Completing Priceable Committees: Utilitarian and Representation Guarantees for Proportional Multiwinner Voting

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Abstract

When selecting committees based on preferences of voters, a variety of different criteria can be considered. Two natural objectives are maximizing the *utilitarian welfare* (the sum of voters' utilities) and *coverage* (the number of represented voters) of the selected committee. Previous work has studied the impact on utilitarian welfare and coverage when requiring the committee to satisfy minimal requirements such as *justified representation* or *weak proportionality*. In this paper, we consider the impact of imposing much more demanding proportionality axioms. We identify a class of voting rules that achieve strong guarantees on utilitarian welfare and coverage when combined with appropriate completions. This class is defined via a weakening of priceability and contains prominent rules such as the Method of Equal Shares. We show that committees selected by these rules (i) can be completed to achieve optimal coverage and (ii) can be completed to achieve an asymptotically optimal approximation to the utilitarian welfare if they additionally satisfy EJR+. Answering an open question of Elkind et al. (2022), we use the *Greedy Justified Candidate Rule* to obtain the best possible utilitarian guarantee subject to proportionality. We also consider completion methods suggested in the participatory budgeting literature and other objectives besides welfare and coverage.

1 Introduction

In multiwinner voting, a subset of candidates (often called a *committee*) needs to be chosen in a way that reflects the preferences of a set of voters over these candidates. The (computational) social choice literature has identified many different, often competing, criteria for committees (Felsenthal and Maoz 1992; Elkind et al. 2017; Lackner and Skowron 2022). In particular, Faliszewski et al. (2017) distinguish between three important desiderata referred to as *individual excellence*, *diversity*, and *proportional representation*.

This paper focuses on *approval-based* multiwinner voting, where voters cast approval ballots over individual candidates (Lackner and Skowron 2022). In this setting, it is commonly assumed that each voter evaluates a committee by counting the number of approved committee members. This number is often referred to as the *utility* that the voter derives from the committee, and a voter is said to be *represented* by a committee if their utility is nonzero. Using these conventions, the first two desiderata of Faliszewski et al. (2017) have straightforward interpretations: Individual excellence is measured by the *utilitarian welfare* of a committee, i.e., the sum over the voters' utilities. Utilitarian welfare is maximized when choosing the candidates with the highest number of approvers, and the voting rule that selects this committee is known as *Approval Voting (AV)*. The diversity of a committee is captured by its *coverage*, i.e., the number of voters that are represented by it. The (computationally intractable) voting rule that selects committees maximizing coverage is known as *Chamberlin–Courant (CC)*. The third desideratum, proportional representation, is much harder to formalize, and the literature has identified a host of different (axiomatic and quantitative) measures to assess the proportionality of a committee (Aziz et al. 2017; Sánchez-Fernández et al. 2017; Skowron 2021; Peters, Pierczyński, and Skowron 2021; Brill and Peters 2023); a prominent example is the *extended justified representation (EJR)* axiom.

The study of trade-offs between the three desiderata has been initiated by Lackner and Skowron (2020), who analyzed how closely classic multiwinner rules such as PAV and Phragmén's rules approximate the optimal utilitarian welfare and coverage. The resulting bounds are referred to as *utilitarian* and *representation guarantees*, respectively. Besides guarantees for specific rules, they also proved upper bounds on the utilitarian and representation guarantees of rules satisfying minimal proportionality requirements. In follow-up work, Elkind et al. (2022) studied which guarantees can be achieved if the committee is required to satisfy the *justified representation (JR)* axiom. While Lackner and Skowron (2020) gave an upper bound of $2/\lfloor\sqrt{k}\rfloor - 1/k$ for the utilitarian guarantee of any committee satisfying JR — with a higher guarantee being better — Elkind et al. (2022) were almost able to close this gap by showing that a utilitarian guarantee of $\frac{2-\varepsilon}{1+\sqrt{k}}$ is possible for any $\varepsilon > 0$ and sufficiently large k.[1] Further, they showed that a representation guarantee of $\frac{3}{4}$ is possible in conjunction with EJR and that

[1]Here, k denotes the size of the committee. Lackner and Skowron (2020) proved the upper bound for an axiom they call *weak proportionality*, but — as Elkind et al. (2022) observe — the bound also holds for JR. Moreover, we note that Elkind et al. (2022) phrase their bounds in terms of the "price of JR," with lower prices being better: a price of P corresponds to a guarantee of $1/P$.

both guarantees can be simultaneously approximated close to optimal while requiring JR. They achieve the bounds for JR by using a variant of the *sequential Chamberlin–Courant* rule, and the bound for EJR is achieved by the somewhat unnatural *GreedyEJR* rule. Both of these rules construct committees by matching committee members to voters approving them (see Section 3 for details), in a way that is reminiscent of *priceability* and rules such as the *Method of Equal Shares (MES)* (Peters and Skowron 2020). These rules suffer from a common drawback: They often select strictly fewer than the desired number of candidates, and thus require a *completion method* to output a full committee.

Our Contribution In this paper, we study which representation and utilitarian guarantees are possible when requiring much more demanding proportionality axioms than those considered by Lackner and Skowron (2020) and Elkind et al. (2022). We identify a class of voting rules that achieve strong guarantees on utilitarian welfare and coverage when combined with appropriate completion methods. This class, defined via a weakening of priceability we call *affordability*, contains prominent voting rules such as MES.

We show that any affordable committee, when completed with the CC rule, achieves an optimal representation guarantee of $\frac{3}{4}$. This general result has several advantages over earlier results on representation guarantees: First, our result implies that any proportionality axiom which is compatible with affordability, such as EJR+ (Brill and Peters 2023) or FJR (Peters, Pierczyński, and Skowron 2021), is also compatible with a representation guarantee of $\frac{3}{4}$. Second, our result uses much simpler rules: Instead of the unnatural and computationally intractable GreedyEJR rule, we can use attractive rules such as MES to achieve this bound.

For utilitarian guarantees, we show that any affordable committee that additionally satisfies EJR+ (which is more demanding than EJR but still satisfied by MES), completed with AV, achieves a utilitarian guarantee of $\Omega(\frac{1}{\sqrt{k}})$. We further show that the recently introduced *Greedy Justified Candidate Rule (GJCR)* (Brill and Peters 2023), completed with AV, achieves a utilitarian guarantee of exactly $\frac{2}{\sqrt{k}} - \frac{1}{k}$, which is the best possible guarantee for proportional rules. This answers an open question by Elkind et al. (2022), who were only able to obtain this bound asymptotically. Moreover, we establish an interesting distinction between EJR and EJR+, by showing that GreedyEJR (which satisfies EJR but not EJR+) does not achieve a utilitarian guarantee of $\Omega(\frac{1}{k^c})$ for any $c < 1$, even on instances where it is exhaustive. Thus, EJR is not sufficient for strong utilitarian guarantees.

We also study trade-offs between utilitarian and representation guarantees and show that one can simultaneously achieve a representation guarantee of $\frac{3}{4} - o(1)$ and a utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{1}{k}$ together with EJR+, improving and simplifying a similar result by Elkind et al. (2022) for JR.

Finally, we consider completion methods that are relevant for specific applications of multiwinner voting: completion by the maximin support method (Sánchez-Fernández et al. 2021), which is relevant for the security of blockchain systems (Cevallos and Stewart 2021), and completion by perturbation or by varying the budget, which have been discussed in the participatory budgeting (PB) literature (Rey and Maly 2023). Omitted proofs can be found in the appendix.

Related Work Besides the papers by Lackner and Skowron (2022) and Elkind et al. (2022), utilitarian and representation guarantees have been studied in the context of PB by Fairstein et al. (2022), who proved some guarantees, but mostly impossibility results. The topic of completing non-exhaustive voting rules is especially relevant in PB, where various ways to complete the Method of Equal Shares have been discussed and analyzed on real-world data (Peters, Pierczyński, and Skowron 2021; Faliszewski et al. 2023; Boehmer et al. 2023). One of the completion methods we study in this paper, "completion by varying the budget," was recently used in a real-world PB election in the Polish city of Wielizka (see https://equalshares.net for details).

In a similar spirit as the paper by Elkind et al. (2022), Michorzewski, Peters, and Skowron (2020) and Tang, Wang, and Zhang (2020) studied the "price of fairness" in the setting of budget division (or probabilistic social choice).

2 Preliminaries

We are given a finite set $C = \{c_1, \dots, c_m\}$ of $m > 0$ *candidates* and a finite set $N = \{1, \dots, n\}$ of $n > 0$ *voters*. Voters cast *approval ballots* over candidates: for each voter $i \in N$, the set $A_i \subseteq C$ consists of the candidates that are approved by voter i. Together $A = (A_i)_{i \in N}$ forms an *approval profile*. For a candidate $c \in C$, we call the members of the set $N_c = \{i \in N : c \in A_i\}$ the *approvers* of c. Further, we are given a *committee size* $k \leq m$. We call any set $W \subseteq C$ of size $|W| \leq k$ a *committee*. If $|W| = k$, we say that W is *exhaustive*. Finally, we call $\mathcal{I} = (A, C, k)$ an *instance*.

A *(multiwinner voting) rule* r is a function mapping instances $\mathcal{I} = (A, C, k)$ to non-empty sets of committees $r(\mathcal{I})$. We often say that a rule "satisfies" a property if, for each instance $\mathcal{I}$, all committees in $r(\mathcal{I})$ satisfy the property.

Proportionality Axioms First, we consider the classical justified representation axioms of Aziz et al. (2017). For an integer $\ell \in \mathbb{N}$, a group $N' \subseteq N$ of voters is said to be *ℓ-large* if $|N'| \geq \ell \frac{n}{k}$ and *ℓ-cohesive* if $\left|\bigcap_{i \in N'} A_i\right| \geq \ell$. The *justified representation (JR)* axiom requires that for any 1-large and 1-cohesive group N', there exists some $i \in N'$ with $|A_i \cap W| \geq 1$. *Extended Justified Representation (EJR)* requires that for all $\ell \in [k]$ and for any ℓ-large and ℓ-cohesive group N' there exists some $i \in N'$ with $|A_i \cap W| \geq \ell$. Finally, the recently introduced *EJR+* axiom requires that for any ℓ-large and 1-cohesive group N', there either exists some $i \in N'$ with $|A_i \cap W| \geq \ell$ or it holds that $\bigcap_{i \in N'} A_i \subseteq W$, for all $\ell \in [k]$. It is easy to see that EJR+ implies EJR which in turn implies JR. We also use the following quantitative notion (Brill and Peters 2023).

Definition 1. *Given a committee W, a candidate $c \in C \setminus W$ is (t, ℓ)*-represented *if there is no ℓ-large group $N' \subseteq N_c$ with $\frac{1}{|N'|} \sum_{i \in N'} |A_i \cap W| < t$. For a given function $f : \mathbb{N} \to \mathbb{R}$, a committee is f*-representative *if every unchosen candidate $c \in C \setminus W$ is $(f(\ell), \ell)$-represented for any $\ell \in [k]$.*

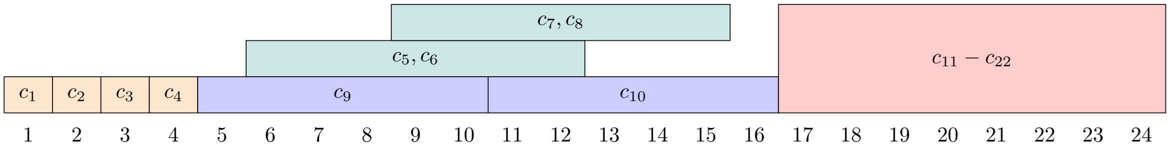

Figure 1: Instance with $n = 24$ voters and $m = 22$ candidates considered in Example 1. Voters correspond to integers and approve candidates placed above them, e.g., voter 1 approves candidate c_1 and voter 9 approves candidates c_5, c_6, c_7, c_8, c_9.

This is a stronger notion than the *proportionality degree* (Skowron 2021), in the sense that an f-representative committee (or rule) also has a proportionality degree of f, as the definition just removes the cohesiveness requirement. The representativeness of a committee is polynomial-time computable, and as we show later, contrary to the proportionality degree, it allows us to obtain bounds on the utilitarian guarantee of a voting rule.

Rules Given an instance $\mathcal{I} = (A, C, k)$ and a committee $W \subseteq C$, we let $\mathsf{uw}(W) = \sum_{i \in N} |A_i \cap W|$ denote the *utilitarian welfare* of W and $\mathsf{cov}(W) = |\{i \in N \colon A_i \cap W \neq \emptyset\}|$ the *coverage* of W. The *Approval Voting (AV)* rule selects a committee of size k maximizing the utilitarian welfare. The *Chamberlin–Courant (CC)* rule selects a committee of size k maximizing the coverage. The *sequential Chamberlin–Courant (seq-CC)* rule starts with an empty committee $W = \emptyset$ and greedily adds candidates $c \notin W$ maximizing $\mathsf{cov}(W \cup \{c\})$ until W has size k.

Next, we define three rules we study throughout the paper. The *Method of Equal Shares* (MES) (Peters and Skowron 2020) starts with an empty committee $W = \emptyset$ and with a budget $b_i = \frac{k}{n}$ for each voter $i \in N$. Then, for each unselected candidate $c \notin W$, it calculates the value ρ such that $\sum_{i \in N_c} \min(b_i, \rho) = 1$. If such a ρ does not exist for any unselected candidate, MES terminates. Otherwise, it selects the candidate c with the minimum ρ, adds it to W, and sets $b_i = b_i - \min(b_i, \rho)$ for each $i \in N_c$.

The *Greedy Justified Candidate Rule* (GJCR) (Brill and Peters 2023) also starts with an empty committee $W = \emptyset$. GJCR then repeatedly checks if there is a candidate c, set $N' \subseteq N_c$, and $\ell \in [k]$ with $|N'| \geq \frac{\ell n}{k}$ and $|A_i \cap W| < \ell$ for all $i \in N'$. If there is such a candidate, it picks the candidate maximizing $|N'|$ and adds it to W.

The *GreedyEJR* rule (Bredereck et al. 2019; Peters, Pierczyński, and Skowron 2021) works similarly to GJCR, but iterates over voter groups rather than candidates (which makes the rule computationally intractable). Starting with an empty committee W, GreedyEJR finds the largest ℓ such that there is an ℓ-cohesive and ℓ-large group N', adds ℓ candidates from $\bigcup_{i \in N'} A_i$ to W, and deletes all voters in N'; this is repeated until no further group can be found.

MES and GJCR satisfy EJR+ (Brill and Peters 2023), whereas GreedyEJR satisfies EJR but not EJR+. All three rules may return non-exhaustive committees (Example 1).

Utilitarian and Representation Guarantees For a given instance $\mathcal{I}$, we define the *utilitarian ratio* of a committee W as the fraction of the highest achievable utilitarian welfare in this instance, i.e., $\frac{\mathsf{uw}(W)}{\mathsf{uw}(AV(\mathcal{I}))}$. Analogously, $\frac{\mathsf{cov}(W)}{\mathsf{cov}(CC(\mathcal{I}))}$ is called the *representation ratio* of W.

A utilitarian or representation *guarantee* is a lower bound on the utilitarian or representation ratio of a committee or a set of committees (such as those output by a rule).

Completion Methods For a non-exhaustive committee W, we let $\overline{AV}(W)$ denote the completion of W with the highest approval score. Formally, $\overline{AV}(W) = W \cup AV(\mathcal{I}')$, where $\mathcal{I}' = (A, C \setminus W, k - |W|)$. Further, $\overline{CC}(W)$ is the completion of W with the highest coverage. Formally, $\overline{CC}(W) = W \cup CC(\mathcal{I}'')$, where $\mathcal{I}'' = (A', C \setminus W, k - |W|)$ and A' results from A by deleting all voters represented by W. Completion via seq-CC is defined analogously.

Example 1. *Consider the instance depicted in Figure 1 with $n = 24, k = 12$, and 22 candidates. In this instance, AV selects the candidates $\{c_{11}, \dots, c_{22}\}$ to achieve a utilitarian welfare of $12 \cdot 8 = 96$. This committee, however, does not satisfy EJR (or even JR), as for instance voters 5 to 10 together with candidate c_9 witness a violation of JR (these voters deserve $6\frac{k}{n} = 3$ seats on the committee). The optimal coverage is $n = 24$ and can be achieved, e.g., by $\{c_1, \dots, c_{12}\}$.*

***MES** begins by assigning everyone a budget of $\frac{1}{2}$. Four candidates out of c_{11} to c_{22} would be bought first, with a ρ of $\frac{1}{8}$ each. Then, c_5, c_6, c_7 can be bought with $\rho = \frac{1}{7}$ each. The approvers of c_8 would then have a budget of $4(\frac{1}{2} - \frac{3}{7}) + 3(\frac{1}{2} - \frac{2}{7}) = \frac{13}{14} < 1$, which is not enough to buy c_8. Both the approvers of c_9 and of c_{10}, on the other hand, have enough budget left to buy their candidate. The approvers of any of c_1 to c_4 have a budget of $\frac{1}{2}$ each and therefore, can not buy the candidate they approve. Hence, MES would (up to tie-breaking) return the committee $W = \{c_5, c_6, c_7, c_9, \dots, c_{14}\}$ with a utilitarian welfare of $2 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 12 \cdot 4 = 65$ and a utilitarian ratio of $\frac{65}{96}$. Since it covers all but 4 voters, its representation ratio is $\frac{20}{24} = \frac{5}{6}$. The completion $\overline{CC}(W)$ additionally contains three of c_1, c_2, c_3, leading to a representation ratio of $\frac{23}{24}$ and a utilitarian ratio of $\frac{68}{96}$, while the completion $\overline{AV}(W)$ would contain three of c_{14} to c_{22}, leading to a utilitarian ratio of $\frac{86}{96}$ and an (unchanged) representation ratio of $\frac{5}{6}$.*

***GJCR** could select $\{c_5, c_6, c_7, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}\}$ (notably leaving out c_9, but still selecting c_{10} since the voters 13 to 16 deserve two seats in the committee). Thus, GJCR completed with CC selects $c_1, \dots, c_4$, leading to a perfect representation ratio of 1, while GJCR completed with AV selects four out of c_{15} to c_{22}, reaching a utilitarian ratio of $\frac{91}{96}$.*

***GreedyEJR** could select $\{c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}\}$ by first considering the 4-cohesive group 17 to 24, then the 2-cohesive group 6 to 12 with candidates c_5 and c_6, and fi-*

nally the 1*-cohesive group* 13 *to* 16 *with candidate* c_{10}. *Notably, this committee violates EJR+ since the approvers of* c_9 *would deserve* 3 *seats, but get at most* 2*. Since they do not form a cohesive group, they could not get selected by GreedyEJR and therefore GreedyEJR does not satisfy EJR+.*

3 Affordable Committees

The three rules that play an important role in this paper are the Method of Equal Shares, the Greedy Justified Candidate Rule, and GreedyEJR. A common feature of these three rules is that they — explicitly or implicitly — construct (fractional) matchings between committee members and their approvers: each voter starts off with a "voting weight" of k/n (so that the total voting weight per committee member is 1), and each time a candidate is added to the committee under one of these rules, there is 1 unit of (previously unused) voting weight supporting this candidate, which then gets used up by the candidate.

This is reminiscent of the notion of *priceability* (Peters and Skowron 2020), which assumes that each voter owns an equal amount of "money" that can be spent on candidates they approve. A committee is called priceable if it is possible to allocate the money in such a way that each committee member is paid for and there is not enough leftover money to pay for the inclusion of further candidates.[2]

To formalize this, consider an instance $\mathcal{I} = (A, C, k)$. A *payment system* is a collection $(p_i)_{i\in N}$ containing a function $p_i\colon C \to \mathbb{R}_{\geq 0}$ for each voter $i \in N$. A committee $W \subseteq C$ is said to be *priceable* if there exists a payment system $(p_i)_{i\in N}$ satisfying the following constraints:

- **C1** $p_i(c) = 0$ if $c \notin A_i$ for all $c \in C$ and $i \in N$
- **C2** $\sum_{c\in C} p_i(c) \leq \frac{k}{n}$ for all $i \in N$
- **C3** $\sum_{i\in N} p_i(c) = 1$ for all $c \in W$
- **C4** $\sum_{i\in N} p_i(c) = 0$ for all $c \notin W$
- **C5** $\sum_{i\in N_c} \left(\frac{k}{n} - \sum_{c\in C} p_i(c)\right) \leq 1$ for all $c \notin W$.

Note that the left-hand side of constraint **C5** exactly corresponds to the money leftover with approvers of candidate c.

MES is probably the most prominent example of a priceable rule (i.e., a rule that always outputs priceable committees). GJCR and GreedyEJR, on the other hand, do not necessarily output priceable committees as they might violate **C5** (while still satisfying **C1–C4**). As it turns out, constraints **C1–C4** are sufficient to prove strong guarantees. We call such committees affordable.

Definition 2. *A committee $W \subseteq C$ is* affordable *if there is a payment system for W satisfying constraints **C1–C4**.*

Clearly, every priceable committee is affordable. Affordability is a rather weak requirement: it simply requires that each committee member can be associated with a sufficient amount of voter support. Every subcommittee of an affordable committee is affordable, and so is the empty committee.

[2]Priceability has been studied in various forms (Peters et al. 2021; Munagala, Shen, and Wang 2022; Brill et al. 2023; Lackner and Maly 2023; Brill and Peters 2023).

Affordability (and thus priceability) is not generally compatible with exhaustiveness. For example, consider an instance with two voters and two candidates such that each candidate is approved by exactly one voter, and let $k = 1$.

GJCR and GreedyEJR output affordable committees. Another example of a rule ensuring affordability is the variant of sequential CC employed by Elkind et al. (2022). In light of the positive results on affordable committees in Section 4, it is perhaps not surprising that both rules used by Elkind et al. (2022) satisfy this notion. However, as we will show, affordable rules beyond those considered by Elkind et al. (2022) can be used to obtain strictly stronger guarantees.

Rules returning priceable or affordable committees are among the most popular rules studied in the approval-based committee voting literature, as they are able to achieve strong proportionality guarantees with often quite simple (greedy) approaches. However, since priceable rules cannot be exhaustive, they require a completion step. The effects of this completion step are largely unexplored. By providing utilitarian and representation guarantees for the completions of affordable committees, our work contributes to the ongoing discussion on how to make priceable rules exhaustive.[3]

We begin our analysis of affordable committees with the following simple observation, which was already implicit in the work of Elkind et al. (2022, Theorem 3.6) and Peters, Pierczyński, and Skowron (2021, Lemma 1).

Observation 1. *Let $W \subseteq C$ be an affordable committee. Then the coverage of W is at least $\frac{|W|}{k}n$.*

Proof. If the coverage of W was less than $\frac{|W|}{k}n$, then there must exist a represented voter who pays more than $\frac{|W|}{\frac{|W|}{k}n} = \frac{k}{n}$, and thus W could not have been affordable. □

In particular, this implies that an exhaustive affordable committee covers all voters and thus has a coverage of n. We can also show that any exhaustive affordable committee has a utilitarian guarantee of $\Omega(\frac{1}{k})$.

Observation 2. *Let W be an exhaustive affordable committee. Then W has a representation guarantee of* 1 *and a utilitarian guarantee of at least $\frac{1}{k}$.*

Proof. The representation guarantee immediately follows from Observation 1. For the utilitarian guarantee, we observe that each candidate in W must have at least $\frac{n}{k}$ ap-

[3]There is variant of priceability that replaces k in **C2** and **C5** with an arbitrary budget $B > 0$. This weakening of priceability is compatible with exhaustiveness. Examples of exhaustive rules satisfying this property include Phragmén's sequential rule (Phragmén 1895), leximax-Phragmén (Brill et al. 2017), the Maximin Support Method (Sánchez-Fernández et al. 2021), and Phragmms (Cevallos and Stewart 2021). Since these rules do not necessarily output affordable committees, our results in Section 4 do not apply to them. This explains why, e.g., the representation guarantee of Phragmén's sequential rule is only $\frac{1}{2}$ (Lackner and Skowron 2020), whereas $\frac{3}{4}$ is possible by completing affordable committees. In **??**, we prove that every exhaustive rule satisfying this variant of priceability has a representation guarantee of exactly $\frac{1}{2}$.

provals, while each candidate outside W can have at most n, hence the utilitarian ratio is at least $\frac{n}{nk} = \frac{1}{k}$. □

While the representation guarantee is optimal, the utilitarian guarantee is far from: Lackner and Skowron (2020) have shown that proportional rules can achieve a utilitarian guarantee of $\Theta(\frac{1}{\sqrt{k}})$. In Section 4.1, we extend Observation 2 and show that any f-representative — with $f \in \Omega(\ell)$— affordable committee containing at least a constant fraction of candidates has a utilitarian guarantee of $\Omega(\frac{1}{\sqrt{k}})$. In particular, this will imply that both MES and GJCR have a utilitarian guarantee of $\Omega(\frac{1}{\sqrt{k}})$ when completed with AV.[4]

4 Utilitarian and Representation Guarantees

We show that optimal utilitarian and representation guarantees can be achieved via completing affordable committees.

4.1 Utilitarian Guarantees

We begin with a general statement, lower bounding the utilitarian ratio of a committee based on the highest number of approvals of any unselected candidate.

For a given committee W, let $\overline{\mathrm{s}}(W) = \max_{c \notin W} |N_c|$ denote the largest approval score of an unselected candidate.

Lemma 3. *Let $f : \mathbb{N} \to \mathbb{R}$ and let W be an f-representative committee with $\overline{\mathrm{s}}(W) \geq t\frac{n}{k}$ for some $t \in \mathbb{N}$. Then, W has a utilitarian guarantee of* $\min\left(\frac{1}{2}, \frac{f(t)}{2k}\right)$.

Hence, if there is an unselected candidate with $\omega(\frac{n}{\sqrt{k}})$ approvals left and our committee is $\Theta(\ell)$-representative, we have a utilitarian guarantee of $\omega(\frac{1}{\sqrt{k}})$.

For affordable committees, we can also show that a similar bound can be obtained for the case $\overline{\mathrm{s}}(W) \in o(\frac{n}{\sqrt{k}})$ (independently of W being f-representative).

Lemma 4. *Let W be an affordable committee with $\overline{\mathrm{s}}(W) \leq t\frac{n}{k}$. Then, W has a utilitarian guarantee of* $\min\left(\frac{1}{2}, \frac{|W|}{2tk}\right)$.

Thus, in this case, we also obtain a bound better than $\Omega(\frac{1}{\sqrt{k}})$ in case $\overline{\mathrm{s}}(W) \in o(\frac{n}{\sqrt{k}})$ and $\frac{|W|}{k} \in \Omega(1)$.

Piecing these two lemmas together, we obtain a bound for both cases.

Theorem 5. *Any affordable and f-representative committee W has a utilitarian guarantee of* $\frac{\min\left(|W|/\sqrt{k}, f(\sqrt{k})\right)}{2k}$.

As a consequence, since any committee satisfying EJR+ is $\frac{\ell-1}{2}$-representative (Brill and Peters 2023), MES and GJCR both have a utilitarian guarantee of $\Omega(\frac{1}{\sqrt{k}})$ in case they select at least a constant fraction of candidates. The same proof idea applies to completions of non-exhaustive committees.

Corollary 6. *Let $f \in \Omega(\ell)$ and W be an affordable and f-representative committee. Then $\overline{AV}(W)$ has a utilitarian guarantee of $\Omega(\frac{1}{\sqrt{k}})$.*

[4] Since MES and GJCR have been proposed relatively recently (and they are not exhaustive), these rules have not been considered in the paper by Lackner and Skowron (2020).

For committees satisfying EJR+, we give a precise bound.

Corollary 7. *Let W be an affordable committee that satisfies EJR+. Then, $\overline{AV}(W)$ has a utilitarian guarantee of $\frac{1}{4\sqrt{k}} - \frac{1}{2k}$.*

Interestingly, these strong guarantees for affordable EJR+ committees do not hold for EJR.

Theorem 8. *For any $0 < c < 1$, there are exhaustive affordable committees satisfying EJR with a utilitarian ratio of $\mathcal{O}(\frac{1}{k^{1-c}})$.*

We point out that the committee witnessing the utilitarian ratio of $\mathcal{O}(\frac{1}{k^{1-c}})$ can be selected by the GreedyEJR rule, since no cohesive groups with an ℓ larger than k^c exists. Thus, the GreedyEJR rule, even when it is exhaustive (and thus cannot be completed any further), can select committees with a very suboptimal utilitarian ratio.

We now show that a utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{1}{k}$ can be achieved via completing the Greedy Justified Candidate Rule with AV, thus meeting the lower bound proven by Lackner and Skowron (2020) and answering an open question by Elkind et al. (2022). For our proof, we use that GJCR behaves very similarly to AV at the beginning of its iterations, picking the candidates with the highest approval score, until one of them would not be "proportional" anymore.

Theorem 9. *The Greedy Justified Candidate Rule completed with AV has a utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{1}{k}$.*

Proof. Fix an instance and let W be the committee output by GJCR. Let $c_1, \ldots, c_k$ be the candidates selected by AV in order of their approval scores. Further, let c_{i+1} be the first among them which was not selected by $\overline{AV}(W)$. Without loss of generality, we assume that all other candidates selected by $\overline{AV}(W)$ with the same approval score as c_{i+1} come before it. Let $\alpha > 0$ such that $|N_{c_{i+1}}| = \alpha\frac{n}{k}$. We distinguish two cases. If $\alpha < 1$, we have $|N_{c_{i+1}}| < \frac{n}{k}$. Thus, c_{i+1} could not have been selected by GJCR, since all candidates selected by it have an approval score of at least $\frac{n}{k}$. Hence, $\overline{AV}(W)$ and AV select candidates with the same approval scores, and therefore the utilitarian guarantee is 1.

For the case of $\alpha \geq 1$, we first want to show that $i \geq \lceil \alpha \rceil$. Since the candidate c_{i+1} was not selected, we know that there is at least one voter $j \in N_{c_{i+1}}$ with $|A_j \cap W| \geq \lfloor \alpha \rfloor$ such that the candidates selected for j (meaning that $j \in N'$ when this candidate was chosen by GJCR) must have at least as many approvals as c_{i+1}. Therefore, $i \geq \lfloor \alpha \rfloor$ immediately holds (and thus also $i \geq \lceil \alpha \rceil$ if $\alpha \in \mathbb{N}$ or if there are at least $\lceil \alpha \rceil$ candidates selected for j). If $\alpha \notin \mathbb{N}$ and if there are less than $\lceil \alpha \rceil$ selected for j, we know that in the payment system constructed by GJCR, the budget of j and thus also the entire budget cannot be fully spent, since the voter pays at most $\frac{\lfloor \alpha \rfloor k}{\alpha n} < \frac{k}{n}$. Therefore, the approval voting step selects at least one candidate, which must have at least an approval score of $\alpha\frac{n}{k}$. Hence, in this case, $i \geq \lfloor \alpha \rfloor + 1 = \lceil \alpha \rceil$. Finally, we know that the other $k - i$ candidates selected by GJCR must have at least $\frac{n}{k}$ approvals since they were selected by GJCR, while we know that the candidates selected by AV afterwards must have an approval score of at most $\alpha\frac{n}{k}$.

We can therefore lower-bound the utilitarian guarantee as

$$\frac{\mathsf{uw}(\overline{AV}(W)))}{\mathsf{uw}(AV)} \geq \frac{\mathsf{uw}(\{c_1,\ldots,c_i)\}+(k-i)\frac{n}{k}}{\mathsf{uw}(\{c_1,\ldots,c_i\})+(k-i)\alpha\frac{n}{k}}$$
$$\geq \frac{i\alpha\frac{n}{k}+(k-i)\frac{n}{k}}{i\alpha\frac{n}{k}+(k-i)\alpha\frac{n}{k}} = \frac{\alpha i+(k-i)}{\alpha i+(k-i)\alpha} = \frac{\alpha i+(k-i)}{k\alpha}$$
$$= \frac{(\alpha-1)i+k}{k\alpha} \geq \frac{(\alpha-1)\alpha+k}{k\alpha} = \frac{\alpha^2+k}{k\alpha} - \frac{1}{k}.$$

By the AM-GM inequality we know that $(\alpha^2+k)/2 \geq \alpha\sqrt{k}$ and thus we obtain a utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{1}{k}$. □

In the same vein, we can obtain a very similar bound for MES completed by AV.

Theorem 10. *The Method of Equal Shares completed by AV has a utilitarian guarantee of* $\frac{2}{\sqrt{k}} - \frac{2}{k}$.

4.2 Representation Guarantees

We begin the analysis of representation guarantees with a simple lemma: Generalizing the proof of Theorem 3.6 by Elkind et al. (2022) gives us a general bound on the representation guarantee of $\overline{CC}(W)$.

Lemma 11. *Let W be a committee with a representation ratio of ρ. Then, $\overline{CC}(W)$ has a representation guarantee of* $\rho + (1-\rho)(\frac{k-|W|}{k})$.

This allows us to derive the same representation guarantee as Elkind et al. (2022), but for any affordable committee completed with CC.

Corollary 12. *Let W be an affordable committee. Then $\overline{CC}(W)$ has a representation guarantee of* $\frac{3}{4}$.

As a consequence, the proportionality notions EJR+ and FJR (Peters, Pierczyński, and Skowron 2021) [5] are compatible with a representation guarantee of $\frac{3}{4}$.

Corollary 13. *For every instance, there exist committees W and W', each with representation ratio of at least $\frac{3}{4}$, such that W satisfies EJR+ and W' satisfies FJR.*

Next, we turn to completing committees via the *sequential* Chamberlin–Courant (seq-CC) rule. It is well known that seq-CC achieves a representation guarantee of $1-\frac{1}{e} \approx 0.632$ and that this is the best possible for any polynomial-time rule (assuming P $\neq$ NP) (Feige 1998). Perhaps surprisingly, we show that the analysis of seq-CC does not depend on the first steps being taken optimally, but that *any* affordable committee, completed with seq-CC, has a representation guarantee of $1-\frac{1}{e}$. Thus, additionally imposing strong proportionality notions (e.g., EJR+) does not lower the best possible representation guarantee of a polynomial-time computable rule.

Theorem 14. *Let W be an affordable committee. Then, $\overline{\text{seq-CC}}(W)$ has a representation guarantee of* $1-\frac{1}{e}$.

[5]FJR or fully justified representation is a strengthening of EJR, which relaxes the cohesiveness requirement and thereby strengthens the axiom. It is still satisfiable via a generalization of the GreedyEJR rule.

4.3 Combining Utilitarian and Representation Guarantees

Finally, we turn to combining both types of guarantees. For both ratios, we show that they can be approximated optimally, while still maintaining a close-to-optimal approximation to the other ratio. We first show that a representation guarantee of exactly $\frac{3}{4}$ is compatible with an asymptotically optimal utilitarian guarantee and EJR+.

Theorem 15. *For any $c > 2$, there exists a committee satisfying EJR+ with a representation guarantee of $\frac{3}{4}$ and a utilitarian guarantee of* $\min\left(\frac{2}{\sqrt{ck}} - \frac{2}{k}, \frac{(c-2)^2}{4c^2} - \frac{1}{k}\right) \in \Omega(\frac{1}{\sqrt{k}})$.

Moreover, we show that one can achieve a representation guarantee of $\frac{3}{4} - \mathcal{O}(\frac{1}{\sqrt{k}})$ and a utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{1}{k}$ simultaneously with EJR+, thereby strengthening the result of Elkind et al. (2022, Theorem 3.3) with a simpler proof.

Theorem 16. *There exists a committee satisfying EJR+ with a representation guarantee of $\frac{3}{4} - \frac{2}{\sqrt{k}}$ and a utilitarian guarantee of* $\frac{2}{\sqrt{k}} - \frac{1}{k}$.

5 Other Completion Methods

There are other ways to complete committees besides AV and (seq-)CC. In this section, we discuss three of them.

5.1 Maximin Support

The *maximin support* of a committee is a quantitative notion measuring the overrepresentation of voters. Cevallos and Stewart (2021) argued that this measure is relevant for the security of a blockchain consensus mechanism known as *Nominated Proof-of-Stake*. They also showed that the maximin support method, a voting rule introduced by Sánchez-Fernández et al. (2021), approximates this measure.

To formally define these concepts, we adopt the formulation of Cevallos and Stewart (2021). The maximin support $\mathrm{mms}(W)$ of a committee W is given by $\min_{S\subseteq W, S\neq\emptyset} \frac{1}{|S|}|\{i \in N\colon A_i \cap S \neq \emptyset\}|$, and the goal is to maximize this value. The *maximin support method (MMS)* starts with an empty committee W and iteratively adds a candidate $c \notin W$ maximizing $\mathrm{mms}(W \cup \{c\})$. This rule achieves a $\frac{1}{2}$-approximation to the maximin support objective (Cevallos and Stewart 2021), but does not satisfy EJR (Sánchez-Fernández et al. 2021).

First, we give an example of an instance in which every committee satisfying EJR approximates the optimal maximin support value by a factor of at most $\frac{2}{3}$.

Example 2. *Consider an instance with $2k$ voters and $2k$ candidates $c_1,\ldots,c_{2k}$. Voters $i = 1,\ldots,k$ each only approve candidate c_i, while voters $i = k+1,\ldots,2k$ approve all of $c_{k+1},\ldots,c_{2k}$ and c_{i-k}. Now, the optimal committee with regard to the maximin support would include the candidates $c_1,\ldots,c_k$ with an optimal maximin support of $\frac{n}{k} = 2$. On the other hand, to satisfy EJR we need to include at least $\frac{k}{2}-1$ of the candidates $c_{k+1},\ldots,c_{2k}$. Hence, there must be at least $\frac{k}{2}-1$ voters from 1 to k without a candidate they approve in the committee. The maximin support is thus at least*

$\frac{1}{k}(k + \frac{k}{2} + 1) = 1.5 + \frac{1}{k}$ *and therefore a maximin support approximation of* $\frac{2}{3} + \varepsilon$ *is not possible.*

Second, we show that a simple application of a lemma from Cevallos and Stewart (2021) shows that any affordable committee, completed with the maximin support rule, achieves an approximation to the maximin support of $\frac{1}{2}$.

Theorem 17. *Let W be an affordable committee. Then the committee obtained by completing W by running the maximin support rule approximates the maximin support objective by a factor of $\frac{1}{2}$.*

This approximation factor is currently the best known polynomial-time approximation to the maximin support. Interesting question for future work include (i) whether there is a polynomial-time rule with a better approximation factor than $\frac{1}{2}$, and (ii) whether there always exists a committee satisfying EJR and providing an approximation to the maximin support value of better than $\frac{1}{2}$.

5.2 Other Completions

Next, we consider two completion methods that have been suggested in the PB literature (Rey and Maly 2023) for the Method of Equal Shares: "completion by varying the budget" and "completion by perturbation."

Completion by Varying the Budget This completion method finds the first committee size $k' \geq k$ for which MES would select at least k candidates. If, for committee size k', MES selects a committee of size exactly k, then this committee is output. Otherwise, the completion method outputs the committee selected for committee size $k' - 1$. This, however, does not need not be exhaustive (or even select any candidates) as the following simple example shows.[6]

Example 3. *Consider an instance with $n = 6, k = 2$ and three candidates c_1, c_2, c_3 such that each candidate is approved by exactly two voters. For $k = 2$, MES would output nothing, while for $k = 3$ it would choose all candidates.*

This shows that a second completion step is necessary. Boehmer et al. (2023) and Faliszewski et al. (2023) suggest to use (a PB analog of) AV for this second step. We show that this has the same guarantees as MES completed with AV.

Theorem 18. *MES completed by varying the budget and AV achieves a representation guarantee of $\frac{1}{k}$ and a utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{2}{k}$. Both bounds are asymptotically tight.*

Completion by Perturbation This variant, suggested by Peters, Pierczyński, and Skowron (2021), is also known as the "epsilon method." We give it here in the formulation of Faliszewski et al. (2023). When MES terminates, the approvers of each unselected candidate $c \notin W$ do not have enough budget left to fund c. The completion method calculates, for each $c \notin W$, the ρ such that $\sum_{i \in N_c} b_i + \sum_{i \notin N_c} \min(b_i, \rho) = 1$. It selects the candidate minimizing ρ and updates the budgets accordingly. Thus, voters *not* approving the candidate help fund it, but should spend as little as possible. It is easy to see that this completion method makes MES exhaustive. We show that it achieves a representation guarantee of $\frac{1}{2}$, but a utilitarian guarantee of only $\frac{1}{k}$.

Theorem 19. *MES completed by perturbation has a utilitarian guarantee of $\Theta(\frac{1}{k})$ and a representation guarantee of $\frac{1}{2}$. Both bounds are asymptotically tight.*

[6]We note that, due to the non-monotonicity of MES, this method does not necessarily select a superset of the committee selected by MES (Rey and Maly 2023), and is thus not a completion method in the technical sense.

6 Conclusion and Future Work

We studied trade-offs between proportionality, coverage (aka diversity), and utilitarian welfare (aka individual excellence) in the setting of approval-based multiwinner voting. We showed that very good compromises, even under strong proportionality notions, can be achieved by completing so-called *affordable* committees. Rules that always output affordable committees include the Method of Equal Shares (Peters and Skowron 2020) and the Greedy Justified Candidate Rule (Brill and Peters 2023). For the latter, we showed that it can be completed to achieve an optimal utilitarian guarantee of $\frac{2}{\sqrt{k}} - \frac{1}{k}$, thereby answering an open question of Elkind et al. (2022). For diversity, we showed that any affordable committee can be completed to satisfy a representation guarantee of $\frac{3}{4}$. We also studied trade-offs between both guarantees and showed that they can be approximated close to optimal simultaneously, while still satisfying EJR+.

We highlight two directions for future work that go beyond approval-based multiwinner voting. For elections based on ordinal (rank-order) preferences, trade-offs between (the ordinal analogue of) the Chamberlin–Courant rule and the Borda rule, as well as the Monroe rule, have been studied (Kocot et al. 2019; Faliszewski and Talmon 2018). It would be interesting to extend this line of research by analyzing the "price of fairness" for axioms such as *proportionality for solid coalitions* (Dummett 1984) or the recently introduced notions of Brill and Peters (2023). Furthermore, ordinal rules such as STV (Tideman 1995) or the Expanding Approvals Rule (Aziz and Lee 2020) may require completion methods as well, in the case of truncated ballots.

Moreover, it would be interesting to revisit the participatory budgeting setting. Although the results by Fairstein et al. (2022) were mostly negative, there remain several possibilities for more positive results. As a first example, we note that Fairstein et al. (2022) exclusively look at the utility function that counts the approved projects in the outcome. This, however, is not the only way to measure utility. For instance, in real-world applications of MES, the utility of a voter is often measured by the *cost* of approved projects in the outcome (rather than their number); and several other utility functions have been discussed in the PB literature (Brill et al. 2023). It is, therefore, natural to ask whether the impossibility results of Fairstein et al. (2022) extend to these functions, or whether positive results are possible.

As a second example, a lot of axiomatic work has focused on "relaxed" notions of fairness, such as *EJR up to one project* (Peters, Pierczyński, and Skowron 2021). An intriguing question is whether meaningful utilitarian and representation guarantees *up to one project* can be achieved.

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