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A Formulation of Composition for Cellular Automata on Groups

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SUMMARY We introduce the notion of 'Composition', 'Union' and 'Division' of cellular automata on groups. A kind of notions of compositions was investigated by Sato [10] and Manzini [6] for linear cellular automata, we extend the notion to general cellular automata on groups and investigated their properties. We observe the all unions and compositions generated by one-dimensional 2-neighborhood cellular automata over $\mathbf{Z}_2$ including non-linear cellular automata. Next we prove that the composition is right-distributive over union, but is not left-distributive. Finally, we conclude by showing reformulation of our definition of cellular automata on group which admit more than three states. We also show our formulation contains the representation using formal power series for linear cellular automata in Manzini [6].
key words: *cellular automata, groups, models of computation, automata*

1. Introduction

The study of cellular automata was initiated by [11] and have been developed by many researchers as a good computational model for physical systems simulation. Recently cellular automata have been investigated in various fields including computer science, biology, physics, since they provide simple and powerful models for parallel computation and natural phenomena.

In this paper, we investigate cellular automata on groups as a formal model of computation. To compose simple cellular automata into a complex cellular automaton, we introduce the notion of 'Composition' of cellular automata on groups. The notion of automata on groups was first treated as a special case for automata on graphs (Caley graphs) which represent groups in [8], [9]. Watanabe and Noguchi investigated the decomposition of finite automata from the view point of spatial networks using groups [12]. Pries et al. investigated cellular automata as a tool for implementing hardware algorithms in VLSI [7]. They considered configurations decided by a cellular automaton as a group and divided configurations into simple configurations using group properties. Sato introduced group structured linear cellular automata and the star operation of local transition rules [10]. The star operation is a kind of composition of cellular automata but the definition of it is different from ours. Manzini also investigated the linear cellular automata using the formal power series and their product to find inverse local transition functions [6]. The product of formal power series are equal to our composition of cellular automata for linear cases. An abstract collision system in [5] is considered as an extension of a cellular automaton, the notion of 'composition' for an abstract collision system on G-sets is investigated in [4]. Further investigation about collision sets related to the set of all connected subsets of a topological space are studied in [3].

This paper follows on from [2]. He introduced the composition of cellular automata on groups in order to reduce a complex behaved dynamics into simpler ones. We introduce a formal definition of cellular automata on group over $\mathbf{Z}_2$. In our framework, operations on cellular automata 'Union', 'Division' and 'Composition' are introduced. Unions of all 2-neighborhood cellular automata are investigated. Compositions of all 2-neighborhood cellular automata are also investigated and determined the subset of 3-neighborhood cellular automata which are generated by composing two 2-neighborhood cellular automata. Next we prove that the composition is right-distributive over union, but is not left-distributive. Finally, we conclude by showing reformulation of our definition of cellular automata on group which admit more than three states. We also show our formulation contains the representation using formal power series for linear cellular automata in [6].

2. Notion of Cellular Automata on Groups and Their Basic Properties

In this section we introduce a notion of cellular automata on groups and show some examples. First we define cellular automata on a group.

Definition 1: Let G be a group with operator $\cdot$ and identity element e. A cellular automaton on G is a triple $C = (G, V, \mathfrak{L})$ of a group G, subsets $V \subset G$ and $\mathfrak{L} \subset 2^V$. An element of 2^G is called a configuration and 2^G is the configuration space of C. V is the neighborhood of C and we define a local transition function $l_{\mathfrak{L}} : 2^V \to \{\phi, \{e\}\}$ by $\mathfrak{L} \subset 2^V$;

$$l_{\mathfrak{L}}(X) = \begin{cases} \phi & (X \notin \mathfrak{L}) \\ \{e\} & (X \in \mathfrak{L}), \end{cases}$$

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and the global transition function $F_C : 2^G \to 2^G$ by

$$F_C(\mathbf{c}) = \bigcup_{g\in G} g l_{\mathfrak{L}}(g^{-1}\mathbf{c} \cap V).$$

Note that $x \cdot Y = \{x \cdot y \mid y \in Y\}$ for $x \in G$ and $Y \subset G$. The equation $F_{C_1} = F_{C_2}$ means that $F_{C_1}(\mathbf{c}) = F_{C_2}(\mathbf{c})$ for any $\mathbf{c} \in 2^G$. In the following proposition we show a necessary and sufficient condition for $F_{C_1} = F_{C_2}$.

Proposition 2: Let $C_1 = (G, V_1, \mathfrak{L}_1)$ and $C_2 = (G, V_2, \mathfrak{L}_2)$ be cellular automata. The equation $F_{C_1} = F_{C_2}$ holds if and only if

$$e \in F_{C_1}(\mathbf{c}) \quad \Leftrightarrow \quad e \in F_{C_2}(\mathbf{c}) \ \ (\text{for any } \mathbf{c} \in 2^G).$$

Proof. Since $F_{C_1}(\mathbf{c}) = \{g \in G \mid l_{\mathfrak{L}_1}(g^{-1}\mathbf{c} \cap V_1) = \{e\}\} = \{g \in G \mid g^{-1}\mathbf{c} \cap V_1 \in \mathfrak{L}_1\}$, we have $g \in F_{C_1}(\mathbf{c}) \Leftrightarrow g^{-1}\mathbf{c} \cap V_1 \in \mathfrak{L}_1 \Leftrightarrow e \in F_{C_1}(g^{-1}\mathbf{c}) \Leftrightarrow e \in F_{C_2}(g^{-1}\mathbf{c}) \Leftrightarrow g^{-1}\mathbf{c} \cap V_2 \in \mathfrak{L}_2 \Leftrightarrow g \in F_{C_2}(\mathbf{c})$. □

Lemma 3: Let $C = (G, V, \mathfrak{L})$ be a cellular automaton. For $\forall X \subset V$ the followings are equivalent;

1. $X \in \mathfrak{L}$
2. $\forall Y \in 2^G$ if $X = Y \cap V$, then $e \in F_C(Y)$.

Proof. (1. $\Rightarrow$ 2.) We assume that $X \in \mathfrak{L}$ and for any $Y' \in 2^{G\setminus V}$ we let $Y = X \cup Y'$. Trivially $X = Y \cap V$ and $l_{\mathfrak{L}}(Y \cap V) = \{e\}$. Then

$$\begin{aligned} F_C(Y) &= \bigcup_{g\in G} g l_{\mathfrak{L}}(g^{-1}Y \cap V) \\ &\supset e l_{\mathfrak{L}}(e^{-1}Y \cap V) \\ &= e\{e\} \\ &= \{e\} \end{aligned}$$

Hence we have $e \in F_C(Y)$.
(1. $\Leftarrow$ 2.) For $\forall g \in G$ and $\forall Y \in 2^G$ we have $g l_{\mathfrak{L}}(g^{-1}Y \cap V) \in \{\phi, \{g\}\}$ by definition $l_{\mathfrak{L}}$ and $e \notin \bigcup_{g\in G\setminus\{e\}} g l_{\mathfrak{L}}(g^{-1}Y \cap V)$. Now we let $X = Y \cap V$ and $e \in F_C(Y)$, and assume that $X \notin \mathfrak{L}$. Then

$$\begin{aligned} F_C(Y) &= \bigcup_{g\in G} g l_{\mathfrak{L}}(g^{-1}Y \cap V) \\ &= \bigcup_{g\in G\setminus\{e\}} g l_{\mathfrak{L}}(g^{-1}Y \cap V) \cup e l_{\mathfrak{L}}(e^{-1}Y \cap V) \\ &= \bigcup_{g\in G\setminus\{e\}} g l_{\mathfrak{L}}(g^{-1}Y \cap V) \cup l_{\mathfrak{L}}(X) \\ &= \bigcup_{g\in G\setminus\{e\}} g l_{\mathfrak{L}}(g^{-1}Y \cap V) \cup \phi \\ &= \bigcup_{g\in G\setminus\{e\}} g l_{\mathfrak{L}}(g^{-1}Y \cap V) \\ &\not\ni e. \end{aligned}$$

This is contradiction. □

In the followings, we consider the set of all integers $\mathbf{Z}$ as an additive group $\mathbf{Z} = (\mathbf{Z}, +, 0)$. So usual one dimensional cellular automata with 2-states are represented as cellular automata on the group $\mathbf{Z}$. We define 2-neighborhood and 3-neighborhood 2-states cellular automata in the next definition and introduce some examples.

Definition 4: For $k \geq 1$ and $n \in \{0, 1, \cdots, 2^{2^k} - 1\}$, we define cellular automata $CA(k, n)$ on $\mathbf{Z}$ by $CA(k, n) = (\mathbf{Z}, V, \mathfrak{L}_n)$ where $V = \{0, 1, \cdots, k-1\}$, and $\mathfrak{L}_n$ is the subset of 2^V which satisfies $n = \sum_{X\in\mathfrak{L}_n} 2^{\sum_{i\in X} 2^i}$.

We note $CA(1, 0) = (\mathbf{Z}, \{0\}, \phi)$ and $CA(1, 2) = (\mathbf{Z}, \{0\}, \{\{0\}\})$.

Example 5: Since $6 = 2 + 2^2 = 2^{2^0} + 2^{2^1}$, we have $CA(2, 6) = (\mathbf{Z}, \{0, 1\}, \{\{0\}, \{1\}\})$. Since $90 = 2 + 2^3 + 2^4 + 2^6 = 2^{2^0} + 2^{2^0+2^1} + 2^{2^2} + 2^{2^1+2^2}$, we have $CA(3, 90) = (\mathbf{Z}, \{0, 1, 2\}, \{\{0\}, \{2\}, \{0, 1\}, \{1, 2\}\})$. The elements X in $\mathfrak{L}_n$ represents the state of neighborhood which induce the next states '1'. For a rule number 90, we have the following table:

Neighborhood	111	110	101	100
$X \in \mathfrak{L}_n$	$\{0, 1, 2\}$	$\{1, 2\}$	$\{0, 2\}$	$\{2\}$
$l_{\mathfrak{L}}(X)$	ϕ	$\{e\}$	ϕ	$\{e\}$
Neighborhood	011	010	001	000
$X \in \mathfrak{L}_n$	$\{0, 1\}$	$\{1\}$	$\{0\}$	ϕ
$l_{\mathfrak{L}}(X)$	$\{e\}$	ϕ	$\{e\}$	ϕ

The configuration $\mathbf{c} \subset \mathbf{Z}$ represents places where the state is 1. Since $n \in F_C(\mathbf{c}) \Leftrightarrow l_{\mathfrak{L}}(n^{-1}\mathbf{c} \cap V) = \{e\} \Leftrightarrow n^{-1}\mathbf{c} \cap V \in \mathfrak{L} \Leftrightarrow \mathbf{c} \cap nV \in n\mathfrak{L}$, the next state at n is 1 if $\mathbf{c} \cap nV \in n\mathfrak{L}$. For 3-neighborhood case we are choosing $V = \{0, 1, 2\}$, the left-hand side of the state is changing. It seems to be better that we choose $V = \{-1, 0, 1\}$ but it is not convenient for even-neighborhood case. Our numbered 3-neighborhood cellular automata $CA(3, n)$ is a shifted version of usual numbered elementary cellular automata. Later, we define a cellular automaton SHIFT which represent a shift operation and an operator 'composition' ($\diamond$) of two cellular automata. After that the usual numbered elementary cellular automata are represented as SHIFT$\diamond CA(3, n)$.

Example 6: SHIFT $= (\mathbf{Z}, \{-1, 0\}, \{\{-1\}, \{-1, 0\}\})$ is a cellular automata on group $\mathbf{Z}$.

$\mathbf{Z}^2$ is also considered as a group, so it is easy to represent a multi dimensional cellular automata such as The Game of Life ([1]) as a cellular automata on a group.

Example 7: LIFE $= (\mathbf{Z}^2, V_{\text{LIFE}}, \mathfrak{L}_{\text{LIFE}})$ is a cellular automata on group $\mathbf{Z}^2$, where

$$V_{\text{LIFE}} = \{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}+1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}+1\\0\end{pmatrix}, \begin{pmatrix}-1\\+1\end{pmatrix}, \begin{pmatrix}0\\+1\end{pmatrix}, \begin{pmatrix}+1\\+1\end{pmatrix}\}, \text{ and}$$

$$\mathfrak{L}_{\text{LIFE}} = \{v \in 2^V \mid (\#v = 3) \vee (\#v = 4 \wedge \begin{pmatrix}0\\0\end{pmatrix} \in v)\}.$$

We note that $\#v$ is the number of elements in a set v.

One dimensional cellular automaton on $\mathbf{Z}$ is embedded into the two dimensional cellular automaton on $\mathbf{Z}^2$. We define two natural embeddings EX and EY in the following.

Definition 8: For a cellular automata $C = (\mathbf{Z}, V, \mathfrak{L})$, we define a cellular automata $EX(C)$ on $\mathbf{Z}^2$ by $EX(C) = (\mathbf{Z}^2, V_{EX(C)}, \mathfrak{L}_{EX(CA)})$ where

$$V_{EX(C)} = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in V\}, \text{ and}$$

$$\mathfrak{L}_{EX(C)} = \{\{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in X\} \mid X \in \mathfrak{L}\}.$$

We also define a cellular automata $EY(C)$ on $\mathbf{Z}^2$ by $EY(C) = (\mathbf{Z}^2, V_{EY(C)}, \mathfrak{L}_{EY(C)})$ where

$$V_{EY(C)} = \{\begin{pmatrix} 0 \\ x \end{pmatrix} \mid x \in V\}, \text{ and}$$

$$\mathfrak{L}_{EY(C)} = \{\{\begin{pmatrix} 0 \\ x \end{pmatrix} \mid x \in X\} \mid X \in \mathfrak{L}\}.$$

Definition 9: Let $1 \leq k < k'$, $0 \leq x \leq k' - k$ and $CA(k, n) = (\mathbf{Z}, V, \mathfrak{L})$. $CA(k, n)^{k'}_x$ is defined by $CA(k, n)^{k'}_x = (\mathbf{Z}, \{0, 1, \cdots, k' - 1\}, \mathfrak{L}')$ where

$$\mathfrak{L}' = \{s_1 \cup (x + v) \cup s_2 \mid s_1 \in S_1, s_2 \in S_2, v \in \mathfrak{L}\},$$

$$S_1 = \begin{cases} \{\phi\} & (x = 0) \\ 2^{\{0,\cdots,x-1\}} & (x > 0) \end{cases},$$

$$S_2 = \begin{cases} \{\phi\} & (k + x = k') \\ 2^{\{k+x,\cdots,k'-1\}} & (k + x < k') \end{cases}$$

We note that $F_{CA(k,n)^{k'}_0} = F_{CA(k,n)}$ and $F_{CA(k,n)^{k'}_1} = SHIFT \diamond F_{CA(k,n)}$.

3. Operations for Cellular Automata on Groups

In this section we introduce operations, union and composition, for cellular automata on groups. First we define the operation of union and show some examples for union of 2-neighborhood cellular automata.

Definition 10 (Union): Let $C_1 = (G, V_1, \mathfrak{L}_1)$ and $C_2 = (G, V_2, \mathfrak{L}_2)$ be cellular automata on G. The union $C_1 \cup C_2$ of C_1 and C_2 is defined by $C_1 \cup C_2 = (G, V_1 \cup V_2, \mathfrak{L}_1 \cup \mathfrak{L}_2)$.

Definition 11 (Division): Let $C = (G, V, \mathfrak{L})$ be a cellular automaton on G. If there exist $C_1 = (G, V_1, \mathfrak{L}_1)$ and $C_2 = (G, V_2, \mathfrak{L}_2)$ be cellular automata on G such that $V = V_1 \cup V_2$ and $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2$, then we call C_1 and C_2 are division of C and C is dividable by C_1 and C_2.

Example 12: The class of all 2-neighborhood cellular automata $\{CA(2, n) \mid n = 0, .., 15\}$ is generated by $\{CA(2, 0)$, $CA(2, 1)$, $CA(2, 2)$, $CA(2, 4)$, $CA(2, 8)\}$ using 'union' operations. For example, $CA(2, 13)$ is dividable by $CA(2, 1)$, $CA(2, 4)$, and $CA(2, 8)$. Fig. 1 is the table of unions for $CA(2, n)$ $(n = 0, .., 15)$.

For a cellular automaton $C = (G, V, \mathfrak{L})$ we define two cellular automata for expansion and restriction of V.

Definition 13: Let $C = (G, V, \mathfrak{L})$ be a cellular automaton on G and $W \subset G$. We define two cellular automata $C_W = (G, W, \mathfrak{L}_W)$ and $C^W = (G, W, \mathfrak{L}^W)$ where $\mathfrak{L}_W = \{X \cap W \mid X \in \mathfrak{L}\}$ and $\mathfrak{L}^W = \{Y \in 2^W \mid Y \cap V \in \mathfrak{L}\}$.

$n\backslash m$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	5	5	7	7
2	2	3	2	3	6	7	6	7
3	3	3	3	3	7	7	7	7
4	4	5	6	7	4	5	6	7
5	5	5	7	7	5	5	7	7
6	6	7	6	7	6	7	6	7
7	7	7	7	7	7	7	7	7
8	8	9	10	11	12	13	14	15
9	9	9	11	11	13	13	15	15
10	10	11	10	11	14	15	14	15
11	11	11	11	11	15	15	15	15
12	12	13	14	15	12	13	14	15
13	13	13	15	15	13	13	15	15
14	14	15	14	15	14	15	14	15
15	15	15	15	15	15	15	15	15

$n\backslash m$	8	9	10	11	12	13	14	15
0	8	9	10	11	12	13	14	15
1	9	9	11	11	13	13	15	15
2	10	11	10	11	14	15	14	15
3	11	11	11	11	15	15	15	15
4	12	13	14	15	12	13	14	15
5	13	13	15	15	13	13	15	15
6	14	15	14	15	14	15	14	15
7	15	15	15	15	15	15	15	15
8	8	9	10	11	12	13	14	15
9	9	9	11	11	13	13	15	15
10	10	11	10	11	14	15	14	15
11	11	11	11	11	15	15	15	15
12	12	13	14	15	12	13	14	15
13	13	13	15	15	13	13	15	15
14	14	15	14	15	14	15	14	15
15	15	15	15	15	15	15	15	15

Fig. 1 Table of unions: $CA(2, n) \cup CA(2, m)$.

Next we prove the following proposition for expansion and restriction of V to show a necessary and sufficient condition for $F_{C_1} = F_{C_2}$ using the operation of union in theorem 15.

Proposition 14: Let $C = (G, V, \mathfrak{L})$ be a cellular automaton on G, $W \subset V$, $C_W = (G, W, \mathfrak{L}_W)$ and $(C_W)^V = (G, V, (\mathfrak{L}_W)^V)$. Then, $F_C = F_{C_W}$ if and only if $\mathfrak{L} = (\mathfrak{L}_W)^V$.

Proof. We assume $\mathfrak{L} = (\mathfrak{L}_W)^V$. For $\mathbf{c} \in 2^G$, we have $e \in F_C(\mathbf{c}) \Leftrightarrow \mathbf{c} \cap V \in \mathfrak{L}(= (\mathfrak{L}_W)^V) \Leftrightarrow \mathbf{c} \cap V \cap W \in \mathfrak{L}_W \Leftrightarrow \mathbf{c} \cap W \in \mathfrak{L}_W \Leftrightarrow e \in F_{C_W}(\mathbf{c})$. So we have $F_C = F_{C_W}$.
Conversely, we assume $F_C = F_{C_W}$. We have

$$\begin{aligned} \mathfrak{L} &= \{V \in 2^V \mid V \in \mathfrak{L}\} \\ &= \{\mathbf{c} \cap V \mid \mathbf{c} \in 2^G \text{ and } \mathbf{c} \cap V \in \mathfrak{L}\} \\ &= \{\mathbf{c} \cap V \mid \mathbf{c} \in 2^G \text{ and } e \in F_C(\mathbf{c})\} \\ &= \{\mathbf{c} \cap V \mid \mathbf{c} \in 2^G \text{ and } e \in F_{C_W}(\mathbf{c})\} \\ &= \{\mathbf{c} \cap V \mid \mathbf{c} \in 2^G \text{ and } \mathbf{c} \cap V \in \mathfrak{L}_W\} \\ &= (\mathfrak{L}_W)^V. \end{aligned}$$

□

Theorem 15: Let $C_1 = (G, V_1, \mathfrak{L}_1)$ and $C_2 = (G, V_2, \mathfrak{L}_2)$ be cellular automata on G. We have $F_{C_1} = F_{C_2}$ if and only if the following conditions hold.

1. $(\mathfrak{L}_1)_{V_1 \cap V_2} = (\mathfrak{L}_2)_{V_1 \cap V_2}$,
2. $\mathfrak{L}_1 = ((\mathfrak{L}_1)_{V_1 \cap V_2})^{V_1}$,
3. $\mathfrak{L}_2 = ((\mathfrak{L}_2)_{V_1 \cap V_2})^{V_2}$.

Proof. First, we assume that $F_{C_1} = F_{C_2}$. For the first equality in the statement of Theorem,

$$
\begin{aligned}
&(\mathfrak{L}_1)_{V_1 \cap V_2}\\
&= \{X \cap V_1 \cap V_2 \mid X \in \mathfrak{L}_1\}\\
&= \{X \cap V_1 \cap V_2 \mid\\
&\qquad \exists Y \in 2^G \text{ s.t. } X = Y \cap V_1 \text{ and } e \in F_{C_1}(Y)\}\\
&= \{Y \cap V_1 \cap V_2 \mid Y \in 2^G \text{ and } e \in F_{C_1}(Y)\}\\
&\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 3)}\\
&= \{Y \cap V_1 \cap V_2 \mid Y \in 2^G \text{ and } e \in F_{C_2}(Y)\}\\
&= (\mathfrak{L}_2)_{V_1 \cap V_2}.
\end{aligned}
$$

For the second equality, the inclusion relationship $\mathfrak{L}_1 \subset ((\mathfrak{L}_1)_{V_1 \cap V_2})^{V_1}$ holds a-priorily. For the converse inclusion $((\mathfrak{L}_1)_{V_1 \cap V_2})^{V_1} \subset \mathfrak{L}_1$, assume $X \in ((\mathfrak{L}_1)_{V_1 \cap V_2})^{V_1}$. Then $X \in 2^{V_1}$ and $X \cap (V_1 \cap V_2) = Y \cap (V_1 \cap V_2)$ for some $Y \in \mathfrak{L}_1$. Since both of X and Y are in 2^{V_1}, $X \cap (V_1 \cap V_2) = Y \cap (V_1 \cap V_2)$ means $X \cap V_2 = Y \cap V_2$. Then by the locality of F_{C_2} on V_2 at e, we have $F_{C_2}(X) \cap \{e\} = F_{C_2}(Y) \cap \{e\}$. But by the assumption $F_{C_1} = F_{C_2}$, $F_{C_1}(X) \cap \{e\} = F_{C_1}(Y) \cap \{e\} = \{e\}$. This means that $e \in F_{C_1}(X)$. Hence we have $X \in \mathfrak{L}_1$.
Conversely, let us assume that the three equalities hold. It follows form the second and third equalities that $F_{C_1} = F_{(C_1)_{V_1 \cap V_2}}$ and $F_{C_2} = F_{(C_2)_{V_1 \cap V_2}}$ using Proposition. 14. Further, from the first equality we have $(C_1)_{V_1 \cap V_2} = (G, V_1 \cap V_2, (\mathfrak{L}_1)_{V_1 \cap V_2}) = (G, V_1 \cap V_2, (\mathfrak{L}_2)_{V_1 \cap V_2}) = (C_2)_{V_1 \cap V_2}$ and $F_{(C_1)_{V_1 \cap V_2}} = F_{(C_2)_{V_1 \cap V_2}}$. Hence we have $F_{C_1} = F_{C_2}$. □

Colorally 16: For a cellular automaton $C = (G, V, \mathfrak{L})$, we have $F_C = \mathbf{id}$ if and only if $e \in V$ and $\mathfrak{L} = \{X \in 2^V \mid e \in \mathfrak{L}\}$.

Next we introduce composition of cellular automata on a group by defining the operation $\diamond$ for $\mathfrak{L}$, and we show that the composition $C_1 \diamond C_2$of cellular automata C_1 and C_2 is equivalent to the cellular automaton defined by the composition of the transition functions F_{C_1} and F_{C_2}.

Definition 17 (Composition): Let $C_1 = (G, V_1, \mathfrak{L}_1)$ and $C_2 = (G, V_2, \mathfrak{L}_2)$ be cellular automata on G. The composition $C_1 \diamond C_2$ of C_1 and C_2 is defined by $C_1 \diamond C_2 = (G, V_1 \cdot V_2, \mathfrak{L}_1 \diamond \mathfrak{L}_2)$ where

$$V_1 \cdot V_2 = \{v_1 v_2 \in G \mid v_1 \in V_1, v_2 \in V_2\} \text{ and}$$
$$\mathfrak{L}_1 \diamond \mathfrak{L}_2 = \{X \in 2^{V_1 \cdot V_2} \mid \{v \in V_1 \mid v^{-1}X \cap V_2 \in \mathfrak{L}_2\} \in \mathfrak{L}_1\}.$$

Example 18: We calculate $CA(2,6) \diamond CA(2,6)$. Let $V = \{0,1\}$, $\mathfrak{L} = \{\{0\},\{1\}\}$ then $CA(2,6) = (\mathbf{Z}, V, \mathfrak{L})$. We have $V + V = \{0,1,2\}$. We let $X = \{0,1\} \in 2^{V+V}$ then

$$
\begin{aligned}
(1^{-1} + X) \cap V &= (1^{-1} + \{0,1\}) \cap \{0,1\}\\
&= \{-1+0, -1+1\} \cap \{0,1\}\\
&= \{-1, 0\} \cap \{0,1\}\\
&= \{0\}\\
&\in \mathfrak{L}
\end{aligned}
$$

and

$$
\begin{aligned}
(0^{-1} + X) \cap V &= (0^{-1} + \{0,1\}) \cap \{0,1\}\\
&= \{0+0, 0+1\} \cap \{0,1\}\\
&= \{0,1\} \cap \{0,1\}\\
&= \{0,1\}\\
&\notin \mathfrak{L}.
\end{aligned}
$$

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	255	236	209	192	139	136	129	128
2	0	16	34	48	68	68	66	64
3	255	252	243	240	207	204	195	192
4	0	2	12	12	48	34	24	8
5	255	238	221	204	187	170	153	136
6	0	18	46	60	116	102	90	72
7	255	254	255	252	255	238	219	200
8	0	1	0	3	0	17	36	55
9	255	237	209	195	139	153	165	183
10	0	17	34	51	68	85	102	119
11	255	253	243	243	207	221	231	247
12	0	3	12	15	48	51	60	63
13	255	239	221	207	187	187	189	191
14	0	19	46	63	116	119	126	127
15	255	255	255	255	255	255	255	255

	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0
1	55	36	17	0	3	0	1	0
2	8	24	34	48	12	12	2	0
3	63	60	51	48	15	12	3	0
4	64	66	68	68	48	34	16	0
5	119	102	85	68	51	34	17	0
6	72	90	102	116	60	46	18	0
7	127	126	119	116	63	46	19	0
8	128	129	136	139	192	209	236	255
9	183	165	153	139	195	209	237	255
10	136	153	170	187	204	221	238	255
11	191	189	187	187	207	221	239	255
12	192	195	204	207	240	243	252	255
13	247	231	221	207	243	243	253	255
14	200	219	238	255	252	255	254	255
15	255	255	255	255	255	255	255	255

Fig. 2 Table of compositions: $CA(2,n) \diamond CA(2,m)$.

So $\{v \in V \mid v^{-1}X \cap V \in \mathfrak{L}\} = \{0\} \in \mathfrak{L}$, that is, $X = \{0,1\} \in \mathfrak{L} \diamond \mathfrak{L}$. Similarly we can calculate for other elements of 2^{V+V} and we have $\mathfrak{L} \diamond \mathfrak{L} = \{\{0\},\{2\},\{0,1\},\{1,2\}\}$. So we have $CA(2,6) \diamond CA(2,6) = (\mathbf{Z}, \{0,1,2\}, \{\{0\},\{2\},\{0,1\},\{1,2\}\})$, that is, $CA(2,6) \diamond CA(2,6) = CA(3,90)$.

Example 19: The rule numbers of the 3-neighborhood cellular automata generated by composing 2-neighborhood cellular automata is $\{0, 1, 2, 3, 8, 12, 15, 16, 17, 18, 19, 24, 34, 36, 46, 48, 51, 55, 60, 63, 64, 66, 68, 72, 85, 90, 102, 116, 119, 126, 127, 128, 129, 136, 139, 153, 165, 170, 183, 187, 189, 191, 192, 195, 200, 204, 207, 209, 219, 221, 231, 236, 237, 238, 239, 240, 243, 247, 252, 253, 254, 255\}$. There are 62 kinds of 3-neighborhood cellular automata. Figure 2 is the table of compositions for $CA(2,n)$ $(n = 0, .., 15)$.

Lemma 20: Let $C = (G, V, \mathfrak{L})$ be a cellular automaton and $V_0 \subset G$. For any $\mathbf{c} \in 2^G$,

$$F_C(\mathbf{c}) \cap V_0 = F_C(\mathbf{c} \cap (V_0 \cdot V)) \cap V_0$$

Proof. We have $F_C(\mathbf{c}) \cap V_0 = \{v_0 \in V_0 \mid v_0^{-1}\mathbf{c} \cap V \in \mathfrak{L}\}$ $= \{v_0 \in V_0 \mid \mathbf{c} \cap v_0 V \in v_0 \mathfrak{L}\} = \{v_0 \in V_0 \mid (\mathbf{c} \cap V_0 \cdot V) \cap v_0 V \in v_0 \mathfrak{L}\}$ $= F_C(\mathbf{c} \cap (V_0 \cdot V)) \cap V_0$. □

The composition of cellular automata corresponds to find a cellular automaton which global transition function is the composition of global transition functions of original cellular automata.

Theorem 21 (Fujio [2]): Let $C_1 = (G, V_1, \mathfrak{L}_1)$ and $C_1 = (G, V_2, \mathfrak{L}_2)$ be cellular automata on G. Then

$$F_{C_1} \circ F_{C_2} = F_{C_1 \diamond C_2}$$

holds.

Proof. By virtue of Proposition 2, it is sufficient to show that

$$e \in F_{C_1} \circ F_{C_2}(\mathbf{c}) \iff e \in F_{C_1 \diamond C_2}(\mathbf{c}) \ (\forall \mathbf{c} \in 2^G).$$

Let $\mathbf{c} \in 2^G$. Then we have

$$\begin{aligned} & e \in F_{C_1}(F_{C_2}(\mathbf{c})) \\ \Leftrightarrow\ & F_{C_2}(\mathbf{c}) \cap V_1 \in \mathfrak{L}_1 \ \text{(by Lemma 3)} \\ \Leftrightarrow\ & F_{C_2}(\mathbf{c} \cap V_1 \cdot V_2) \cap V_1 \in \mathfrak{L}_1 \ \text{(by Lemma 20)} \end{aligned}$$

On the other hand, since $F_{C_2}(\mathbf{c}') \cap V_1 = \{v \in V_1 \mid v^{-1}\mathbf{c}' \cap V_2 \in \mathfrak{L}_2\}$,

$$\begin{aligned} & F_{C_2}(\mathbf{c} \cap V_1 \cdot V_2) \cap V_1 \\ =\ & \{v \in V_1 \mid v^{-1}(\mathbf{c} \cap V_1 \cdot V_2) \cap V_2 \in \mathfrak{L}_2\} \end{aligned}$$

Hence by the definition of composition (Definition 17),

$$\begin{aligned} & F_{C_2}(\mathbf{c} \cap V_1 \cdot V_2) \cap V_1 \in \mathfrak{L}_1 \\ \Leftrightarrow\ & \mathbf{c} \cap V_1 \cdot V_2 \in \mathfrak{L}_1 \diamond \mathfrak{L}_2 \\ \Leftrightarrow\ & e \in F_{C_1 \diamond C_2}(\mathbf{c}) \end{aligned}$$

which establishes the assertion. □

We prove that the right distributive law holds for union and composition of cellular automata on groups.

Theorem 22: Let $C_1 = (G, V, \mathfrak{L}_1)$, $C_2 = (G, V, \mathfrak{L}_2)$ and $C_3 = (G, V_3, \mathfrak{L}_3)$ be cellular automata on a group G. Then,

$$(C_1 \cup C_2) \diamond C_3 = (C_1 \diamond C_3) \cup (C_2 \diamond C_3)$$

Proof. First, we note

$$(C_1 \cup C_2) \diamond C_3 = (G, V \cdot V_3, (\mathfrak{L}_1 \cup \mathfrak{L}_2) \diamond \mathfrak{L}_3),$$

and

$$(C_1 \diamond C_3) \cup (C_2 \diamond C_3) = (G, V \cdot V_3, (\mathfrak{L}_1 \diamond \mathfrak{L}_3) \cup (\mathfrak{L}_2 \diamond \mathfrak{L}_3)).$$

Next, we have

$$\begin{aligned} & (\mathfrak{L}_1 \cup \mathfrak{L}_2) \diamond \mathfrak{L}_3 \\ =\ & \{X \in 2^{V \cdot V_3} \mid \{v \in V \mid v^{-1}X \cap V_3 \in \mathfrak{L}_3\} \in \mathfrak{L}_1 \cup \mathfrak{L}_2\} \\ =\ & \{X \in 2^{V \cdot V_3} \mid \{v \in V \mid v^{-1}X \cap V_3 \in \mathfrak{L}_3\} \in \mathfrak{L}_1\} \\ & \cup \{X \in 2^{V \cdot V_3} \mid \{v \in V \mid v^{-1}X \cap V_3 \in \mathfrak{L}_3\} \in \mathfrak{L}_2\} \\ =\ & (\mathfrak{L}_1 \diamond \mathfrak{L}_3) \cup (\mathfrak{L}_2 \diamond \mathfrak{L}_3) \end{aligned}$$

□

We note that $C_1 \diamond (C_2 \cup C_3) = (C_1 \diamond C_2) \cup (C_1 \diamond C_3)$ does not always holds for cellular automata C_1, C_2 and C_3. For example $CA(2,6) \diamond (CA(2,2) \cup CA(2,4)) = CA(2,6) \diamond CA(2,6) = CA(3,90)$, and $(CA(2,6) \diamond CA(2,2)) \cup (CA(2,6) \diamond CA(2,4)) = CA(3,46) \cup CA(3,116) = CA(3,126)$.

Proposition 23: Let $CA(1,n)_x^{k_1}$, $CA(k_2,n_2)$ and $CA(k_2,n_3)$ be cellular automata on $\mathbf{Z}$, where $0 \le x < k_1$, and $n = 0, 1$. Then,

$$\begin{aligned} & CA(1,n)_x^{k_1} \diamond (CA(k_2,n_2) \cup CA(k_2,n_3)) \\ =\ & (CA(1,n)_x^{k_1} \diamond CA(k_2,n_2)) \cup (CA(1,n)_x^{k_1} \diamond CA(k_2,n_3)). \end{aligned}$$

Fig. 3 A configuration of $CA(3,3)$.

Proof. Let $V_1 = \{0, \cdots, k_1 - 1\}$, $\mathfrak{L}_1 = \{X \in 2^V \mid x \in X\}$, $\bar{\mathfrak{L}}_1 = \{X \in 2^V \mid x \notin X\}$, $CA(k_2,n_2) = (\mathbf{Z}, V_2, \mathfrak{L}_2)$, and $CA(k_2,n_3) = (\mathbf{Z}, V_2, \mathfrak{L}_3)$. First, we note $CA(1,0)_x^{k_1} = (\mathbf{Z}, V_1, \bar{\mathfrak{L}}_1)$, $CA(1,1)_x^{k_1} = (\mathbf{Z}, V_1, \mathfrak{L}_1)$, $CA(1,0)_x^{k_1} \diamond (CA(k_2,n_2) \cup CA(k_2,n_3)) = (\mathbf{Z}, V_1 \cdot V_2, \mathfrak{L}_1 \diamond (\mathfrak{L}_2 \cup \mathfrak{L}_3))$, and $(CA(1,n)_x^{k_1} \diamond CA(k_2,n_2)) \cup (CA(1,n)_x^{k_1} \diamond CA(k_2,n_3)) = (\mathbf{Z}, V_1 \cdot V_2, (\mathfrak{L}_1 \diamond \mathfrak{L}_2) \cup (\mathfrak{L}_1 \diamond \mathfrak{L}_3))$. Since

$$\begin{aligned} & \mathfrak{L}_1 \diamond (\mathfrak{L}_2 \cup \mathfrak{L}_3) \\ =\ & \{X \in 2^{V_1 \cdot V_2} \mid \{v \in V \mid v^{-1}X \cap V_2 \in (\mathfrak{L}_2 \cup \mathfrak{L}_3)\} \in \mathfrak{L}_1\} \\ =\ & \{X \in 2^{V_1 \cdot V_2} \mid x^{-1}X \cap V_2 \in (\mathfrak{L}_2 \cup \mathfrak{L}_3)\}, \end{aligned}$$

and

$$\begin{aligned} & (\mathfrak{L}_1 \diamond \mathfrak{L}_2) \cup (\mathfrak{L}_1 \diamond \mathfrak{L}_3) \\ =\ & \{X \in 2^{V_1 \cdot V_2} \mid \{v \in V \mid v^{-1}X \cap V_2 \in \mathfrak{L}_2\} \in \mathfrak{L}_1\} \\ & \cup \{X \in 2^{V_1 \cdot V_2} \mid \{v \in V \mid v^{-1}X \cap V_2 \in \mathfrak{L}_3\} \in \mathfrak{L}_1\} \\ =\ & \{X \in 2^{V_1 \cdot V_2} \mid x^{-1}X \cap V_2 \in \mathfrak{L}_2\} \\ & \cup \{X \in 2^{V_1 \cdot V_2} \mid x^{-1}X \cap V_2 \in \mathfrak{L}_3\}, \end{aligned}$$

we have $\mathfrak{L}_1 \diamond (\mathfrak{L}_2 \cup \mathfrak{L}_3) = (\mathfrak{L}_1 \diamond \mathfrak{L}_2) \cup (\mathfrak{L}_1 \diamond \mathfrak{L}_3)$, and $CA(1,1)_x^{k_1} \diamond (CA(k_2,n_2) \cup CA(k_2,n_3)) = (CA(1,1)_x^{k_1} \diamond CA(k_2,n_2)) \cup (CA(1,1)_x^{k_1} \diamond CA(k_2,n_3))$. Similarly, we can prove $CA(1,0)_x^{k_1} \diamond (CA(k_2,n_2) \cup CA(k_2,n_3)) =$
$(CA(1,0)_x^{k_1} \diamond CA(k_2,n_2)) \cup (CA(1,0)_x^{k_1} \diamond CA(k_2,n_3))$. □

Example 24: We note $CA(3,3) = (\mathbf{Z}, \{0,1,2\}, \{\phi, \{0\}\})$, $CA(3,102) = (\mathbf{Z}, \{0,1,2\}, \{\{0\},\{1\},\{0,2\},\{1,2\}\})$ and $CA(3,18) = (\mathbf{Z}, \{0,1,2\}, \{\{0\},\{2\}\})$. The composition of cellular automata $CA(3,3)$ and $CA(3,102)$ is

$$\begin{aligned} & CA(3,3) \diamond CA(3,102) \\ =\ & (\mathbf{Z}, \{0,1,2,3,4\}, \\ & \{\{1\},\{0,1\},\{1,4\},\{0,1,4\},\{3\},\{0,3\},\{3,4\},\{0,3,4\}\}). \end{aligned}$$

Since $CA(3,18)_1^5 = (\mathbf{Z}, \{0,1,2,3,4\}, \mathfrak{L})$ and

$$\begin{aligned} \mathfrak{L} =\ & \{s_1 \cup (1+v) \cup s_2 \\ & \quad \mid s_1 \in 2^{\{0\}},\ s_2 \in 2^{\{4\}},\ v \in \{\{0\},\{2\}\}\} \\ =\ & \{\{1\},\{0,1\},\{1,4\},\{0,1,4\},\{3\},\{0,3\},\{3,4\},\{0,3,4\}\}), \end{aligned}$$

we have $CA(3,3) \diamond CA(3,102) = CA(3,18)_1^5$.
(cf. Fig. 3, Fig. 4, Fig. 5)

Example 25: We can consider a 2-neighborhood cellular automaton as a 3-neighborhood cellular automaton and also a 3-neighborhood cellular automaton as a 5-neighborhood cellular automaton. The followings is an observation of the embeddings and compositions.

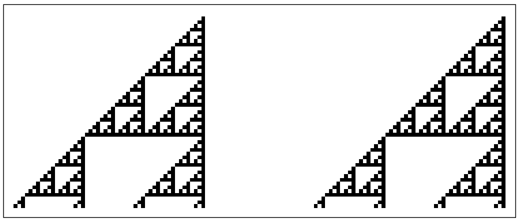

Fig. 4 A configuration of $CA(3, 102)$.

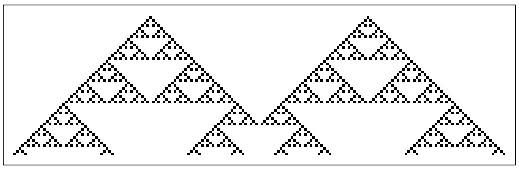

Fig. 5 An example of configurations of $CA(3, 18) = CA(3, 3)\Diamond CA(3, 102)$.

- $CA(2, 1) = (\mathbf{Z}, \{0, 1\}, \{\phi\})$
- $CA(2, 1)_0^3 = (\mathbf{Z}, \{0, 1, 2\}, \{\phi, \{2\}\}) = CA(3, 17)$
- $CA(2, 1)\Diamond CA(2, 1)$
  $= (\mathbf{Z}, \{0, 1, 2\}, \{\{0, 1\}, \{0, 2\}, \{1, 2\}, \{1\}\}) = CA(3, 236)$
- $CA(3, 17)\Diamond CA(3, 17) = (\mathbf{Z}, \{0, 1, 2, 3, 4, 5\}, \mathfrak{L})$
  $= CA(5, 3974950124) = CA(3, 236)_0^5$
- $\mathfrak{L} = \bigcup\{\{s, s \cup \{3\}, s \cup \{4\}, s \cup \{3, 4\}\} \mid s \in CA(3, 236)\}$

4. Generalization

A subset V of G is considered as a characteristic function $V : G \to 2$ where $2 = \{0, 1\}$. That is V is a function which values are

$$V(g) = \begin{cases} 0 & (g \notin V) \\ 1 & (g \in V). \end{cases}$$

Sometimes V is represented as an injection $i_V : V \to G$ where $i_V(g) = g$.

Extending our 2-states cellular automata on groups to many-states cellular automata on groups, we replace the set $2 = \{0, 1\}$ to a finite set S.

Definition 26: Let G be a group, S a finite set. A generalized cellular automaton on G is a four-tuple $C = (G, S, i_V, \mathfrak{L})$ of the group G, an injection $i_v : V \to G$, and a function $\mathfrak{L} : S^V \to S$ where S^V is the set of all functions from V to S. A configuration $\mathbf{c} : G \to S$ is a function. The global transition function $F_C : S^G \to S^G$ is defined by $F_C(\mathbf{c})(g) = \mathfrak{L}(\mathbf{c} \circ g \circ i_V)$.

Proposition 27: Let G be a group, $V \subset G$, and $S = 2 = \{0, 1\}$. And let $F_C : 2^G \to 2^G$ and $F'_C : 2^G \to 2^G$ be the global transition functions of a generalized cellular automaton $C = (G, S, i_V, \mathfrak{L})$ and a cellular automaton $C' = (G, V, \mathfrak{L})$ on G. Then F'_C coincides F_C.

Proof. We will show that for $\forall \mathbf{c} \in 2^G$

$$\begin{aligned} F_C(\mathbf{c}) &= \{g \in G \mid \mathfrak{L}(\mathbf{c} \circ g \circ i_V) = 1\} \\ &= \bigcup_{g \in G} g \cdot l_{\mathfrak{L}}(g^{-1} \cdot \mathbf{c} \cap V) \\ &= F'_C(\mathbf{c}). \end{aligned}$$

For $g \in G$, we have

$$\begin{aligned} & g \in F_{C'}(\mathbf{c}) \\ \Leftrightarrow\ & g \in \bigcup_{g \in G} g \cdot l_{\mathfrak{L}}(g^{-1} \cdot \mathbf{c} \cap V) \\ \Leftrightarrow\ & l_{\mathfrak{L}}(g^{-1} \cdot \mathbf{c} \cap V) = \{e\} \\ \Leftrightarrow\ & g^{-1} \cdot \mathbf{c} \cap V \in \mathfrak{L} \\ \Leftrightarrow\ & g^{-1}\{x \mid \mathbf{c}(x) = 1\} \cap V \in \mathfrak{L} \\ \Leftrightarrow\ & \{g^{-1}x \mid \mathbf{c}(x) = 1\} \cap V \in \mathfrak{L} \\ \Leftrightarrow\ & \{v \mid \mathbf{c}(gv) = 1\} \cap V \in \mathfrak{L} \text{ (cf. } (x = gv)) \\ \Leftrightarrow\ & \{v \mid \mathbf{c}(gv) = 1, v \in V\} \in \mathfrak{L} \\ \Leftrightarrow\ & \{v \mid \mathbf{c} \circ g \circ i_V(v) = 1\} \in \mathfrak{L} \\ \Leftrightarrow\ & \mathfrak{L}(\mathbf{c} \circ g \circ i_V) = 1 \\ \Leftrightarrow\ & g \in F_C(\mathbf{c}). \end{aligned}$$

□

Example 28: Let $G = \mathbf{Z}$, $S = \mathbf{Z}_m$, and $V = \{-r, -r+1, \cdots, 0, \cdots, +r\}$. For a polynomial $f(X) = \sum_{i=-r}^{+r} a_i X^i$, ($a_i \in \mathbf{Z}_m$), we define the function $\mathfrak{L}_{f(X)} : \mathbf{Z}_m^V \to \mathbf{Z}_m$ by

$$\mathfrak{L}(x_{-r}, x_{-r+1}, \cdots, x_0, \cdots, x_{+r}) = \sum_{i=-r}^{+r} a_{-i} x_i,$$

($(x_{-r}, x_{-r+1}, \cdots, x_0, \cdots, x_{+r}) \in \mathbf{Z_m}^V$). A configuration $\mathbf{c} \in \mathbf{Z}_m^{\mathbf{Z}}$ is represented as a formal power series $\sum c_i X^i$ where $c_i = \mathbf{c}(i)$ (cf. [6], [10]). Since $\mathbf{c} \circ j \circ i_V(i) = \mathbf{c}(j + i) = c_{j+i}$, and $\mathbf{c} \circ j \circ i_V = (c_{j-r}, c_{j-r+1}, \cdots, c_j, \cdots, c_{j+r})$, we have

$$\begin{aligned} & (\sum \mathbf{c}(i) X^i) f(X) \\ &= (\sum c_i X^i) f(X) \\ &= (\sum c_i X^i)(\sum_{i'=-r}^{+r} a_{i'} X^{i'}) \\ &= (\sum c_i X^i)(\sum_{i'=-r}^{+r} a_{-i'} X^{-i'}) \\ &= \sum (\sum_{i'=-r}^{+r} c_i a_{-i'} X^{i-i'}) \\ &= \sum ((\sum_{i'=-r}^{+r} a_{-i'} c_{j+i'}) X^j) \text{ (cf. } j = i - i') \\ &= \sum (\mathfrak{L}(c_{j-r}, c_{j-r+1}, \cdots, c_j, \cdots, c_{j+r}) X^j) \\ &= \sum (\mathfrak{L}(\mathbf{c} \circ j \circ i_V) X^j). \\ &= \sum (F_C(\mathbf{c})(j) X^j). \end{aligned}$$

The transition of the cellular automaton $C = (\mathbf{Z}, \mathbf{Z}_m, i_V, \mathfrak{L}_{f(X)})$ is corresponding to the product of polynomials (the formal power series).

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