set*

1. Introduction

A vertex set $F \subseteq V(G)$ of a graph G is a *feedback vertex set* (FVS) if the subgraph of G induced by $V(G)\backslash F$ has no cycles. A *minimum feedback vertex set* (MFVS) is an FVS with minimum cardinality. The *minimum feedback vertex set problem* (MinFVS) is to find an MFVS in a given graph. It is known that MinFVS has many applications in various areas including integrated circuits and optical networks (see [2], [22], for example).

We first consider MinFVS for bipartite graphs (bigraphs). The following relationships between bigraph classes have been known [15] :

{Bipartite Permutation Graphs}
⊂ {Convex Graphs}
⊂ {2-directional Orthogonal Ray Graphs}
⊂ {Chordal Bipartite Graphs},

and

{2-directional Orthogonal Ray Graphs}
⊂ {Orthogonal Ray Graphs}
⊂ {Unit Grid Intersection Graphs}
⊂ {Grid Intersection Graphs}.

It is known that MinFVS is NP-hard for bigraphs [23], while it can be solved in $O(n^5)$ time for chordal bipartite graphs [10], in $O(n^2m)$ time for convex graphs [13], and in $O(nm)$ time for permutation graphs [12], where n and m are the number of vertices and edges of a graph, respectively. We show in Sect. 2 that MinFVS can be solved in $O(n + m)$ time for bipartite permutation graphs. We also show in Sect. 3 that MinFVS is NP-hard for grid intersection graphs.

We next consider MinFVS for degree-constrained graphs. It is known that MinFVS is NP-hard even for planar graphs with maximum degree at most 4 [16], while it can be solved in $O(n^4)$ time for graphs with maximum degree at most 3 [5], [20]. We show in Sect. 4 that MinFVS can be solved in $O(n^2 \log^6 n)$ time for graphs with maximum degree at most 3.

2. A Linear Time Algorithm for Bipartite Permutation Graphs

2.1 Bipartite Permutation Graphs

A graph $G = (V, E)$ with a vertex set $V = \{v_1, \ldots, v_n\}$ is a *permutation graph* if there exists a permutation π on $\{1, \ldots, n\}$ such that for all $i, j \in \{1, \ldots, n\}$, $(v_i, v_j) \in E$ if and only if $(i - j)(\pi(i) - \pi(j)) < 0$. A permutation graph G is a *bipartite permutation graph* (permutation bigraph) if it is bipartite.

A *strong ordering* of a bigraph G with a bipartition (X, Y) is a pair of total orderings $(x_1, \ldots, x_p)$ of X and $(y_1, \ldots, y_q)$ of Y such that for any $i, i', j, j' (1 \le i < i' \le p, 1 \le j < j' \le q)$, $(x_i, y_{j'}) \in E$ and $(x_{i'}, y_j) \in E$ imply $(x_i, y_j) \in E$ and $(x_{i'}, y_{j'}) \in E$. For a bigraph with a strong ordering, the vertices of the bigraph are said to be strongly ordered. It is shown in [18] that a bigraph G is a permutation bigraph if and only if G has a strong ordering, and a strong ordering of G can be obtained in $O(n + m)$ time.

It is also known that a strong ordering of a permutation bigraph G has the adjacency property: For every $x \in X[y \in Y]$, the vertices in $\Gamma_G(x)$ $[\Gamma_G(y)]$ are consecutive. Here $\Gamma_G(v)$ is the set of vertices adjacent to v in G. If no confusion arise, we will omit the index.

2.2 The Algorithm

Let $G = (V, E)$ be a connected permutation bigraph with a bipartition (X, Y) and a strong ordering $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_q)$. Define that

$$V^i_j = \{x_1, \ldots, x_i, y_1, \ldots, y_j\}$$

for $1 \le i \le p$, $1 \le j \le q$, and $G[V^i_j]$ is a subgraph of G induced by V^i_j.

For convenience, we use S_1+S_2, S_1-S_2, $S+x$ and $S-x$

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instead of $S_1 \cup S_2$, $S_1 \setminus S_2$, $S \cup \{x\}$ and $S \setminus \{x\}$, respectively. We also use $\max\{S_1, S_2, \ldots, S_k\}$ to denote S_i with maximum cardinality.

A *cycle-free set* (CFS) is the complement of an FVS in a graph. Our algorithm computes a maximum CFS (MCFS) instead of an MFVS.

Our algorithm applies a dynamic programming scheme and computes the following for each $(x_i, y_j) \in E$.

A^i_j: an MCFS of $G[V^i_j]$,
B^i_j: an MCFS of $G[V^i_j]$ including x_i and y_j,
C^i_j: an MCFS of $G[V^i_j]$ including x_i and y_j, and excluding the vertices in $\Gamma(x_i) - y_j$,
D^i_j: an MCFS of $G[V^i_j]$ including x_i and y_j, and excluding the vertices in $\Gamma(y_j) - x_i$.

Note that $A^0_0 = B^0_0 = C^0_0 = D^0_0 = \emptyset$, and A^p_q is an MCFS of G.

Let $l(i)$ and $r(i)$ be the smallest and largest index of a vertex in $\Gamma(x_i)$, respectively, and let $l(j)$ and $r(j)$ be the smallest and largest index of a vertex in $\Gamma(y_j)$, respectively. We use $\tilde{A}^i_j$ $(1 \le i \le p, 1 \le j \le q)$ defined as follows.

$$\tilde{A}^i_j = \begin{cases} A^i_j & \text{if } (x_i, y_j) \in E, \\ A^i_{r(i)} + \{y_{r(i)+1}, \ldots, y_j\} & \text{if } r(i) < j, \\ A^{r(j)}_j + \{x_{r(j)+1}, \ldots, x_i\} & \text{if } r(j) < i. \end{cases}$$

Note that $\tilde{A}^i_j$ is an MCFS of $G[V^i_j]$ even if $(x_i, y_j) \notin E$, since if $r(i) < j$ $[r(j) < i]$ then $y_{r(i)+1}, \ldots, y_j$ $[x_{r(j)+1}, \ldots, x_i]$ are isolated vertices in $G[V^i_j]$.

We can compute A^i_j, B^i_j, C^i_j, and D^i_j for all $(x_i, y_j) \in E$ in linear time by the following relationship among these data structures.

Lemma 1. *$A^i_j = \max\{B^i_j, \tilde{A}^{i_1}_j, \tilde{A}^i_{j_1}\}$, where $i_1 = i - 1$ and $j_1 = j - 1$.*

Proof. Consider the following four cases.
(1) If $x_i, y_j \in A^i_j$ then $A^i_j = B^i_j$.
(2) If $x_i \notin A^i_j$ and $y_j \in A^i_j$ then $A^i_j = \tilde{A}^{i_1}_j$.
(3) If $x_i \in A^i_j$ and $y_j \notin A^i_j$ then $A^i_j = \tilde{A}^i_{j_1}$.
(4) If $x_i, y_j \notin A^i_j$ then $A^i_j = \max\{\tilde{A}^{i_1}_j, \tilde{A}^i_{j_1}\}$. □

Lemma 2. *$B^i_j = \max\{C^i_j, D^i_j\}$.*

Proof. Let

$$X_1 = \{x_{l(j)}, \ldots, x_{i_1}\} \text{ and } Y_1 = \{y_{l(i)}, \ldots, y_{j_1}\}.$$

Notice that $X_1 \subseteq \Gamma(y_i)$ and $Y_1 \subseteq \Gamma(x_i)$. Suppose $B^i_j \cap X_1 \neq \emptyset$ and $B^i_j \cap Y_1 \neq \emptyset$. Let $\hat{x} \in B^i_j \cap X_1$ and $\hat{y} \in B^i_j \cap Y_1$. Since $(x_i, \hat{y}), (\hat{x}, y_j) \in E$, we have $(\hat{x}, \hat{y}) \in E$ by the definition of the strong ordering. It follows that B^i_j contains a cycle $(\hat{x}, \hat{y}, x_i, y_i)$, a contradiction. Thus $B^i_j \cap X_1 = \emptyset$ or $B^i_j \cap Y_1 = \emptyset$. If $B^i_j \cap X_1 = \emptyset$ then we have $B^i_j = D^i_j$. If $B^i_j \cap Y_1 = \emptyset$ then we have $B^i_j = C^i_j$. □

We also have the following two lemmas, which are proved in the next section.

Lemma 3. *C^i_j is*

1. *$\tilde{A}^{i_1}_{j_2} + \{x_i, y_j\}$ if $l(j) \ge i_1$,*
2. *$C^{i_1}_j + x_i$ if $l(j) < i_1$ and $(x_{i_1}, y_{j_2}) \notin E$,*
3. *$\max\{\tilde{A}^{i_2}_{j_2} + \{x_i, y_j\},\ C^{i_1}_j + x_i,\ D^{i_1}_{j_2} + \{x_i, y_j\}\}$ if $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, and $(x_{i_2}, y_{j_2}) \notin E$,*
4. *$\max\{\tilde{A}^{i_2}_{j_2} + \{x_i, y_j\},\ C^{i_1}_j + x_i,\ D^{i_1}_{j_2} + \{x_i, y_j\},\ B^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}\}$ if $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, and $l(i_1) = j_2$,*
5. *$\max\{\tilde{A}^{i_2}_{j_2} + \{x_i, y_j\},\ C^{i_1}_j + x_i,\ D^{i_1}_{j_2} + \{x_i, y_j\}\}$ otherwise.*

Here $i_2 = l(j) - 1$ and $j_2 = l(i) - 1$. □

Lemma 4. *D^i_j is*

1. *$\tilde{A}^{i_2}_{j_1} + \{x_i, y_j\}$ if $l(i) \ge j_1$,*
2. *$D^i_{j_1} + y_j$ if $l(i) < j_1$ and $(x_{i_2}, y_{j_1}) \notin E$,*
3. *$\max\{\tilde{A}^{i_2}_{j_2} + \{x_i, y_j\},\ D^i_{j_1} + y_j,\ C^{i_2}_{j_1} + \{x_i, y_j\}\}$ if $l(i) < j_1$, $(x_{i_2}, y_{j_1}) \in E$, and $(x_{i_2}, y_{j_2}) \notin E$,*
4. *$\max\{\tilde{A}^{i_2}_{j_2} + \{x_i, y_j\},\ D^i_{j_1} + y_j,\ C^{i_2}_{j_1} + \{x_i, y_j\},\ B^{i_2}_{j_2} + \{x_i, y_j, y_{j_1}\}\}$ if $l(i) < j_1$, $(x_{i_2}, y_{j_1}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, and $l(j_1) = i_2$,*
5. *$\max\{\tilde{A}^{i_2}_{j_2} + \{x_i, y_j\},\ D^i_{j_1} + y_j,\ C^{i_2}_{j_1} + \{x_i, y_j\}\}$ otherwise.* □

The lemmas above establish an algorithm using dynamic programming technique for computing A^i_j, B^i_j, C^i_j, and D^i_j for each edge (x_i, y_j) in an increasing order from (x_1, y_1) to (x_p, y_q) so that $A^{i'}_{j'}, B^{i'}_{j'}, C^{i'}_{j'}$, and $D^{i'}_{j'}$ for every i', j' $(i' + j' < i + j)$ are available when computing the data for edge (x_i, y_j). Our algorithm is shown in Fig. 1.

Theorem 1. *Algorithm 1 solves MinFVS in $O(n + m)$ time for permutation bigraphs, where n and m are the number of vertices and edges of a graph, respectively.* □

2.3 Proof of Lemmas 3 and 4

We show a proof of Lemma 3. Lemma 4 can be proved by similar arguments. We distinguish five cases.

Case 1. $l(j) \ge i_1$:
We show $C^i_j = \tilde{A}^{i_1}_{j_2} + \{x_i, y_j\}$. Notice that $l(j) \ge i_1$ implies that $\tilde{A}^{i_1}_{j_2} + \{x_i, y_j\}$ is an MCFS of $G[V^i_j]$ that contains no vertex in Y_1, since there exists at most one vertex in $V^{i_1}_{j_2}$ adjacent to x_i or y_j.

Case 2. $l(j) < i_1$ and $(x_{i_1}, y_{j_2}) \notin E$:
We show $C^i_j = C^{i_1}_j + x_i$. Notice that $l(j) < i_1$ implies $(x_{i_1}, y_j) \in E$, and $(x_{i_1}, y_{j_2}) \notin E$ implies $l(i_1) = l(i)$. Thus $C^{i_1}_j + x_i$ is an MCFS of $G[V^i_j]$ that contains no vertex in Y_1.

Case 3. $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, and $(x_{i_2}, y_{j_2}) \notin E$:

Obtain a strong ordering of G.
$A_0^0, B_0^0, C_0^0, D_0^0 \leftarrow \emptyset$.
Compute $l(i), l(j), r(i), r(j)$ for i and j, $1 \le i \le p, 1 \le j \le q$.

for all $(x_i, y_j) \in E$ **do**
    $i_1 \leftarrow i-1$, $j_1 \leftarrow j-1$, $i_2 \leftarrow l(j)-1$, and $j_2 \leftarrow l(i)-1$.
    if $i_1 \le l(j)$ **then**
        $C_j^i \leftarrow \tilde{A}_{j_2}^{i_1} + \{x_i, y_j\}$.
    else if $(x_{i_1}, y_{j_2}) \notin E$ **then**
        $C_j^i \leftarrow C_j^{i_1} + x_i$.
    else if $(x_{i_2}, y_{j_2}) \in E$ and $l(j_1) = i_2$ **then**
        $C_j^i \leftarrow \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\}, C_j^{i_1} + x_i, D_{j_2}^{i_1} + \{x_i, y_j\}, B_{j_2}^{i_2} + \{x_i, y_j, x_{i_1}\}\}$.
    else
        $C_j^i \leftarrow \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\}, C_j^{i_1} + x_i, D_{j_2}^{i_1} + \{x_i, y_j\}\}$.
    end if
    if $j_1 \le l(i)$ **then**
        $D_j^i \leftarrow \tilde{A}_{j_1}^{i_2} + \{x_i, y_j\}$.
    else if $(x_{i_2}, y_{j_1}) \notin E$ **then**
        $D_j^i \leftarrow D_{j_1}^i + y_j$.
    else if $(x_{i_2}, y_{j_2}) \in E$ and $l(j_1) = i_2$ **then**
        $D_j^i \leftarrow \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\}, D_{j_1}^i + y_j, C_{j_1}^{i_2} + \{x_i, y_j\}, B_{j_2}^{i_2} + \{x_i, y_j, y_{j_1}\}\}$.
    else
        $D_j^i \leftarrow \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\}, D_{j_1}^i + y_j, C_{j_1}^{i_2} + \{x_i, y_j\}\}$.
    end if
    $B_j^i \leftarrow \max\{C_j^i, D_j^i\}$.
    $A_j^i \leftarrow \max\{\tilde{A}_j^{i_1}, \tilde{A}_{j_1}^i, B_j^i\}$.
end for
print $V - A_q^p$

Fig. 1 Algorithm 1.

We show

$$C_j^i = \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\},\ C_j^{i_1} + x_i,\ D_{j_2}^{i_1} + \{x_i, y_j\}\}$$

by a series of claims.

Let $C_j^i(x_{i_1})$ be an MCFS of $G[V_j^i]$ that contains x_i, y_j, and x_{i_1}, and let $C_j^i(x_{i_1}, y_{j_2})$ be an MCFS of $G[V_j^i]$ that contains x_i, y_j, x_{i_1}, and y_{j_2}. Let

$$Y_2 = \{y_{l(i_1)}, \ldots, y_{j_2}\}.$$

Note that $C_j^i(x_{i_1})$ contains no vertex in Y_1, and $C_j^i(x_{i_1}, y_{j_2})$ contains no vertex in $X_1 - x_{i_1}$, since the vertices are strongly ordered.

Claim 1. *If $l(j) < i_1$ and $(x_{i_1}, y_{j_2}) \in E$ then*

$$C_j^i = \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\},\ C_j^i(x_{i_1})\}.$$

Proof. Let $\hat{x} \in X_1 - x_{i_1}$, and $\hat{C}$ be an MCFS of $G[V_j^i]$ that contains x_i and y_j, and no vertex in Y_1. If $\hat{C}$ contains $\hat{x}$ but not x_{i_1} then $\hat{C} - \hat{x} + x_{i_1}$ is also an MCFS of $G[V_j^i]$, since $\Gamma_{G[V_j^i]}(x_{i_1}) \subseteq \Gamma_{G[V_j^i]}(\hat{x})$. Thus we have the claim. □

Claim 2. *If $l(j) < i_1$ and $(x_{i_1}, y_{j_2}) \in E$ then*

$$C_j^i(x_{i_1}) = \max\{C_j^{i_1} + x_i,\ C_j^i(x_{i_1}, y_{j_2})\}.$$

Proof. The proof is similar to that of Claim 1, and is omitted. □

Claim 3. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, and $(x_{i_2}, y_{j_2}) \notin E$ then*

$$C_j^i(x_{i_1}, y_{j_2}) = D_{j_2}^{i_1} + \{x_i, y_j\}.$$

Proof. Notice that $(x_{i_2}, y_{j_2}) \notin E$ implies $l(j_2) = l(j)$. Thus $D_{j_2}^{i_1} + \{x_i, y_j\}$ is a CFS of $G[V_j^i]$, since no vertex in $V_{j_2}^{i_2}$ is adjacent to x_i or y_j. Notice that $D_{j_2}^{i_1} + \{x_i, y_j\}$ contains x_i, y_j, x_{i_1}, and y_{j_2}. Notice also that any CFS containing x_i, y_j, x_{i_1}, and y_{j_2} contains no vertex in $V_j^i - V_{j_2}^{i_2} - \{x_i, y_j, x_{i_1}, y_{j_2}\}$, since the vertices are strongly ordered. Thus $D_{j_2}^{i_1} + \{x_i, y_j\}$ is an MCFS of $G[V_j^i]$ that contains x_i, y_j, x_{i_1}, and y_{j_2}. □

Case 4. $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, and $l(i_1) = j_2$:
We show

$$C_j^i = \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\},\ C_j^{i_1} + x_i,\ D_{j_2}^{i_1} + \{x_i, y_j\},\ B_{j_2}^{i_2} + \{x_i, y_j, x_{i_1}\}\}$$

by the following two claims together with Claims 1 and 2.

Let $C_j^i(x_{i_1}, y_{j_2}, x_{i_2})$ be an MCFS of $G[V_j^i]$ that contains x_i, y_j, x_{i_1}, y_{j_2}, and x_{i_2}. Let

$$X_2 = \{x_{l(j_2)}, \ldots, x_{i_2}\}.$$

Note that $C_j^i(x_{i_1}, y_{j_2}, x_{i_2})$ contains no vertex in $Y_2 - y_{j_2}$, since the vertices are strongly ordered.

Claim 4. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, and $(x_{i_2}, y_{j_2}) \in E$ then*

$$C_j^i(x_{i_1}, y_{j_2}) = \max\{D_{j_2}^{i_1} + \{x_i, y_j\},\ C_j^i(x_{i_1}, y_{j_2}, x_{i_2})\}.$$

Proof. The proof is similar to that of Claim 1, and is omitted. □

Claim 5. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, and $l(i_1) = j_2$ then*

$$C_j^i(x_{i_1}, y_{j_2}, x_{i_2}) = B_{j_2}^{i_2} + \{x_i, y_j, x_{i_1}\}.$$

Proof. Notice that $l(i_1) = j_2$ implies that $B_{j_2}^{i_2} + \{x_i, y_j, x_{i_1}\}$ is a CFS of $G[V_j^i]$, since no vertex in $V_{j_2}^{i_2} - y_{j_2}$ is adjacent to x_i, y_j, or x_{i_1}. Notice that $B_{j_2}^{i_2} + \{x_i, y_j, x_{i_1}\}$ contains x_i, y_j, x_{i_1}, y_{j_2}, and x_{i_2}. Notice also that any CFS containing x_i, y_j, x_{i_1}, y_{j_2}, and x_{i_2} contains no vertex in $V_j^i - V_{j_2}^{i_2} - \{x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}\}$, since the vertices are strongly ordered. Thus $B_{j_2}^{i_2} + \{x_i, y_j, x_{i_1}\}$ is an MCFS of $G[V_j^i]$ that contains x_i, y_j, x_{i_1}, y_{j_2}, and x_{i_2}. □

Now we consider the remaining case.

Case 5. $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, and $l(i_1) < j_2$:
We show

$$C_j^i = \max\{\tilde{A}_{j_2}^{i_2} + \{x_i, y_j\},\ C_j^{i_1} + x_i,\ D_{j_2}^{i_1} + \{x_i, y_j\}\}$$

by the following claims together with Claims 1, 2, and 4.

Let

$$i_3 = l(j_2) - 1,\ j_3 = l(i_1) - 1, \text{ and } Y_3 = \{y_{l(i_2)}, \ldots, y_{j_3}\}.$$

Let $C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3})$ be an MCFS of $G[V^i_j]$ that contains $x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}$, and y_{j_3}. Note that $C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3})$ contains no vertex in $X_2 - x_{i_2}$, since the vertices are strongly ordered.

We distinguish two cases.

Case 5-1. $(x_{i_2}, y_{j_3}) \notin E$:

Claim 6. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, $l(i_1) < j_2$, and $(x_{i_2}, y_{j_3}) \notin E$ then*

$$C^i_j(x_{i_1}, y_{j_2}, x_{i_2}) = C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}.$$

Proof. Notice that $(x_{i_2}, y_{j_3}) \notin E$ implies $l(i_2) = l(i_1)$. Thus $C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}$ is a CFS of $G[V^i_j]$, since no vertex in $V^{i_2}_{j_3}$ is adjacent to x_i, y_j, or x_{i_1}. Notice that $C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}$ contains x_i, y_j, x_{i_1}, y_{j_2}, and x_{i_2}. Notice also that any CFS containing x_i, y_j, x_{i_1}, y_{j_2}, and x_{i_2} contains no vertex in $V^i_j - V^{i_2}_{j_3} - \{x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}\}$, since the vertices are strongly ordered. Thus $C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}$ is an MCFS of $G[V^i_j]$ that contains $x_i, y_j, x_{i_1}, y_{j_2}$, and x_{i_2}. □

Claim 7. *If $l(j) < i_1$ then*

$$|C^{i_1}_j + x_i| \geq |C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}|.$$

Proof. Let $\hat{C} = C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}$. There exists a vertex $\hat{x} \in X_1$ such that $\hat{x} \notin \hat{C}$, since $l(j) < i_1$. Thus $\hat{C} - y_{j_2} + \hat{x}$ contains no vertex of $Y_1 + Y_2$, since the vertices are strongly ordered. Thus $\hat{C} - y_{j_2} + \hat{x}$ is a CFS of $G[V^i_j]$, and $|C^{i_1}_j + x_i| \geq |\hat{C} - y_{j_2} + \hat{x}| = |C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\}|$. □

Case 5-2. $(x_{i_2}, y_{j_3}) \in E$:

Claim 8. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, $l(i_1) < j_2$, and $(x_{i_2}, y_{j_3}) \in E$ then*

$$C^i_j(x_{i_1}, y_{j_2}, x_{i_2}) = \max\{C^{i_2}_{j_2} + \{x_i, y_j, x_{i_1}\},\ C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3})\}.$$

Proof. The proof is similar to that of Claim 1, and is omitted. □

We further distinguish two cases.

Case 5-2-1. $(x_{i_3}, y_{j_3}) \notin E$:

Claim 9. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, $l(i_1) < j_2$, $(x_{i_2}, y_{j_3}) \in E$, and $(x_{i_3}, y_{j_3}) \notin E$ then*

$$C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}) = D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}.$$

Proof. Notice that $(x_{i_3}, y_{j_3}) \notin E$ implies $l(j_3) = l(j_2)$. Thus $D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}$ is a CFS of $G[V^i_j]$, since no vertex in $V^{i_3}_{j_3}$ is adjacent to x_i, y_j, x_{i_1}, or y_{j_2}. Notice that $D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}$ contains $x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}$, and y_{j_3}. Notice also that any CFS containing x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}, and y_{j_3} contains no vertex in $V^i_j - V^{i_3}_{j_3} - \{x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}, y_{j_2}\}$, since the vertices are strongly ordered. Thus $D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}$ is an MCFS of $G[V^i_j]$ that contains x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}, and y_{j_3}. □

Claim 10. *If $l(i_1) < j_2$ then*

$$|D^{i_1}_{j_2} + \{x_i, y_j\}| \geq |D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}|.$$

Proof. Let $\hat{D} = D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}$. There exists a vertex $\hat{y} \in Y_2$ such that $\hat{y} \notin \hat{D}$, since $l(i_1) < j_2$. Thus $\hat{D} - x_{i_2} + \hat{y}$ contains no vertex of $X_1 + X_2 - x_{i_1}$, since the vertices are strongly ordered. Thus we conclude that $\hat{D} - x_{i_2} + \hat{y}$ is a CFS of $G[V^i_j]$, and $|D^{i_1}_{j_2} + \{x_i, y_j\}| \geq |\hat{D} - x_{i_2} + \hat{y}| = |D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\}|$. □

Case 5-2-2. $(x_{i_3}, y_{j_3}) \in E$:

Let $C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}, x_{i_3})$ be an MCFS of $G[V^i_j]$ that contains $x_i, y_j, x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}$, and x_{i_3}.

Claim 11. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, $l(i_1) < j_2$, $(x_{i_2}, y_{j_3}) \in E$, and $(x_{i_3}, y_{j_3}) \in E$ then*

$$C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}) = \max\{D^{i_2}_{j_3} + \{x_i, y_j, x_{i_1}, y_{j_2}\},\ C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}, x_{i_3})\}.$$

Proof. The proof is similar to that of Claim 1, and is omitted. □

Claim 12. *If $l(j) < i_1$, $(x_{i_1}, y_{j_2}) \in E$, $(x_{i_2}, y_{j_2}) \in E$, $l(i_1) < j_2$, $(x_{i_2}, y_{j_3}) \in E$, and $(x_{i_3}, y_{j_3}) \in E$ then*

$$|C^{i_1}_j + x_i| \geq |C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}, x_{i_3})|.$$

Proof. Let $\hat{C} = C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}, x_{i_3})$. There exists a vertex $\hat{x} \in X_1$ such that $\hat{x} \notin \hat{C}$, since $l(j) < i_1$. Thus $\hat{C} - y_{j_2} + \hat{x}$ contains no vertex of $Y_1 + Y_2 + Y_3 - y_{j_3}$, since the vertices are strongly ordered. Thus we conclude that $\hat{C} - y_{j_2} + \hat{x}$ is a CFS of $G[V^i_j]$, and $|C^{i_1}_j + x_i| \geq |\hat{C} - y_{j_2} + \hat{x}| = |C^i_j(x_{i_1}, y_{j_2}, x_{i_2}, y_{j_3}, x_{i_3})|$ by the definition of $C^{i_1}_j$. □

3. NP-Hardness for Grid Intersection Graphs

3.1 Grid Intersection Graphs

A bigraph G with a bipartition (X, Y) is a *grid intersection graph* if X and Y correspond to sets of horizontal and vertical line segments in the plain, respectively, such that for any $x \in X$ and $y \in Y$, $(x, y) \in E(G)$ if and only if a line segment corresponding to x and a line segment corresponding to y intersect. The following is shown in [8].

Theorem I. *Any planar bigraph is a grid intersection graph.* □

3.2 NP-Hardness

We consider a decision problem associated with MinFVS defined as follows.

FEEDBACK VERTEX SET
INSTANCE: Graph G, positive integer k.
QUESTION: Is there an FVS of size k in G?

It is known that **FEEDBACK VERTEX SET** is NP-complete for planar graphs [11] and bigraphs [23]. We show the following.

Theorem 2. FEEDBACK VERTEX SET *is NP-complete even for planar bigraphs.*

Proof. Our proof is similar to that used in [11] and [23]. We show a polynomial time reduction from **VERTEX COVER** for planar graphs to **FEEDBACK VERTEX SET** for planar bigraphs. It is well-known that **VERTEX COVER** is NP-complete for planar graphs [7].

VERTEX COVER is defined as follows.

VERTEX COVER
INSTANCE: Graph H, positive integer h.
QUESTION: Is there a vertex cover of size h in H, i.e., a subset $S \subseteq V(H)$ with $\lvert S\rvert = h$ such that for each edge $(u, v) \in E$ at least one of u and v belongs to S?

Let H be a planar graph as an instance of **VERTEX COVER**. Let G be a graph obtained from H by replacing each edge (u, v) by a cycle (u, x_{uv}, v, y_{uv}), where x_{uv} and y_{uv} are new vertices. It is easy to see that G is a planar bigraph and can be constructed in linear time.

It is also easy to see that H has a vertex cover of size h if and only if G has an FVS of size h. □

From Theorems I and 2, we have the following.

Theorem 3. *MinFVS is NP-hard for grid intersection graphs.* □

4. A Polynomial Time Algorithm for Graphs with Maximum Degree at most Three

A vertex set $S \subseteq V(G)$ of a graph G is a *separating set* if the number of connected components of the subgraph of G induced by $V(G)\backslash S$ is more than that of G. A vertex set $S \subseteq V(G)$ of a graph G is an *independent set* if no two vertices of S are adjacent. A *maximum nonseparating independent set* (MNIS) is a maximum independent set that contains no separating set. The *maximum nonseparating independent set problem* (MaxNIS) is to find an MNIS in a given graph.

Like MinFVS, MaxNIS is also NP-hard even for planar graphs with maximum degree at most 4 [6], while it can be solved in $O(n^4)$ time for graphs with maximum degree at most 3 [5], [20], where n is the number of vertices of a graph.

A graph is said to be k-regular if the degree of every vertex is k. Let $\eta(G)$ and $\nu(G)$ be the number of vertices in an MFVS and an MNIS of G, respectively. It is shown in [20] that for any graph H with maximum degree at most 3, we can construct 3-regular graphs G and G' in linear time such that $\eta(G) = \eta(H)$ and $\nu(G') = \nu(H)$, respectively. It is also shown that for a 3-regular graph G,

$$\nu(G) + \eta(G) = \mu(G).$$

Here $\mu(G) = m - n + c$, where n, m, and c are the number of vertices, edges, and connected components of G, respectively. $\mu(G)$ is known as the nullity, cyclomatic number, and first Betti number of G.

An embedding of a graph G in S_k, a sphere with k handles, is a continuous one-to-one mapping. The components of $S_k - G$ are called regions. An embedding is said to be cellular if each region is homeomorphic to an open disk. $\gamma_M(G)$ is the maximum-genus of G, which is the maximum value of k such that G is cellular embeddable in S_k. It is shown in [9] that

$$\gamma_M(G) = \nu(G),$$

for a 3-regular graph G. Moreover, it is known that computing $\gamma_M(G)$ can be reduced to the cographic matroid parity problem [3], which can be solved in $O(nm \log^6 n)$ time [4], [5], where n and m are the number of vertices and edges of a graph, respectively. Thus we have the following.

Theorem 4. *MinFVS and MaxNIS can be solved in $O(n^2 \log^6 n)$ time for graphs with maximum degree at most 3.* □

5. Concluding Remarks

It should be noted that our linear time algorithm, Algorithm 1, for permutation bigraphs is similar to an $O(n^2 m)$ time algorithm for convex graphs proposed in [13]. The difference in the time complexity is due to the strong ordering.

It is known that the class of grid intersection graphs is a subclass of the boxicity-2 graphs [1], [8]. Thus, from Theorem 3, we conclude that MinFVS is NP-hard for boxicity-2 graphs, settling an open question posed in [17].

A vertex cover $S \subseteq V(G)$ of a connected graph G is a *connected vertex cover* if the subgraph of G induced by S is connected. A *minimum connected vertex cover problem* (MinCVC) is to find a connected vertex cover with minimum cardinality in a given graph. It is shown in [14], [21] that MinCVC for quasi-wheels, which is a subclass of 3-connected graphs, can be reduced to the problem to find an MNIS that consists of only vertices of degree 3. It is also shown that this problem can be reduced to the cographic matroid parity problem in linear time by the reduction similar to that shown in Sect. 4. It follows that MinCVC for quasi-wheels can be solved in $O(n^2 \log^6 n)$ time, where n is the number of vertices of a graph.

The time complexity of MinFVS for orthogonal ray graphs and unit grid intersection graphs remains open.

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